TCC - HILBERT SCHEMES AND MODULI SPACES - LECTURE 4

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1. Connectness

Our next goal is:

Theorem 1 (Hartshorne [Har66])**.** *The Hilbert scheme* $\mathrm{Hilb}_P(\mathbb{P}^n)$ *is connected.*

We will actually show that it is *rationally chain connected*: for any closed point p on $\mathrm{Hilb}_P(\mathbb{P}^n)$ there is a sequence of rational curves $C_1, \dots, C_s$, where $p \in C_1$, $C_i \cap C_{i+1} \neq \emptyset$, and the point $[I_{lex}]$ corresponding to the lexicographic ideal lives in C_s. We will follow the proof given in [PS05]. One notable aspect of this proof is that one does not actually need any details of the existence of the Hilbert scheme.

1.1. Gröbner degenerations.

The key technique is that of *Gröbner degeneration*. Given a homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$ and a weight vector $\mathbf{w} \in \mathbb{N}^n$, we construct the ideal

$$I_t = \langle \tilde{f} : f \in I \rangle \subseteq K[x_0, \dots, x_n, t],$$

where for $f = \sum c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ we have $\tilde{f} = t^{\max \mathbf{w} \cdot \mathbf{u}} \sum c_u t^{-\mathbf{w} \cdot \mathbf{u}} \mathbf{x}^{\mathbf{u}}$.

Example 1. Let $I = \langle x_0 x_2 - x_1^2 \rangle \subseteq K[x_0, x_1, x_2]$. For $\mathbf{w} = (10, 5, 1)$, and $f = x_0 x_2 - x_1^2$ we have $\tilde{f} = t^{11}(t^{-11} x_0 x_2 - t^{-10} x_1^2) = x_0 x_2 - t x_1^2$. While for arbitrary ideals we do not have $\tilde{I} = \langle \tilde{f} : f$ is a minimal generator of $I \rangle$, for principal ideals it is true, so $\tilde{I} = \langle x_0 x_2 - t x_1^2 \rangle$. For $\mathbf{w} = (1, 5, 1)$ we have $\tilde{I} = \langle t^1 0 (t^{-2} x_0 x_2 - t^{-10} x_1^2 \rangle = \langle t^8 x_0 x_2 - x_1^2 \rangle$.

The ideal $\tilde{I}$ defines a subscheme of $\mathbb{P}^n \times \mathbb{A}^1$, and the inclusion $K[t] \to K[x_0, \dots, x_n, t]/\tilde{I}$ induces a morphism to $\mathbb{A}^1$:

$$\begin{array}{c} \mathrm{Proj}(K[x_0, \dots, x_n, t]/I) \subseteq \mathbb{P}^n \times \mathbb{A}^1 \\ \big\downarrow \pi \\ \mathbb{A}^1 \end{array}$$

Here we take Proj with respect to the grading $\deg(x_i) = 1$, $\deg(t) = 0$. The key fact is that π is a flat family, with all fibers over $t \neq 0$

isomorphic to $\mathrm{Proj}(K[x_0, \ldots, x_n]/I)$, and fiber over $t = 0$ equal to $\mathrm{Proj}(K[x_0, \ldots, x_n/\operatorname{in}_{\mathbf{w}}(I))$, where $\operatorname{in}_{\mathbf{w}}(I)$ is the initial ideal in the sense of Gröbner theory. See [Eis95, Chapter 15] for details on this. More information about computing these degenerations can be found in [Stu96]. The classic reference for the basics of Gröbner bases is [CLO15].

Example 2. Continuing Example 1, note that I defines the image of the Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^2$. The first degeneration breaks that into the union of the two coordinate lines $x_0 = 0$ and $x_2 = 0$, while the second degenerates it to a double line supported on $x_1 = 0$. To see that these are flat, note that in the first case the quotient $K[x_0, x_1, x_2, t]/\tilde{I}$ is a free $K[t]$-module with basis the monomials in x_0, x_1, x_2 not divisible by x_0x_2, while in the second case the quotient is a free $K[t]$-module with basis the monomials not divisible by x_1^2.

