TCC - HILBERT SCHEMES AND MODULI SPACES - LECTURE 6

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In the last lecture we saw that $\mathrm{Hilb}^N(\mathbb{A}^2)$ is smooth. In this lecture we discuss pathologies of Hilbert schemes.

1. The smoothable component of the Hilbert scheme of points

We first show that the closure of the locus of N distinct points is an irreducible component of $\mathrm{Hilb}^N(\mathbb{A}^d)$ for any d. This locus has dimension Nd, and is contained in an irreducible component, so it suffices to find a point in this locus with tangent space of dimension Nd. That point will then be smooth and the dimension of the component is then Nd.

We will show this for a complete intersection of N points in $\mathbb{A}^d$. Let

$$I = \langle f, x_2, \dots, x_d \rangle \subseteq S := K[x_1, \dots, x_d],$$

where f is a polynomial of degree N in x_1 with distinct roots. As discussed last time, the dimension of the tangent space to $[I]$ in $\mathrm{Hilb}^N(\mathbb{A}^d)$ is

$$\dim_K \mathrm{Hom}_S(I, S/I).$$

An element $\phi \in \mathrm{Hom}(I, S/I)$ is determined by d elements of S/I, giving by $\phi(f), \phi(x_2), \dots, \phi(x_d)$. Since $\dim_K S/I = N$, the space of such choices has dimension Nd, so this is an upper bound for the dimension of $\mathrm{Hom}_S(I, S/I)$. As Nd is also our lower bound for the dimension of the tangent space, we must have $\mathrm{Hom}_S(I, S/I) = Nd$, and so I is a smooth point of $\mathrm{Hilb}^N(\mathbb{A}^d)$. Note that this implies that the syzygy constraints on the choices of $\phi(f)$ and $\phi(x_i)$ for $2 \leq i \leq n$ are trivial. This can be seen directly in examples: for example, the constraint that $x_2\phi(x_3) = x_3\phi(x_2)$ is immediate from the fact that both are zero in S/I.

The irreducible component containing the locus of N reduced points is called the *smoothable component* of the Hilbert scheme of points. It has an explicit description as a blow-up of $(\mathbb{A}^d)^N/S_N$ [ES14].

2. The Hilbert scheme of points in $\mathbb{A}^3$

We first observe that the Hilbert scheme $\mathrm{Hilb}^N(\mathbb{A}^3)$ can be singular.

Example 1. Consider $N = 4$. For a monomial ideal M, we can generalise the description we saw in the last lecture of the basis for the tangent space $\mathrm{Hom}_S(M, S/M)$ given by nonzero equivalence classes of arrows from minimal generators for M to monomials not in M. Let $M = \langle x^2, xy, xz, y^2, yz, z^2\rangle \subseteq S := K[x, y, z]$. The equivalence classes are determined by monomial syzygies on the generators. A direct calculation shows that the arrows (generator, monomial not in M)

$$\begin{array}{ccc}
(x^2, x) & (x^2, y) & (x^2, z) \\
(xy, x) & (xy, y) & (xy, z) \\
(xz, x) & (xz, y) & (xz, z) \\
(y^2, x) & (y^2, y) & (y^2, z) \\
(yz, x) & (yz, y) & (yz, z) \\
(z^2, x) & (z^2, y) & (z^2, z)
\end{array}$$

are all nonzero and not equivalent, so $\dim_S(M, S/M) = 18$. However M is on the smoothable component of $Hilb^4(\mathbb{A}^3)$ (for example, it is the initial ideal of the ideal of the 4 points $\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$), which has dimension $12 = (4)(3)$, so $[M]$ is not a smooth point of $\mathrm{Hilb}^4(\mathbb{A}^3)$.

We next show that the Hilbert scheme $\mathrm{Hilb}^N(\mathbb{A}^3)$ is reducible for $N \gg 0$. We will show this by showing that a large Grassmannian of dimension greater than $3N$ embeds into $\mathrm{Hilb}^N(\mathbb{A}^3)$ for $N \gg 0$.

