TCC - HILBERT SCHEMES AND MODULI SPACES - LECTURE 3

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Overview of construction. Last time we introduced the Hilbert functor

$$\begin{aligned}\text{Hilb}_P(\mathbb{P}^n)(B) = &\{\text{flat families over} B \text{ of subschemes of } \mathbb{P}^n_B \text{ with Hilbert} \\ &\text{polynomial of fibres equal to } P\}.\end{aligned}$$

We now show that $\text{Hilb}_P(\mathbb{P}^n)$ is representable. The key idea is to find a degree D, depending only on P, for which

(1) Every saturated ideal I with Hilbert polynomial P is generated in degree at most D, so $I_{\geq D} = \langle I_D \rangle$, and
(2) If I is an ideal generated in degree D, and has $h_I(d) = P(d)$ for $d = D, D+1$, then I has Hilbert polynomial P.

By sending I to I_D, together these mean that we get a correspondence between saturated ideals $I \subseteq K[x_0, \dots, x_n]/I$ with Hilbert polynomial P, and a closed subscheme of the Grassmannian $\text{Gr}(\binom{n+D}{n} - P(D), \binom{n+D}{n})$, cut out by equations that ensure that $\langle I_D \rangle_{D+1}$ has the correct Hilbert function.

0.1. **Castelnuovo-Mumford regularity.** The *Castelnuovo-Mumford regularity* of a subscheme $Z \subseteq \mathbb{P}^n$ is an invariant introduced by Mumford in [Mum66] to make the construction of the Hilbert scheme more explicit.

Definition 1. Let Z be a subscheme of $\mathbb{P}^n_K$. The *regularity* of Z is

$$\text{reg}(Z) = \min\{j : H^i(\mathbb{P}^n, \mathcal{O}_Z(l-i)) = 0 \text{ for all } l \geq j, i > 0\}.$$

Equivalently, we can associate to Z a homogeneous ideal $I \subseteq S := K[x_0, \dots, x_n]$, saturated with respect to $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$. In that case the regularity of S/I is

$$\min\{j : H^i_{\mathfrak{m}}(S/I)_l = 0 \text{ for all } l + i > j, i \geq 0\},$$

where $H^i_{\mathfrak{m}}(S/I)$ is the local cohomology module, which inherits a grading from S/I. This can also be calculated as follows. We write $S(-b)$ for the polynomial ring with the grading shifted, so that

$$S(-b)_d = S_{d-b}.$$

In particular, $1 \in S$ has degree b. Suppose

$$0 \leftarrow S/I \leftarrow F_0 \leftarrow \cdots \leftarrow F_s \leftarrow 0$$

is the minimal free resolution of S/I as an S-module, where $F_i = \oplus_j S(-\beta_{ij})$, then

$$\operatorname{reg}(S/I) = \max_{i,j}(\beta_{ij} - i).$$

We will use two key facts about regularity:

(1) $F_1 = \oplus_j S(-\beta_{ij})$ surjects onto the *generators* of I, so β_{1j} are the degrees of generators. Thus $\operatorname{reg}(S/I) + 1$ is an upper bound on the degrees of generators of I.

A proof of the corresponding fact in the geometric context (if the regularity of Z is k, then $\mathcal{I}(k)$ is generated by global sections) is at the start of Chapter 14 of [Mum66], where Mumford attributes it to Castelnuovo.

(2) If I is a saturated ideal, then the Hilbert function agrees with the Hilbert polynomial from the regularity:

$$h_I(d) = p_I(d) \text{ for } d \geq \operatorname{reg}(S/I).$$

We can give a *uniform bound* on the regularity of all subschemes Z with Hilbert polynomial P. Such a bound was first shown to exist in Mumford [Mum66, Chapter 14], simplifying Grothendieck's construction of the Hilbert scheme. We will follow the treatment of Gotzmann[Got78], which uses Macaulay's characterisation of possible Hilbert polynomials.
Sample question: Is there a subscheme of $\mathbb{P}^n$ for some n with Hilbert polynomial $P(t) = t^2$?