1.2. Step 1: Reduce to the case that I is Borel-fixed.

Definition 1. The group $\mathrm{GL}(n+1, K)$ acts on $K[x_0, \ldots, x_n]$ by linear change of coordinates:

$$x_i \mapsto \sum_{j=0}^{n} a_{ji} x_j.$$

For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot x_0^2 + 2x_0x_1 + x_1^2 = (x_0 + 3x_1)^2 + 2(x_0 + 3x_1)(2x_0 + 4x_1) + (2x_0 + 4x_1)^2$$
$$= 9x_0^2 + 42x_0x_1 + 49x_1^2.$$

We consider the induced action of the *Borel group* of upper triangular matrices.

Lemma 2. *When* $\mathrm{char}(K) = 0$*, an ideal* $I \subseteq K[x_0, \ldots, x_n]$ *is fixed by the action of the Borel group if and only if*

(1) *I is a monomial ideal, and*
(2) *for all monomials $\mathbf{x}^{\mathbf{u}} \in I$, and x_i dividing $\mathbf{x}^{\mathbf{u}}$, we have $x_j/x_i \mathbf{x}^{\mathbf{u}} \in I$ for all $j < i$.*

Ideals satisfying the second condition of the lemma are called *strongly stable*. The fact that I is a monomial ideal is actually a consequence of the fact that I is fixed by the algebraic torus of diagonal matrices contained in the Borel group. Diagonal matrices act on the polynomial ring by scaling variables. As an example to see that this implies that the ideal must be monomial, consider an ideal I containing $x_0 + x_1$ that is fixed by the torus action. Then $(1, t) \cdot (x_0 + x_1) = x_0 + tx_1 \in I$ for all t, so $(x_0 + x_1) - (x_0 + tx_1) = (1 - t)x_1 \in I$, so $x_1 \in I$, and so $x_0 \in I$.

Making a set-theoretic argument like this does assume that the field is infinite; to handle arbitrary fields one needs to be more careful about the definitions of these actions.

Example 3. The ideal $\langle x_0^2, x_0x_1, x_1^2\rangle \subseteq K[x_0,x_1,x_2]$ is Borel fixed, while $\langle x_0^2, x_1^2, x_2^2\rangle$ is not. In the second case, note that $0 < 1$, but $x_0/x_1(x_1^2) \notin \langle x_0^2, x_1^2, x_2^2\rangle$.

We use the following two facts about Borel-fixed ideals:

(1) The saturation of a Borel-fixed ideal is Borel-fixed, and
(2) There are only finitely many saturated Borel-fixed ideals with a fixed Hilbert polynomial.

Proposition 3. *For a fixed homogeneous ideal $I \subseteq K[x_0,\dots,x_n]$ and general $\mathbf{w}$ there is an open set $U \subseteq \mathrm{GL}(n+1,K)$ for which $\mathrm{in}_{\mathbf{w}}(gI)$ is constant for $g \in U$. This is Borel-fixed if $w_0 > w_1 > \cdots > w_n$.*

The ideal $\mathrm{in}_{\mathbf{w}}(gI)$ for $g \in U$ is called the *generic initial ideal* of I with respect to $\mathbf{w}$. For details, see [Eis95, Chapter 15].

Example 4. Let $I = \langle x_1 + x_2\rangle \subseteq K[x_0,x_1,x_2]$. For $g = (g_{ij}) \in \mathrm{GL}(3,K)$,

$$\begin{aligned} gI &= \langle g_{01}x_0 + g_{11}x_1 + g_{21}x_2 + g_{02}x_0 + g_{12}x_1 + g_{22}x_2\rangle \\ &= \langle (g_{01}+g_{02})x_0 + (g_{11}+g_{12})x_1 + (g_{21}+g_{22})x_2\rangle. \end{aligned}$$

Thus for $\mathbf{w} = (10,5,1)$, and the open set $U = \{g_{01}+g_{02} \neq 0\} \subseteq \mathrm{GL}(3,K)$ we have $\mathrm{in}_{\mathbf{w}}(gI) = \langle x_0\rangle$. Note that this is Borel-fixed.

