TCC - HILBERT SCHEMES AND MODULI SPACES - LECTURE 7

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1. Multigraded Hilbert schemes

The multigraded Hilbert scheme of Haiman and Sturmfels [HS04] parameterises subschemes of affine space invariant under the action of an abelian group.

Definition 1. A grading of a polynomial ring $S = K[x_1, \dots, x_n]$ by an abelian group A is given by a semigroup homomorphism $\deg\colon \mathbb{N}^n \to A$. This defines $\deg(\mathbf{x}^{\mathbf{u}})$ for $\mathbf{u} \in \mathbb{N}^n$, and induces a decomposition

$$S \cong \oplus_{a \in A} S_a$$

with $S_a \cdot S_b \subseteq S_{a+b}$ for all $a, b \in A$. We will always assume that A is finitely generated by $\deg(x_1), \dots, \deg(x_n)$, as the subgroup generated by these elements contains all degrees $a \in A$ for which $S_a \neq 0$.

Note that every polynomial $f \in S$ can be written uniquely as $f = \sum_{a \in A} f_a$, where $f_a \in S_a$ for each a, and all but finitely many of the f_a are zero.

Example 2.
1. $A = \mathbb{Z}$, and $\deg\colon \mathbb{N}^n \to \mathbb{Z}$ is given by $\deg(\mathbf{e}_i) = 1$ for $1 \leq i \leq n$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{N}^n$. This is the "standard grading".
2. $A = \{0\}$, and $\deg\colon \mathbb{N}^n \to A$ is given by $\deg(\mathbf{u}) = 0$ for all $\mathbf{u} \in \mathbb{N}^n$. This is the trivial grading.
3. $A = \mathbb{Z}/3\mathbb{Z}$, and $\deg\colon \mathbb{N}^2 \to A$ is given by
$$\deg((1,0)) = 1 \mod 3, \ \deg((0,1)) = 2 \mod 3.$$
4. $A = \mathbb{Z}^2$, and $\deg\colon \mathbb{N}^4 \to \mathbb{Z}^2$ is given by
$$\deg((1,0,0,0)) = (3,0), \ \deg((0,1,0,0)) = (2,1), \\ \deg((0,0,1,0)) = (1,2), \ \deg((0,0,0,1)) = (0,3).$$

An ideal $I \subseteq S$ is homogeneous with respect to a grading by A if I is generated by homogeneous elements (there is a generating set where each generator is an element of S_a for some a; the generators do not all have to have the same degree). As with usual $\mathbb{Z}$-gradings, an ideal I is homogeneous if and only if whenever $f \in I$, each of its graded pieces $\{f_a : a \in A\}$ is also in I.

The key lemma that motivates discussing unusual gradings is the following.

Lemma 3. *A grading of $S = K[x_1, \dots, x_n]$ by an abelian group A corresponds to a linear action of $A^* = \mathrm{Hom}_{gp}(A, K^*)$ on $\mathbb{A}^n$. An ideal $I \subseteq S$ is homogeneous with respect to the grading if and only if* $\mathrm{Spec}(S/I)$ *is invariant under the group action.*

The first example of this is the correspondence between the standard grading and the diagonal action of K^* on $\mathbb{A}^n$. Given a grading deg on S, we construct an action on S by $\phi \cdot x_i = \phi(\deg(\mathbf{e}_i)x_i$ for $\phi \in A^* = \mathrm{Hom}(A, K^*)$. For homogeneous polynomials $f \in S_a$, $\phi \cdot f = \phi(a)f$ for all $\phi \in A^*$, so if I is homogeneous we have $\phi \cdot I = I$.

Recall that an R-module M is *locally free* if there are $f_1, \dots, f_r \in R$ generating the unit ideal for which the localization M_{f_i} is a free R_{f_i} module for $1 \leq i \leq r$. For example, $\mathbb{Z}^2/\mathbb{Z}(2,3)$ is a locally free $\mathbb{Z}$-module.

Definition 4. Fix a grading $\deg\colon \mathbb{N}^n \to A$ and a commutative ring R. A homogeneous ideal $I \subseteq R[x_1, \dots, x_n]$ is *admissible* if $(S/I)_a$ is a locally free R-module of finite rank, constant on $\mathrm{Spec}(R)$, for all $a \in A$.

Example 5.
(1) Let $A = \{0\}$. Then I is admissible if S/I is a locally free R-module of constant finite rank.
(2) Let $A = \mathbb{Z}/3\mathbb{Z}$ and $n = 2$, with $\deg(x) = 1 \mod 3$ and $\deg(y) = 2 \mod 3$. Let $R = \mathbb{C}$. Then $I = \langle x^2, xy, y^2 \rangle$ is admissible, as $(S/I)_0$ is free with basis 1, $(S/I)_1$ is free with basis x, and $(S/I)_2$ is free with basis y. However $J = \langle x \rangle$ is not admissible, a $(S/I)_0$ is not finite rank as a $\mathbb{C}$-module (finite dimensional as a vector space over $\mathbb{C}$).

