TCC - HILBERT SCHEMES AND MODULI SPACES - LECTURE 5

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1. The tangent space to $\mathrm{Hilb}_P(\mathbb{P}^n)$

Recall that the Zariski tangent space to a scheme X at a K-rational point p is

$$\mathrm{Hom}_K(\mathfrak{m}/\mathfrak{m}^2, K),$$

where $\mathfrak{m} = \mathfrak{m}_{X,p}$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$, and $K = \kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$.

Lemma 1. *A $K[\epsilon]/\langle \epsilon^2 \rangle$-valued point of X is a K-rational closed point p of X together with an element of the Zariski tangent space to X at p.*

Proof. The K-algebra homomorphism $K[\epsilon]/\langle \epsilon^2 \rangle \to K$ sending ϵ to 0 induces a morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(K[\epsilon]/\langle \epsilon^2 \rangle)$. Thus a $K[\epsilon]/\langle \epsilon^2 \rangle$ -valued point of X, which is an element of $\mathrm{Hom}(\mathrm{Spec}(K[\epsilon]/\langle \epsilon^2 \rangle), X)$ determines a morphism $\mathrm{Spec}(K) \to X$, and so a K-rational closed point. An element of $\mathrm{Hom}(\mathrm{Spec}(K[\epsilon]/\langle \epsilon^2 \rangle), X)$ also gives a local homomorphism $\mathcal{O}_{X,p} \to K[\epsilon]/\langle \epsilon^2 \rangle$, so gives a map $\mathfrak{m}_p \to \langle \epsilon \rangle$ that sends $\mathfrak{m}^2$ to 0. This induces a map $\mathfrak{m}_p/\mathfrak{m}_p^2 \to \langle \epsilon \rangle \cong K$, which an element of the Zariski tangent space.

Conversely, given $p \in X$ and $t : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \langle \epsilon \rangle \cong K$, note that

$$\mathcal{O}_{X,p}/\mathfrak{m}_p^2 \cong \mathcal{O}_{X,p}/\mathfrak{m}_p \oplus \mathfrak{m}_p/\mathfrak{m}_p^2.$$

Define $\phi \colon \mathcal{O}_{X,p}/\mathfrak{m}^2 \to K[\epsilon]/\langle \epsilon^2 \rangle$ as the identity on $\mathcal{O}_{X,p}/\mathfrak{m}_p \cong K$, and t on $\mathfrak{m}_p/\mathfrak{m}_p^2$. This induces a homomorphism $\mathcal{O}_{X,p} \to K[\epsilon]/\langle \epsilon^2 \rangle$, so a morphism $\mathrm{Spec}(K[\epsilon]/\langle \epsilon^2 \rangle) \to X$. $\square$

Thus the tangent space to $\mathrm{Hilb}_P(\mathbb{P}^n)$ at a point $[X]$ is an element of $\mathrm{Hom}(\mathrm{Spec}(K[\epsilon]/\langle \epsilon^2 \rangle, \mathrm{Hilb}_P(\mathbb{P}^n))$ that maps $\mathrm{Spec}(K)$ to $[X]$. This is in natural correspondence with the set of flat families

$$\begin{array}{c} \mathcal{X} \subseteq \mathbb{P}^n_{K[\epsilon]/\langle \epsilon^2 \rangle} \\ \downarrow \\ \mathrm{Spec}(K[\epsilon]/\langle \epsilon^2 \rangle). \end{array}$$

where the fiber over $\langle \epsilon \rangle$ is X. The space of such flat families is called the space of *first order deformations of* X *in* $\mathbb{P}^n_K$.

Definition 2. The *normal sheaf* $\mathcal{N}_{X/Y}$ to a closed subscheme X of a scheme Y is the sheaf

$$\mathcal{N}_{X/Y} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_X)$$

where $\mathcal{I}$ is the ideal sheaf of X in Y.