This means that given a point $[I] \in \mathrm{Hilb}_P(\mathbb{P}^n)$ corresponding to a homogeneous ideal I, we can choose a path from I to $g \in U$, which gives a path from $[I]$ to $[gI]$ in $\mathrm{Hilb}_P(\mathbb{P}^n)$. This is part of the power of the functor language; giving a family over $\mathbb{A}^1$ is the same as giving a morphism from $\mathbb{A}^1$ to the Hilbert scheme. We can then take the Gröbner degeneration that takes $[gI]$ to $[\mathrm{in}_{\mathbf{w}}(gI)]$ for general $\mathbf{w}$. Thus $[\mathrm{gin}_{\mathbf{w}}(I)]$, which is Borel fixed, is in the same connected component of $\mathrm{Hilb}_P(\mathbb{P}^n)$ as I.

1.3. Step 2: Move towards the lexicographic ideal.

If I is a Borel fixed ideal, but is not the lexicographic ideal, then we construct an ideal J with exactly two monomial initial ideals:

$$\mathrm{in}_{\mathbf{w}}(J) = I, \;\; \mathrm{in}_{-\mathbf{w}}(J) = I'.$$

Here $\mathrm{in}_{-\mathbf{w}}(J)$ really means $\mathrm{in}_{N(1,\dots,1)-\mathbf{w}}(J)$ for $N \gg 0$; note that adding a multiple of $(1,\dots,1)$ to $\mathbf{w}$ does not change the initial ideal of a homogeneous ideal. The ideal J is chosen so that the other initial ideal I' is closer to the lexicographic ideal, in the sense that the list of monomials in I' of degree d in lexicographic order is lexicographically greater than the list of monomials in I of degree d.

Example 5. Let $P(t) = 4$ and $n = 2$. The ideals $I = \langle x_0^2, x_0x_1, x_1^3\rangle$ and $I' = \langle x_0, x_1^4\rangle \subseteq K[x_0, x_1, x_2]$ are both Borel-fixed. The monomials in I of degree 3, listed in descending lexicographic order, are

$$x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_1x_2, x_1^3,$$

while the monomials in I' are

$$x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1^2, x_0x_1x_2, x_0x_2^2.$$

Note that the these differ only in the last position, where I' is larger.

Fix a saturated Borel-fixed ideal I. If I is the lexicographic ideal I_{lex} then we are done. Otherwise let d be the smallest degree in which I_d is not a lex-segment. In [PS05] Peeva and Stillman construct a binomial ideal J as follows. Let $m \in S_d$ be the largest monomial in lexicographic order in $(I_{lex})_d \setminus I_d$. Let f be the largest monomial in lexicographic order in I_d that is smaller than m. The fact that I is Borel-fixed implies that f is a minimal generator of I. We construct the binomial $f - m$. The first draft of the construction of J is to replace f in the generating set of I by $f - m$. This doesn't quite work as desired, as there may be other generators of I that imply $f \in I$ once we know that I is Borel-fixed (in the sense that if $x_2^2 \in I$, then x_1x_2 must be). The construction in [PS05] also replaces these generators by modifications of the binomial $f - m$.

The ideal J is been constructed so that $\mathrm{in}_{(1,N,N^2,\dots,N^n)}(J) = I$ for $N \gg 0$, and there is only one other monomial initial ideal I', which is closer to I_{lex} in the above order. The saturation I'' of $\mathrm{gin}_{(N^n,\dots,N,1)}(I')$ for $N \gg 0$ is Borel-fixed, and closer to I_{lex} than I. If $I'' \neq I_{lex}$ then we repeat. Since there are only a finite number of saturated Borel-fixed ideals with Hilbert polynomial P, this procedure must terminate with $I'' = I_{lex}$ after a finite number of steps. Each step corresponds to a map $\mathbb{A}^1 \to \mathrm{Hilb}_P(\mathbb{P}^n)$, so this shows that $\mathrm{Hilb}_P(\mathbb{P}^n)$ is connected.

Example 6. Continuing Example 5, we can check that $I = \langle x_0^2, x_0x_1, x_1^3\rangle$ and $I' = \langle x_0, x_1^4\rangle$ are the only saturated Borel-fixed ideals with Hilbert polynomial $P(t) = 4$. The ideal I' is the saturated lexicographic ideal with that Hilbert polynomial. We construct

$$J = \langle x_0^2, x_0x_1, x_1^3 - x_0x_2^2\rangle.$$

The other monomial initial ideal of J is

$$\langle x_0^2, x_0x_1, x_0x_2^2, x_1^4\rangle,$$

which has saturation $\langle x_0, x_1^4 \rangle = I'$. This shows that $\mathrm{Hilb}_4(\mathbb{P}^2)$ is connected.