Theorem 1 (Macaulay)**.** *The Hilbert polynomial of a homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$ can be written as*

$$p_I(t) = \sum_{j=1}^{D} \binom{t + a_i - i + 1}{a_i}, \tag{1}$$

where $a_1 \geq a_2 \geq \cdots \geq a_D \geq 0$.

In addition, if J is a homogeneous ideal in $K[x_0, \dots, x_n]$ with $h_J(d) = p_I(t)$, then $h_J(d+1) \leq p_I(d+1)$.

This means that the answer to the sample question is no! The summand $\binom{t+a_i-i+1}{a_i}$ is a polynomial in t of degree a_i, so if there is such a description for t^2, we must have $a_1 = 2$, and the first term is $\binom{t+2}{2} = 1/2(t+2)(t+1)$. Subtracting this from t^2, we are left with another polynomial of degree 2, so we also have $a_2 = 2$. However $\binom{t+2}{2} + \binom{t+1}{2} = (t+1)^2$, so subtracting this from t^2 we get $-2t - 1$.

Since the leading coefficient of $\binom{t+a_i-i+1}{a_i}$ is always positive, there is no way to write $-2t-1$ as a sum of such polynomials, so we conclude that t^2 does not have such a description, and so $p(t) = t^2$ is never a Hilbert polynomial. This means that the Hilbert scheme $\mathrm{Hilb}_{t^2}(\mathbb{P}^n)$ is empty for all n.

Question: Where does this formula come from?

Macaulay actually shows the existence of the *lexicographic ideal* with a given Hilbert function, and shows that the Hilbert function of this ideal grows as fast as possible. The lexicographic term order on monomials in $K[x_0,\dots,x_n]$ has $\mathbf{x}^{\mathbf{u}} = \prod_{i=0}^n x_i^{u_i} \prec \mathbf{x}^{\mathbf{v}}$ if the first nonzero entry of $\mathbf{v}-\mathbf{u}$ is positive. In particular, we have $x_0 \succ x_1 \succ \cdots \succ x_n$. The lexicographic ideal with Hilbert function h is the ideal I for which I_d is the span of the largest $\binom{n+d}{n} - h(d)$ monomials in lexicographic order. Macaulay's theorem is that this ideal exists for any function h that is the Hilbert function of some ideal. The content here is that

$$S_1 I_d \subseteq I_{d+1}.$$

For $d \gg 0$, we have $I_{d+1} = S_1 I_d$.

Example 1. Let $P(t) = 3t+1$, and $n = 3$. Set

$$H(t) = \begin{cases} \binom{t+3}{3} & t < 4 \\ P(t) & t \geq 4 \end{cases}.$$

Then

$$\begin{aligned} I_{lex} =& \langle x_0^4, x_0^3x_1, x_0^3x_2, x_0^3x_3, x_0^2x_1^2, x_0^2x_1x_2, x_0^2x_1x_3, x_0^2x_2^2, x_0^2x_2x_3, x_0^2x_3^2, \\ & x_0x_1^3, x_0x_1^2x_2, x_0x_1^2x_3, x_0x_1x_2^2, x_0x_1x_2x_3, x_0x_1x_3^2, x_0x_2^3, x_0x_2^2x_3, \\ & x_0x_2x_3^2, x_0x_3^3, x_1^4, x_1^3x_2\rangle \end{aligned}$$

Fix $D \gg 0$, so that $S_1(I_{lex})_D = (I_{lex})_{D+1}$, and let $m = \mathbf{x}^{\mathbf{u}}$ be the smallest monomial in $(I_{lex})_D$ with respect to the lexicographic order. For $d > D$, the smallest monomial in $(I_{lex})_d$ is mx_n^{d-D}. The Hilbert function $h(d) = \dim_K(S/I_{lex})_d$ equals the number of monomials of degree d not in I_{lex}.

Example 2. Continuing Example 1, in degree 4, $x_1^3x_2$ is the smallest element of I_{lex} of degree 4. This is $\mathbf{x}^{\mathbf{u}}$ for $\mathbf{u} = (0,3,1,0)$. Monomials not in I_{lex} of degree 4 are exactly the monomials of degree 4 less than $x_1^3x_2$ in the lexicographic term order. These are:

- $x_1^3x_3$,
- x_1^2 times any monomial of degree 2 in x_2, x_3,
- x_1 times any monomial of degree 3 in x_2, x_3, and

- any monomial of degree 4 in x_2, x_3.