Definition 6. The Hilbert function of an admissible ideal $I \subseteq S = R[x_1, \dots, x_n]$ is the function $h_I\colon A \to \mathbb{N}$ given by

$$h_I(a) = \mathrm{rank}(S/I)_a, \quad \text{for all } a \in A,$$

where rank() means the rank as a locally free R-module.

Example 7.
(1) When $A = \{0\}$, $h_I(0)$ is the rank of S/I.
(2) Let $A = \mathbb{Z}/3\mathbb{Z}$, and $\deg\colon \mathbb{N}^2 \to A$ be given by $\deg((1,0)) = 1 \mod 3$, and $\deg((0,1)) = 2 \mod 3$. The Hilbert function of $I = \langle x^2, xy, y^2 \rangle$ is $h_I(0) = h_I(1) = h_I(2) = 1$.
(3) Let $A = \mathbb{Z}^2$, and $\deg\colon \mathbb{N}^4 \to A$ be given by $\deg(\mathbf{e}_1) = (3,0)$, $\deg(\mathbf{e}_2) = (2,1)$, $\deg(\mathbf{e}_3) = (1,2)$, $\deg(\mathbf{e}_4) = (0,3)$. The ideal

$I = \langle x_0x_3 - x_1x_2, x_0x_2 - x_1^2, x_1x_3 - x_2^2\rangle \subseteq \mathbb{C}[x_0, x_1, x_2, x_3]$ is homogeneous with respect to this grading, with Hilbert function $h_I(\mathbf{a}) = 1$ if $a_1, a_2 \geq 0$, $a_1 + a_2 = 0 \mod 3$, and $h_I(\mathbf{a}) = 0$ otherwise.

Definition 8. Fix a function $h\colon A \to \mathbb{N}$. The multigraded Hilbert functor Hilb_S^h is the functor from the category of commutative rings to the category of sets that sends a ring R to

$$\{\text{admissible ideals } I \subseteq R[x_1, \ldots, x_n] \text{ such that} (R[x_1, \ldots, x_n]/I)_a \text{ is locally free of rank } h(a) \text{ for all } a \in A\}.$$

The functor takes a ring homomorphism $\phi\colon R \to S$ to the map of sets that takes $I \subseteq R[x_1, \ldots, x_n]$ to $IS[x_1, \ldots, x_n]$.

Theorem 9 (Haiman-Sturmfels [HS04])**.** *The functor* Hilb_S^h *is representable by a quasiprojective scheme, called the multigraded Hilbert scheme. If the grading* $\deg\colon \mathbb{N}^n \to A$ *is positive (*$\deg^{-1}(0) = \{0\}$*), then the multigraded Hilbert scheme* Hilb_S^h *is projective.*

Example 10.
(1) When $A = \{0\}$, and $h : \{0\} \to \mathbb{N}$ is given by $h(0) = N$, then $\mathrm{Hilb}_S^h = \mathrm{Hilb}^N(\mathbb{A}^n)$ is the Hilbert scheme of N points in $\mathbb{A}^n$.
(2) Fix $A = \mathbb{Z}$, the standard grading $\deg(x_i) = 1$ for $1 \leq i \leq n$, and a Hilbert polynomial P. Let D be the Gotzmann number of P. Define $h : \mathbb{Z} \to \mathbb{N}$ by
$$h(a) = \begin{cases} 0 & a < 0 \\ \binom{n-1+a}{n-1} & 0 \leq a < D \\ P(a) & a \geq D \end{cases}.$$
Then Hilb_S^h is the usual Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^{n-1})$. To see this, we just need to construct a natural isomorphism between the two moduli functors. This takes an ideal with Hilbert function h in $R[x_1, \ldots, x_n]$ to its saturation with respect to $\langle x_1, \ldots, x_n\rangle$. The fact that this ideal defines a flat family over $\mathrm{Spec}(R)$ follows from the local freeness condition.
(3) Let $A = \mathbb{Z}^d$, and fix $\deg\colon \mathbb{N}^n \to A$. Let $h\colon A \to \mathbb{N}$ be given by $h(a) = 1$ if $S_a \neq 0$, and $h(a) = 0$ if $S_a = 0$. Then Hilb_S^h is the *toric Hilbert scheme* [PS02].
(4) Let A be a finite abelian group, and suppose $\deg\colon \mathbb{N}^n \to A$ is surjective. Let $h\colon A \to \mathbb{N}$ be defined by $h(a) = 1$ for all $a \in A$. Then Hilb_S^h is the G-Hilbert scheme for $G = A$ [Nak01].

Remark 11. Since every $\mathrm{Hilb}_P(\mathbb{P}^n)$ is a multigraded Hilbert scheme, every singularity type occurs on some multigraded Hilbert scheme. However multigraded Hilbert schemes can be disconnected. The smallest known example is due to Santos [San05], and is an example of a toric Hilbert scheme ($h(a) = 1$ if $S_a \neq 0$) coming from a grading $\deg \colon \mathbb{N}^{26} \to \mathbb{Z}^6$.