Set $S = K[x, y, z]$. Fix a degree r and $0 < s < \dim_K S_r$. Set $N = \sum_{i=0}^{r-1} \dim_K S_i + s = \binom{r-1+3}{3} + s = \binom{r+2}{3} + s$. Then for every subspace $L \subseteq S_r$ of dimension $\dim_K S_r - s$, the ideal $I_L = \langle L\rangle + S_{\geq r+1}$ has dimension $\dim_K(S/I_L) = N$. This defines a flat family over the Grassmannian $\mathrm{Gr}(\dim_K S_r - s, S_r)$, and thus an embedding of this Grassmannian $\mathrm{Gr}(\binom{r+2}{2} - s, \binom{r+2}{2})$ into $\mathrm{Hilb}^N(\mathbb{A}^3)$.

To show that $\mathrm{Hilb}^N(\mathbb{A}^3)$ is reducible, it suffices to show that we can choose r, s so that the dimension of this Grassmannian, $s(\binom{r+2}{2} - s)$, is larger than the dimension $3N = 3\binom{r+2}{3} + 3s$ of the smoothable component.

To simplify the computation, choose $r \equiv 3 \mod 4$, so $\dim S_r = \binom{r+2}{2} = 1/2(r + 2)(r + 1)$ is even, and choose $s = 1/2\binom{r+2}{2}$. Then

$$s\left(\binom{r+2}{2} - s\right) = 1/4\binom{r+2}{2}^2$$

and

$$3\binom{r+2}{3} + s = 3\binom{r+2}{3} + 3/2\binom{r+2}{2}.$$

The former is a polynomial of degree 4 in r, while the latter is a polynomial of degree 3. Thus for $r \gg 0$ the dimension of the Grassmannian is larger, so $\mathrm{Hilb}^N(\mathbb{A}^3)$ is reducible.

A careful use of an argument of this form by Iarrobino [Iar72] [Iar84] shows that $\mathrm{Hilb}^d(\mathbb{A}^3)$ is reducible for $d \geq 78$. It is known to be irreducible for $d \leq 11$ [DJNT17], but the precise transition point is unknown. The story is simpler for larger d: $\mathrm{Hilb}^N(\mathbb{A}^d)$ is reducible for $N \geq 8$ for $d \geq 4$, and irreducible for $N < 8$ in this range; see [CEVV09].

For smoothness, the story is (surprisingly!) simpler, thanks to very recent work of Skjelnes and Smith [RSaGGS20], who completely characterise for which Hilbert polynomials $\mathrm{Hilb}_P(\mathbb{P}^n)$ is a smooth variety.

3. Murphy's law for Hilbert schemes

The pathologies we have discussed so far are fairly tame. We now discuss how bad things can be. The informal slogan is:

Murphy's Law for Hilbert Schemes "There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme".

This is a quote from [HM98, §1D]. This expectation goes back to Mumford [Mum62]. Mumford showed that $\mathrm{Hilb}_{14t-23}(\mathbb{P}^3)$ has an irreducible component that is everywhere nonreduced, even though the component generically parameterizes smooth irreducible curves of degree 14 and genus 24. The parameterized curves are those lying on a smooth cubic surface S and linearly equivalent in S to $4H + 2L$, where H is the hyperplane divisor, and L is a line on S. See [EH00] exercises VI-35, VI-36, and VI-37.

The slogan was formalised by Vakil in [Vak06]. A morphism $\phi\colon X \to Y$ of schemes of finite type over K is *smooth* if it is flat, and every fiber is geometrically regular (so still nonsingular when we pass to the algebraic closure). We say $\phi\colon (X, p) \to (Y, q)$ is a smooth morphism of pointed schemes if $p \in X, q \in Y$, ϕ is a smooth morphism, and $\phi(p) = q$.

Definition 2. A *singularity type* is an equivalence class of pointed schemes defined by setting $(X, p) \sim (Y, q)$ if $\phi\colon (X, p) \to (Y, q)$ is a smooth morphism.

For example, the projection map $\phi\colon X \times \mathbb{A}^n \to X$ is smooth, so $(X \times \mathbb{A}^n, (p, 0)) \sim (X, p)$ for any point $p \in X$.