There are

$$1+\binom{2+1}{1}+\binom{3+1}{1}+\binom{4+1}{1}=1+3+4+5=13=h(4)$$

such monomials, as expected. In general $x_1^3x_2x_3^{d-4}$ is the smallest monomial in I_{lex} of degree d for $d \geq 4$, and we replace $x_1^3x_3$ by $x_1^3x_3^{d-3}$, "of degree 2" by "of degree $d-2$" in the second case, and similarly 3 by $d-3$, and 4 by $d-4$. This gives a decomposition

$$\begin{aligned} h(d) &= 1+\binom{d-2+1}{1}+\binom{d-1+1}{1}+\binom{d+1}{1} \\ &= \binom{d+1}{1}+\binom{d-1+1}{1}+\binom{d-2+1}{1}+\binom{d-3+0}{0} \\ &= 3d+1 \end{aligned}$$

which is of the form of Theorem 1 with $a_1 = a_2 = a_3 = 1$, and $a_4 = 0$.

A proof of Macaulay's theorem can be found in [BH93, Chapter 4].

Theorem 2. *Let $I \subseteq K[x_0, \dots, x_n]$ be a homogeneous ideal saturated with respect to $\langle x_0, \dots, x_n\rangle$ (so $(I : \langle x_0, \dots, x_n\rangle^\infty) = I$). Write*

$$p_I = \sum_{i=1}^{D} \binom{t+a_i-i+1}{a_i}$$

where $a_1 \geq a_2 \geq \cdots \geq a_D \geq 0$. Then $\operatorname{reg}(S/I) \leq D-1$. Thus in particular I is generated in degree at most D.

The number D in Theorem 2 is called the Gotzmann number of the polynomial p_I.

Theorem 2 gives a *computable, uniform* upper bound on the regularity of all saturated ideals with a given Hilbert polynomial, and thus on the degrees of their generators.

Write $I_{\geq d} = \langle f \in I : \deg(f) \geq d\rangle$. Since $(I_{\geq d} : \langle x_0, \dots, x_n\rangle^\infty) = (I : \langle x_0, \dots, x_n\rangle^\infty)$ and $I_{\geq d} = \langle I_d\rangle$ if d is at least the maximum degree of a generator of I, if I is a saturated ideal with Hilbert polynomial P, the ideal sheaf $\mathcal{I} = \tilde{I}$ is determined by I_{Got}, where Got is the Gotzmann number of P. We henceforth write D for the Gotzmann number.

Regularity is upper semicontinuous in flat families, since the rank of cohomology is. This means that if $Z \to B$ is a flat family, where $Z \subseteq \mathbb{P}^n_B$, then the locus of $b \in B$ for which the regularity of a fibre is at least l is closed in B. Passing from an ideal to its initial ideal, in the sense of Gröbner bases, is a flat degeneration, so the largest regularity is achieved at a monomial ideal. See [Eis95, Chapter 15] for details on Gröbner degenerations. This means that to prove Theorem 2 we only need to show that $D-1$ bounds the regularity of S/I when I is a monomial ideal. There are only a finite number of saturated monomial ideals with a given Hilbert polynomial, and it turns out that the lexicographic ideal has the largest regularity. These ideas can be turned into a proof of Theorem 2. For other proofs, see [Got78] or [BH93, Chapter 4].

Gotzmann's regularity theorem implies that every subscheme of $\mathbb{P}^n_K$ with Hilbert polynomial P defines a point in $\mathrm{Gr}(\binom{n+D}{n} - h(D), \binom{n+D}{n})$, by associating to a subscheme Z the saturated ideal $I \subseteq K[x_0, \dots, x_n]$, and considering the subspace $I_D \subseteq K[x_0, \dots, x_n]_D$.

We now consider equations for this locus. This uses a second theorem of Gotzmann [Got78].