2. Coarse moduli spaces and the moduli spaces of curves

In the first lecture we discussed moduli functors of the form

$$B \mapsto \{ \text{ a family of objects being parameterised over a base } B\}/\sim$$

where $\sim$ is a notion of equivalence. So far the equivalence relation $\sim$ has been trivial. We now consider an important special case where it is not.

We first do a toy example (taken from [Cav16]). Consider in the category of topological spaces the moduli problem of isomorphism classes of unit length line segments in the plane up to rigid motion (isometry) in the plane.

Formally, this takes a topological space X to the set of isomorphism classes of families

$$\begin{array}{c} Y \subseteq X \times \mathbb{R}^2 \\ \Big\downarrow{\scriptstyle \pi} \\ X \end{array}$$

where π is a surjective continuous function with $\pi^{-1}(x)$ a unit length line segment in the plane. Here two families $\pi \colon Y \to X$ and $\pi' \colon Y' \to X$ are isomorphic if there is a homeomorphism $i : Y \to Y'$ such that

$$\begin{array}{ccc} Y & \xrightarrow{\quad i \quad} & Y' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & X & \end{array}$$

commutes, and $i|_{\pi^{-1}(x)}$ is a rigid motion of $\mathbb{R}^2$ for all $x \in X$. A continuous function $f : X \to X'$ applied to a family $\pi \colon Y \to X$ induces the pullback family $\pi' \colon Y' \to X'$, where $Y' = Y \times_X X' = \{(y, x') : \pi(y) = f(x)\}$.

We first note that if a fine moduli space M with universal family $\pi \colon U \to M$ exists or this problem, then M must be a single point. Indeed, if X is a point, there is only one family up to isomorphism, so for any map $X \to M$ the pullback family $U \times_M X \to M$ is isomorphic. Since the map $X \to M$ corresponding to a family $Y \to X$ is unique for a fine moduli space, M must be a point. This means that every family

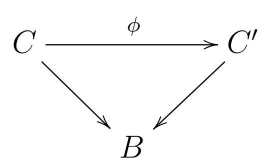
$Y \to X$ would be trivial, as $Y = X \times_M U = X \times U$. However this is not the case, as we can construct nontrivial families. For example, consider the family over S^1, which we parameterize by the angle θ, for which the fiber over the point θ is the line segment with mid-point $(1/2, 0)$ making angle θ with the x-axis. This is not a trivial family.

The same phenomenon occurs for the moduli space of curves. We would like to study smooth curves of genus g up to isomorphism. This is the moduli space M_g. This is a central moduli space in algebraic geometry, with deep roots in the nineteenth century. It is also studied in topology as the moduli space of Riemann surfaces (where it is confusingly often just called "moduli space").

Formally we have the functor

(1)
$$B \mapsto \{ \text{ isomorphism classes of flat proper families } \pi\colon C \to B \text{ whose fibers are smooth complete connected curves of genus } g\}.$$

Here $C \to B$ is isomorphic to $C' \to B$ if there is an isomorphism $\phi\colon C \to C'$ with



However this is not representable. The issue, as in our toy example, is automorphisms of curves. A heuristic is that automorphisms are almost always what causes problems with moduli functors being representable.

For example, suppose that there was a fine moduli space M for smooth proper connected curves of genus one. Then we would have a universal family $\pi\colon U \to M$, and every family $\psi\colon F \to B$ of smooth curves of genus one would come from π by pullback along a unique morphism $B \to M$.

Consider $F = V(y^2z - x^3 - tz^3) \subseteq \mathbb{P}^2 \times (\mathbb{A}^1 \setminus \{0\})$. The projection $\pi\colon F \to (\mathbb{A}^1 \setminus \{0\})$ is a family of smooth connected proper curves of genus one. Note that every fiber is isomorphic to $E = V(y^2z - x^3 - z^3)$ (for example, we may scale x by $t^{1/3}$ and y by $t^{1/2}$), so the map $\mathbb{A}^1 \setminus \{0\} \to M$ must have image a point. However the family π is not the trivial family, so cannot be the pullback.

See [HM98, §2.A] for more on why M_g is not representable.

A first solution to this problem is to ask only for a *coarse moduli space*.

Definition 12. A scheme M is a *coarse moduli space* for a moduli functor F if there is a natural transformation ψ_M from F to $\mathrm{Hom}(-, M)$ such that

1. for all algebraically closed fields K, $\psi_{\mathrm{Spec}(K)}\colon F(\mathrm{Spec}(K)) \to \mathrm{Hom}(\mathrm{Spec}(K), M)$ is a bijection of sets, and
2. given another scheme M' and a natural transformation $\psi_{M'}$ from F to $\mathrm{Hom}(-, M')$ there is a *unique* morphism $M \to M'$ such that the natural transformation $\Pi : \mathrm{Hom}(-, M) \to \mathrm{Hom}(-, M')$ satisfies $\psi_{M'} = \Pi \circ \psi_m$.

The second condition implies that a coarse moduli space is unique.

The alternative (more common now) apprach is to instead consider the *moduli stack*. That would require a whole module by itself, so we will not consider that here.

References

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