Definition 3. We say that *Murphy's law holds* for a moduli space $\mathcal{M}$ if for every singularity type of pointed schemes appears on $\mathcal{M}$. This means that there is a point $q \in \mathcal{M}$ with the completed local ring $\widehat{\mathcal{O}_{\mathcal{M}}, q}$

isomorphic to $\widehat{\mathcal{O}_{X,p}}$ for some representative (X, p) of the singularity type.

Vakil's result is then:

Theorem 4. [Vak06] *The Hilbert scheme* $\mathrm{Hilb}(\mathbb{P}^n)$ *satisfies Murphy's law for large* n. *In particular, this holds for the Hilbert scheme of surfaces in* $\mathbb{P}^4$.

Note that we need to talk about "singularity types" here instead of just asking that a singularity appears. For example, if (X, p) is a double point p, which has local ring $K[x]/\langle x^2\rangle$, for this to appear in some component of some Hilbert scheme we would have to have the local ring to $\mathrm{Hilb}_P(\mathbb{P}^n)$ at some point $[X]$ to be zero-dimensional. Since $\mathrm{PGL}(n+1)$ acts on $\mathrm{Hilb}_P(\mathbb{P}^n)$, X must be fixed by this action, so the only possibility is that $X = \mathbb{P}^n$, or $X = \emptyset$, so $P = \binom{t+n}{n}$, or $P = 0$. In either case $\mathrm{Hilb}_P(\mathbb{P}^n)$ is a single reduced point. Thus (X, p) does not appear in any Hilbert scheme. However we can still look for something else in the equivalence class (such as $(\mathrm{Spec}(K[x_1, \dots, x_n]/\langle x_1^2\rangle), (0, \dots, 0))$ for some $n > 1$).

The Murphy's law result implies the existence of non-reduced points on the Hilbert scheme. To see this, take $(X, p) = (\mathrm{Spec}(K[x]/\langle x^2\rangle), 0)$. Any (Y, q) in the same equivalence class has $\mathcal{O}_{Y,q}$ non-reduced. It also implies the existence of components of the Hilbert scheme that only exist in characteristic $p > 0$. Formally, we treat the Hilbert scheme over $\mathrm{Spec}(\mathbb{Z})$; everything goes through, as wherever we seemed to actually need a field K as opposed $\mathbb{Z}$, we were dealing with fibers. We then want to show the existence of irreducible components that lie entirely over the fiber over $\langle p\rangle \subseteq \mathbb{Z}$. For this, take $(X, p) = (\mathrm{Spec}(\mathbb{Z}/p\mathbb{Z}), 0)$. If $(Y, q) \sim (X, p)$, then $p\mathcal{O}_{Y,q} = 0$, so any point with this singularity type lives only in characteristic p.

One case left open in Vakil's original paper is whether the Hilbert scheme of points satisfies Murphy's law. This has recently been resolved by Jelisiejew.

Theorem 5. [Jel20] *Murphy's law holds for* $\mathrm{Hilb}_{pts}(\mathbb{A}^{16})$.

The key part of the proof is to reduce to another moduli space where Murphy's law also holds. This the moduli space of point-line incidences in $\mathbb{P}^2$.

An *incidence scheme* of points and lines in $\mathbb{P}^2$ is a locally closed subscheme of $(\mathbb{P}^2)^m \times (\mathbb{P}^{2^\vee})^n = \{(p_1, \dots, p_m, l_1, \dots, l_n)\}$ parameterizing $m \geq 4$ distinct marked lines and n distinct lines in $\mathbb{P}^2$ with prescribed incidences and nonincidences (p_i lies on line l_j or p_i does not lie on line l_j). We normalise by setting $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, $p_3 = [0 : 0 : 1]$, and $p_4 = [1 : 1 : 1]$; this is quotienting by the PGL(3) action, under the assumption that no three of these points are collinear. We also require any pair of lines to contain a common marked point, and any line to contain at least three marked points.

Example 6. Consider the line arrangement (where the circle is a line) shown in Figure 1.

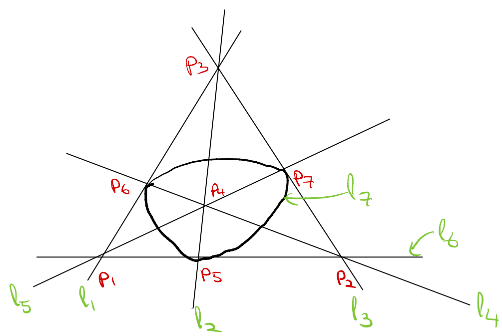

FIGURE 1.