If X is a subscheme of $\mathbb{A}^n$ defined by an ideal $I \subseteq S$, then the normal sheaf is the sheafification of $\mathrm{Hom}_S(I, S/I)$.

Theorem 3. *The space of first-order deformations of a closed subscheme* X *of a scheme* Y *is the space of global sections of the normal sheaf* $\mathcal{N}_{X/Y}$.

We will apply this next when $Y = \mathbb{A}^2$, so we give the key idea there. A family $\mathcal{X} \subseteq \mathbb{A}^2 \times \mathrm{Spec}(K[\epsilon]/\langle \epsilon^2 \rangle$ over $\mathrm{Spec}(K[\epsilon]/\langle \epsilon^2 \rangle$ has

$$\mathcal{X} = \mathrm{Spec}(K[x, y, \epsilon]/\langle \epsilon^2 \rangle + J),$$

where $J = \langle f_1 + \epsilon g_1, \dots, f_k + \epsilon g_k \rangle$ for $f_i, g_j \in K[x, y]$, and we assume that the ideal I of X equals $\langle f_1, \dots, f_k \rangle$. The key idea of the proof is to show that there is an $S := K[x, y]$-module homomorphism $I \to S/I$ given by $\phi(f_i) = g_i$ if and only if the family is flat. See [EH00, Theorem VI-29] for a full proof.

2. Smoothness of $\mathrm{Hilb}^N(\mathbb{A}^2)$

We now show that the Hilbert scheme $\mathrm{Hilb}^N(\mathbb{A}^2)$ of N points in the plane is smooth. This is an open locus in $\mathrm{Hilb}_P(\mathbb{P}^2)$ for $P(t) = N$. This is the main special case of the result of Fogarty [Fog68] that Hilbert schemes of points in smooth surfaces are smooth and irreducible. We follow the approach of Haiman [Hai98]; see also [MS05, Chapter 18].

2.1. **The singular locus (if nonempty) contains a monomial ideal.** This follows from considering the Gröbner degeneration flat family. Given an ideal $I \subseteq K[x, y]$ with $\dim_K S/I = N$, and $\mathbf{w} = (w_1, w_2) \in \mathbb{N}^2$, we construct the ideal $I_t \subseteq K[x, y, t]$ by $I_t = \langle \tilde{f} : f \in I \rangle$, where for $f = \sum c_{ij} x^i y^j$ we have $\tilde{f} = t^{\max(\mathbf{w} \cdot (i,j)} \sum c_{ij} t^{-w_1 i - w_2 j} x^i y^j \in K[x, y, t]$. The family

$$\begin{array}{c} \mathrm{Spec}(K[x, y, t]/I_t) \\ \downarrow \\ \mathbb{A}^1_t \end{array}$$

FIGURE 1.

is a flat family, so gives a morphism $\phi\colon \mathbb{A}^1 \to \mathrm{Hilb}^N(\mathbb{A}^2)$.

Suppose that $[I]$ is a singular point of $\mathrm{Hilb}^N(\mathbb{A}^2)$. For $t \neq 0$ the fiber over $t \in \mathbb{A}^1$ of the Gröbner family is isomorphic to $[I]$, has the same local dimension, and isomorphic tangent space, since $[tI]$ is obtained from I by scaling the variables by powers of t. Thus these are all singular points as well. Since the singular locus of any scheme is closed, the special fiber over $t = 0$ must also be singular. This fiber is given by the initial ideal $\mathrm{in}_{\mathbf{w}}(I)$, which is a monomial ideal for sufficiently general $\mathbf{w}$. Thus if the singular locus is nonempty, it contains a monomial ideal.

This claim is actually true for any Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$, or $\mathrm{Hilb}_P(\mathbb{A}^n)$.

2.2. **The dimension of $\mathrm{Hilb}^N(\mathbb{A}^2)$ at a monomial ideal is at least $2N$.** A monomial ideal $M \subseteq S = K[x, y]$ has minimal generators $\{x^{i_k} y^{j_k} : 0 \leq k \leq s\}$, where $i_k > i_{k+1}, j_k < j_{k+1}$ for $0 \leq k < s$.