Theorem 3. *Let p be a Hilbert polynomial. If I is a homogeneous ideal with $h_I(d) = p(d)$, and $h_I(d+1) = p(d+1)$ then $h_I(m) = p(m)$ for $m \geq d$, so I has Hilbert polynomial p.*

Thus in particular if I is the ideal generated by a subspace of S_D of dimension $\binom{n+D}{n} - h(D)$, and $h_I(D+1) = p(D+1)$ then $p_I = p$. This implies that the saturation of I also has Hilbert polynomial p, and that the saturation of I is generated in degrees at most D. From this we see that $I_{\geq D} = (I : \langle x_0, \dots, x_n \rangle^\infty)_{\geq D}$.

Thus there is a bijection between

$$\{ \text{ homogeneous saturated ideals in } K[x_0, \dots, x_n] \text{ with Hilbert polynomial } p\}$$

and

$$\{ \text{ points } p \in \mathrm{Gr}(\binom{n+D}{n} - h(D), S_D) \text{ for which the ideal } \langle p \rangle \text{ has}$$
$$\text{Hilbert function } h_{\langle p \rangle}(D+1) = p(D+1)\}.$$

This extends in a natural manner to the version where K is replaced by a ring R, using the connection between locally freeness and flatness. One version of this, when B is a connected and reduced, says that a family of subschemes of $\mathbb{P}^n_B$ is flat over the base B if and only if all fibres have the same Hilbert polynomial.

Let I be generated in degree D. As $I_{D+1} = K\{S_1 I_D\}$, $h_I(D+1) = p(D+1)$ if and only if $\dim_K K\{S_1 I_D\} = \binom{n+D+1}{n} - p(D+1)$. Macaulay's theorem implies that $h_I(D+1) \leq p_I(D+1)$, so $\dim_K K\{S_1 I_D\} = \dim_K I_{D+1} \geq \binom{n+D+1}{n} - p(D+1)$. So to guarantee that $h_I(D+1) = p(D+1)$, we only need to check that $\dim_K K\{S_1 I_D\} \leq \binom{n+D+1}{n} - p(D+1)$.

This is a determinantal condition: we form the matrix whose row space is a basis for $K\{S_1 I_D\}$, and set all its minors of size $\binom{n+D+1}{n} - p(D+1) + 1$ equal to zero.

Example 3. Let $n = 1$, and $p(t) = 2$. The Gotzmann number for p is 2, but we will used $D = 3$ to better illustrate this phenomenon. Set $S = K[x_0, x_1]$. Since $\dim_K S_3 = 4$, we are looking for a locus in the Grassmannian $\mathrm{Gr}(2,4)$. In order to have fewer variables in this example, we will also work with the affine chart of the Grassmannian with $p_{x_0^3, x_0^2 x_1} \neq 0$. To complete the example, we could either work with each affine chart in turn, or work with the homogeneous coordinates on the Grassmannian.

An ideal corresponding to a point in this affine chart has the form $\langle x_0^3 + a x_0 x_1^2 + b x_1^3, x_0^2 x_1 + c x_0 x_1^2 + d x_1^3 \rangle$, where a, b, c, d are the coordinates on the affine chart $\mathbb{A}^4$ of $\mathrm{Gr}(2,4)$. The degree 4 part of this ideal is the row space of the following matrix:

$$\begin{array}{c} \begin{array}{ccccc} x_0^4 & x_0^3 x_1 & x_0^2 x_1^2 & x_0 x_1^3 & x_1^4 \end{array} \\ \begin{bmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & 0 & a & b \\ 0 & 1 & c & d & 0 \\ 0 & 0 & 1 & c & d \end{bmatrix} \end{array}$$

Since $\dim_K S_4 = 5$, and $p(5) = 2$, we want this matrix to have rank at most 3, so need all the 4×4 minors to vanish. This the requirement that

$$a + c^2 - d = b + cd = -ad + bc + d^2 = -acd + bc^2 - bd = a^2 d - abc - ad^2 + b^2 + bcd = 0.$$

Minimal generators of the ideal generated by these equations are $b + cd$ and $a + c^2 - d$, so the locus in $\mathbb{A}^4 \subseteq \mathrm{Gr}(2,4)$ with this Hilbert polynomial is a copy of $\mathbb{A}^2$.

References

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