2. Other Hilbert schemes

Let X be a projective scheme. The Hilbert functor for X is defined as for $\mathbb{P}^n$:

$$\mathrm{Hilb}(X)(B) = \{ \text{ closed subschemes } Z \subseteq X \times B \text{ with } Z \to B \text{ flat and proper}\}.$$

We can show that this representable, by fixing an embedding $X \to \mathbb{P}^N$. This defines a Hilbert polynomial for subschemes of X, so we have the functor

$$\mathrm{Hilb}_P(X)(B) = \{ \text{ closed subschemes } Z \subseteq X \times B \subseteq \mathbb{P}^N_B \text{ flat over } B, \\ \text{where the fibres have Hilbert polynomial } P\}.$$

We get a natural transformation of functors $\mathrm{Hilb}_P(X) \to \mathrm{Hilb}_P(\mathbb{P}^N)$ by forgetting X.

To see that $\mathrm{Hilb}_P(X)$ is representable, note that in our construction of $\mathrm{Hilb}_P(\mathbb{P}^N)$ as a subscheme of a Grassmannian, adding the condition that $Z \subseteq X$ (so the ideal of X is contained in the ideal of Z) is a closed condition.

This uses the following lemma.

Lemma 4. *The locus $Z \subseteq \mathrm{Gr}(r, n)$ of r-dimensional subspaces of K^n containing a fixed vector $\mathbf{v} \in K^n$ is closed.*

Proof. Given an r-dimensional subspace V, pick a basis for V, and write this as the rows of an $r \times n$ matrix. If $\mathbf{v} \in V$, the $(r + 1) \times n$ matrix with $\mathbf{v}$ added as the first row also has rank r, so all $(r + 1) \times (r + 1)$ minors vanish. Expand these along the first row. They all have the form $\sum \pm v_{i_j} p_{I_j}$, so Z is the intersection of $\mathrm{Gr}(k, n)$ in its Plücker embedding with a subspace of $\mathbb{P}^{\binom{n}{k}-1}$. □

We apply this by fixing a degree D that is at least the Gotzmann number of P and also at least the maximum degree of a minimal generator of the ideal I_X of X. Fix a basis $f_1, \dots, f_r$ of $(I_X)_D$. We want to guarantee that $f_i \in I_D$ for $1 \leq i \leq$, so $I_X \subseteq I$. The condition that $f_i \in I_D$ is a closed condition on $\mathrm{Gr}(\binom{n+D}{n} - P(D), S_D)$ by the lemma. Intersecting these loci describes $\mathrm{Hilb}_P(X)$ as a subscheme of $\mathrm{Hilb}_P(\mathbb{P}^N)$.
Question: Does this depend on the choice of embedding of X into $\mathbb{P}^N$?
Answer: The decomposition of $\mathrm{Hilb}(X)$ into the disjoint union of $\mathrm{Hilb}_P(X)$ might vary, but the $\mathrm{Hilb}(X)$ does not. This part of the magic of representable functors! Once we show $\mathrm{Hilb}(X)$ is representable, it is unique, so the choices made in its construction disappear. The same

is true about the original construction of $\mathrm{Hilb}_P(\mathbb{P}^N)$; any choice of sufficiently large degree D gives an isomorphic scheme.
Warning: As we discussed, $\mathrm{Hilb}_P(\mathbb{P}^n)$ is always connected. However for other varieties X we might have $\mathrm{Hilb}_P(X)$ disconnected.

Example 7. Let $X = V(f) \subseteq \mathbb{P}^3$ be a smooth cubic surface. The Hilbert scheme $\mathrm{Hilb}_{t+1}(X)$ is 27 points, as a subscheme of $\mathbb{P}^3$ with Hilbert polynomial $t+1$ is a line, and X contains exactly 27 lines.

Example 8. Let X be a smooth conic in $\mathbb{P}^3$. Recall that X is a *ruled surface*. Let $P = t+1$. Then $\mathrm{Hilb}_{t+1}(X)$ has two connected components, each corresponding to one of the rulings (so isomorphic to $\mathbb{P}^1$).

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