If $\dim_K(S/M) = N$, then $j_0 = i_s = 0$. We can represent M by its staircase diagram; see Figure 1.

The number of boxes under the staircase in this picture is N. This is the Young diagram/Ferrers shape of a *partition* of N. For example, for $M = \langle x^4, x^3y, xy^2, y^4\rangle \subseteq S$ we have $N = 9 = 4 + 3 + 1 + 1$. The partition determines the monomial ideal, so the number of monomial ideals in $\mathrm{Hilb}^N(\mathbb{A}^2)$ is the number of partitions of N (see https://oeis.org/A000041).

Fix a monomial ideal M with $\dim_K(S/M) = N$. For ease of exposition we assume that $\mathrm{char}(K) = 0$ in what follows (exercise: generalise this to K an arbitrary infinite field). Consider the set $\{(i, j) \in \mathbb{N}^2 : x^i y^j \notin M\}$ as a collection of N points in $\mathbb{A}^2$, and let I be the radical ideal of all polynomials vanishing at these points. For each generator $x^{i_k} y^{j_k}$ of M, construct the polynomial

$$f_k = \prod_{l=0}^{i_k - 1} (x - l) \prod_{l=0}^{j_k - 1} (y - l).$$

Note that $f_k \in I$ for all $0 \leq k \leq s$. For any generic $\mathbf{w} = (w_1, w_2) \in \mathbb{N}^2$ we can construct the Gröbner degeneration defined by the ideal $\tilde{I} = \langle \tilde{f} : f \in I\rangle \subseteq K[x, y, t]$ as in previous lectures. We have the initial term $\tilde{f}_k|_{t=0} = x^{i_k} y^{j_k}$, so $M \subseteq \tilde{I}_0$. Since M and the initial ideal $\tilde{I}_0$ both have colength N, we must have equality. This means that M is the limit of a family of N distinct points, so lies on the same irreducible component as the locus of N distinct points in $\mathrm{Hilb}^N(\mathbb{A}^2)$. This locus is isomorphic to $\{((x_1,y_1),\dots,(x_N,y_N)) \in (\mathbb{A}^2)^N : (x_i,y_i) \neq (x_j,y_j) \text{ for } i \neq j\}/S_N$, so has dimension $2N$.

2.3. **The tangent space at a monomial ideal is $2N$-dimensional.** An element $\phi \in \mathrm{Hom}(M, S/M)$ is given by choosing where ϕ sends the generators of M, subject to the requirement that these choices should be *compatible*.

Example 4. For $M = \langle x^4, x^3y, xy^2, y^4\rangle$ we need $\phi(x^4y) = y\phi(x^4) = x\phi(x^3y)$. This comes from a *syzygy* of M.

Definition 5. The (first) syzygy module of a graded S-module P is the module of relations among the minimal generators of P. If P is generated by $p_1,\dots,p_m$, then

$$\mathrm{Syz}(P) = \{(r_1,\dots,r_m) \in S^m : \sum_{i=1}^{m} r_i p_i = 0.$$

This fits in the exact sequence:

$$0 \leftarrow P \leftarrow S^m \leftarrow \mathrm{Syz}(P) \leftarrow 0.$$

Example 6. The element $(-y, x, 0, 0) \in \mathrm{Syz}(M) \subseteq S^4$ for the monomial ideal M of Example 4.

For $(f_0,\dots,f_s) \in S^{s+1}$ to define an element $\phi \in \mathrm{Hom}(M, S/M)$ we need $\sum_{i=0}^{s} r_i f_i = 0 \in S/M$ for all elements of $\mathrm{Syz}(M)$.