This diagram encodes the required incidences and non-incidences. A closed point of the incidence scheme has the form

$$\left(\begin{bmatrix} 1 & 0 & 0 & 1 & a_1 & b_1 & c_1 \\ 0 & 1 & 0 & 1 & a_2 & b_2 & c_2 \\ 0 & 0 & 1 & 1 & a_3 & b_3 & c_3 \end{bmatrix}, \begin{bmatrix} d_1 & e_1 & f_1 & g_1 & h_1 & i_1 & j_1 \\ d_2 & e_2 & f_2 & g_2 & h_2 & i_2 & j_2 \\ d_3 & e_3 & f_3 & g_3 & h_3 & i_3 & j_3 \end{bmatrix} \right).$$

Entering the incidence equations, we get the following:

(1) Point p_1 lies on line 1, so $(1, 0, 0) \cdot (d_1, d_2, d_3) = d_1 = 0$. Since p_3 also lies on line 1, $d_3 = 0$, so we may set $d_2 = 1$. Since p_6 lies on line 1, $b_2 = 0$.
(2) Points p_3, p_4, p_5 lie on line 2, so $e_3 = 0 = e_1 + e_2 = 0$, and thus $[e_1 : e_2 : e_3] = [1 : -1 : 0]$. In addition, $a_1 - a_2 = 0$.
(3) Points p_2, p_3, p_7 lie on line 3, so $f_2 = f_3 = 0$, we may set $f_1 = 1$, and $c_1 = 0$.
(4) Points p_2, p_4, p_6 lie on line 4, so $g_2 = g_1 + g_3 = 0$, we may set $g_1 = 1$, and $i_1 - i_3 = 0$.
(5) Points p_1, p_4, p_7 lie on line 5, so $h_1 = h_2 + h_3 = 0$, we may set $h_2 = 1$, and $c_2 - c_3 = 0$.

(6) Points p_1, p_2, p_5 lie on line 6, so $i_1 = i_2 = 0$, and $a_3 = 0$.

This reduces the choices to

$$\left(\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & j_1 \\ 1 & -1 & 0 & 0 & 1 & 0 & j_2 \\ 0 & 0 & 0 & -1 & -1 & 1 & j_3 \end{bmatrix} \right).$$

Finally, points p_5, p_6, p_7 lie on line 7, so $j_1 + j_2 = j_1 + j_3 = j_2 + j_3 = 0$. This is possible if and only if $\mathrm{char}(K) = 2$. So the incidence scheme is one reduced point in characteristic 2, and empty if $\mathrm{char}(K) \neq 2$.

Theorem 7 (Mnëv-Sturmfels universality)**.** *The disjoint union of all incidence schemes satisfies Murphy's law. Specifically, given a singularity (Y, q), there is a point p of an incidence scheme X and a smooth morphism $\pi\colon (X, p) \to (Y, q)$.*

A version is also found in the work of Laffourge. See [LV13] [Car15, §4] for expositions. This is actually saying that realisation spaces of *matroids* satisfy Murphy's law.

The idea of the proof is as follows. First reduce to the case that $Y = \mathrm{Spec}(K[x_1, \ldots, x_n]/\langle f_1, \ldots, f_r \rangle)$. Encode the polynomials f_i in terms of atomic operations:

(1) $x_i = x_j$;
(2) $x_i = -x_j$;
(3) $x_i + x_j = x_k$;
(4) $x_i x_j = x_k$,

by adding extra variables if necessary.

For example,

$$\begin{aligned} K[x_1]/\langle x_1^2 + x_1 + 1 \rangle &\cong K[x_1, x_2]/\langle x_2 - x_1^2, x_2 + x_1 - 1 \rangle \\ &\cong K[x_1, x_2, x_3]/\langle x_2 - x_1^2, x_1 + x_2 - x_3, x_3 + 1 \rangle. \end{aligned}$$

In this last expression the polynomials use the fourth, third, and second atomic operations respectively. We then find point-line configurations that encode these atomic operations individually, and combine.

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