Lemma 7. *The syzygy module of a monomial ideal $M \subseteq K[x,y]$ is generated by*

$$x^{i_{k-1}-i_k}\mathbf{e}_k + (-y^{j_k-j_{k-1}})\mathbf{e}_{k-1}$$

for $1 \leq k \leq s$.

Example 8. For the M of Example 4 the syzygy module $\mathrm{Syz}(M)$ is generated by

$$\{(-y, x, 0, 0), (0, -y, x^2, 0), (0, 0, -y^2, x)\}.$$

As a consequence, we have

(1)
$$\mathrm{Hom}(M, S/M) = \{(f_0,\dots,f_s) \in S^{s+1} \text{ with no terms of any } f_i \text{ in } M, \\ \text{and } x^{i_{k-1}-i_k} f_k = y^{j_k-j_{k-1}} f_{k-1} \in S/M \text{ for } 1 \leq k \leq s\}.$$

Write $m_1,\dots,m_N$ for the monomials not in M. We can write $f_k = x^{i_k}y^{j_k} + \sum_{l=1}^{N} a_{kl} m_l$ for $0 \leq k \leq s$.

Example 9. Let $M = \langle x^3, xy, y^2 \rangle \subseteq K[x,y]$. Then $N = 4$, and we write $m_1 = 1$, $m_2 = x$, $m_3 = x^2$, $m_4 = y$. We set

$$\begin{aligned} f_0 &= a_{01} + a_{02}x + a_{03}x^2 + a_{04}y \\ f_1 &= a_{11} + a_{12}x + a_{13}x^2 + a_{14}y \\ f_2 &= a_{21} + a_{22}x + a_{23}x^2 + a_{24}y \end{aligned}$$

The conditions (1) then say that we must have $yf_0 = x^2 f_1$ in S/M, so $a_{01} = a_{11} = 0$, and $xf_1 = yf_2$, so $a_{11} = a_{21} = a_{22} = 0$, so the choice becomes

$$\begin{aligned} f_0 &= a_{02}x + a_{03}x^2 + a_{04}y \\ f_1 &= a_{12}x + a_{13}x^2 + a_{14}y \\ f_2 &= a_{23}x^2 + a_{24}y \end{aligned}$$

This means that $\dim_K \operatorname{Hom}(I, S/I) = 8 = 2N$.

We represent a_{kl} as an *arrow* from $x^{i_k}y^{j_k}$ to m_l, and write (k, m_l) for the arrow.

The condition (1) says that if an arrow can be moved right and down keeping the tail on the boundary, and the head below the staircase, then the corresponding coefficient a_{kl} is the same. If the head goes past the the x-axis or the y-axis then $a_{kl} = 0$, and we call this a zero arrow. Note that if (k, m_l) is a nonzero arrow, then either the x exponent of m_l is at least i_k, or the y exponent of m_l is at least j_k (but not both). We call the arrow positive in the first case, and negative in the second.

Write

$$\begin{aligned} T(M) = \{(k, m_j) : 0 \le k \le s, 1 \le j \le N, x^{i_{k-1}-i_k} m_j \in M \text{ if } (k, m_j) \\ \text{is positive, and } y^{j_{k+1}-j_k} m_j \in M \text{ if } (k, m_j) \text{ is negative}\}. \end{aligned}$$

This is the set of nonzero arrows moved "as far up or down as possible".

We have

$$\dim_K \operatorname{Hom}(M, S/M) = |T(M)|.$$

We now show that $|T(M)| = 2N$. We will associate to each box of the Young diagram two arrows.

Given a positive arrow (k, m_j), move it right until the head is about to leave the Young diagram. To each such translated arrow we associate the box of the Young diagram with the same x coordinate as the tail, and the same y coordinate as the head. Given a negative arrow (k, m_j), move it up until the head is about to leave the Young diagram. To each translated arrow, we associate the box of the Young diagram with the same x coordinate as the head, and the same y coordinate as the tail. Note that there are exactly two arrows associated to each box of the Young diagram by this procedure, so $|T(M)| = 2N$.

References

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