TCC - HILBERT SCHEMES AND MODULI SPACES - LECTURE 2

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Recall: A moduli functor F : Schemes $\to$ Sets sends a scheme B to the set of families over the base B of objects being parameterised, moduli some form of equivalence relation.

A scheme X is a fine moduli space for this moduli problem if $F \cong h_X = \mathrm{Mor}(-, X)$ (the functor of points of X). The scheme X is unique, if it exists, by Yoneda's lemma.

Example 1. For the Grassmannian we have the moduli functor Schemes $\to$ Sets given by

$$B \mapsto \{\text{subsheaves } \mathcal{F} \subseteq \mathcal{O}_B^n \text{ that are locally summands free of rank } r\}.$$

Restricted to affine schemes, so $B = \mathrm{Spec}(R)$, this becomes

$$R \mapsto \{ \text{ submodules } M \subseteq R^n \text{ that are locally free direct summands of rank } r\}.$$

The property that a scheme X is a fine moduli space for a moduli functor F is equivalent to the existence of a *universal family* $\pi \colon U \to X$ with the property that whenever $\psi \colon Y \to B$ is a family of the required form (so an element of $F(B)$), there is a unique morphism $\phi \colon B \to X$ such that

$$\begin{tikzcd} Y = U \times_X B \arrow[r] \arrow[d, "\psi"'] & U \arrow[d, "\pi"] \\ B \arrow[r, "\phi"'] & X \end{tikzcd}$$

If we know that X represents F (so $F \cong \mathrm{Mor}(-, X)$, we can take $\pi \colon U \to X$ to be the element of $F(X)$ corresponding to $\mathrm{id} \colon X \to X \in \mathrm{Hom}(X, X)$. Given a universal family $\pi \colon U \to X$, for each B we get a function $\alpha_B \colon F(B) \to \mathrm{Mor}(B, X)$, which defines a natural isomorphism.

Example 2. The universal family of $\mathbb{P}^2$ is

$$U = \{([x_0 : x_1 : x_2], (y_0, y_1, y_2) : \mathrm{rk}\left(\begin{smallmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{smallmatrix}\right) = 1\} \subseteq \mathbb{P}^2 \times \mathbb{A}^3.$$

The map $\pi \colon U \to \mathbb{P}^2$ is projection onto the first factor, and the fibre over a point $[x] \in \mathbb{P}^2$ is the line through the origin spanned by x in $\mathbb{A}^3$

In today's lecture, we introduce the star of the module: the Hilbert scheme. This parameterises subschemes of $\mathbb{P}^n$.

Our first attempt at the moduli functor is:

$$\mathrm{Hilb}_{\mathbb{P}^n} : \text{Schemes} \to \text{Sets}$$

given by

(1)
$$\mathrm{Hilb}_{\mathbb{P}^n}(B) = \{Z \text{ a subscheme of } \mathbb{P}^n_B = \mathbb{P}^n \times B \text{ that are flat over } B\}.$$

Here "flat" is the appropriate niceness property for a family in algebraic geometry, which guarantees that the fibres of of the morphism $Z \to B$ are not too different from each other.

In commutative algebra, an R-module M is flat if $- \otimes M$ is an exact functor. If

$$0 \to A \to B \to C \to 0$$

is an exact sequence of R-modules, we always have

$$A \otimes M \to B \otimes M \to C \otimes M \to 0$$

exact. The new criterion is that $A \otimes M \to B \otimes M$ is injective. For example, if $M = R^n$ is free, then M is flat, while $\mathbb{Z}/2\mathbb{Z}$ is not a flat $\mathbb{Z}$-module; this can be seen by considering the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

If A, B are commutative rings, with $\phi^* : A \to B$ making B into an A-module, then ϕ: $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat if B is a flat A-module. In general, $\phi : X \to Y$ is flat if the stalk $\mathcal{O}_{X,x}$ of X at a point x is flat as an $\mathcal{O}_{Y,y}$-module, where $y = \phi(x)$.

One useful fact about flat families: if A is a PID, then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat if and only if B is a torsion-free A-module ([Eis95, Corollary 6.3]). This is useful when $A = K[t]$ with K a field. We'll see some other useful characterizations later.

It turns out that the functor in (1) is too "big" a question, in the sense that the resulting moduli space would have infinitely many components. We remove this problem by specifying the *Hilbert polynomial* of the subscheme Z. We'll work for now for simplicity over a base field K (so the functor should be restricted to K-schemes) but everything works over $\mathbb{Z}$.

We use the following commutative algebra notation. We write S for the polynomial ring $K[x_0, \dots, x_n]$. This is graded by $\deg(x_i) = 1$ for $0 \le i \le n$, so $S = \oplus_{d=0}^{\infty} S_d$, where S_d is the vector space of homogeneous polynomials of degree d. An ideal $I \subseteq S$ is homogeneous if it is generated by homogeneous elements of S. In that case, $S/I \cong \oplus_{d=0}^{\infty}(S/I)_d \cong \oplus_{d=0}^{\infty} S_d/I_d$ is graded.

The key fact that we will use repeatedly is that a subscheme Z of $\mathbb{P}^n_K$ is determined by a homogeneous ideal $I \subseteq S$:

$$\mathcal{O}_Z = \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_Z$$

where I_Z is the ideal sheaf of Z, and $\mathcal{I}_Z$ is the sheafification $\tilde{I}$ of I. To see this on affine charts, let $I_i = (IS[x_i^{-1}])_0 \subseteq S[x_i^{-1}]_0 = K[x_0/x_i, \dots, x_n/x_i]$. Then the intersection of Z with the affine chart $D(x_i)$, where the ith coordinate is nonzero is $\operatorname{Spec}(K[x_0/x_i, \dots, x_n/x_i]/I_i)$.

This correspondence between subschemes of $\mathbb{P}^n$ and homogeneous ideals in $S = K[x_0, \dots, x_n]$ is not one-to-one, as two different ideals can have the same sheafification.

Ideals $I, J \subseteq S$ correspond to the same subscheme if their *saturations* agree:

$$(I : \langle x_0, \dots, x_n \rangle^\infty) = (J : \langle x_0, \dots, x_n \rangle^\infty).$$

Here $(I : \langle x_0, \dots, x_n \rangle^\infty) = \{f \in S : \exists k > 0 \text{ such that} fm \in I \text{ for all } m \in \langle x_0, \dots, x_n \rangle^k\}$. For example, in $K[x_0, x_1]$, the two ideals $\langle x_1 \rangle$ and $\langle x_1^2, x_0x_1 \rangle = \langle x_1 \rangle \cap \langle x_0, x_1^2 \rangle$ both have saturation $\langle x_1 \rangle$. The associated subscheme of $\mathbb{P}^1$ is the point $[1:0]$ with the reduced structure. In general, two ideals $I, J \subseteq S$ have the same saturation if and only if $I_d = J_d$ for all $d \gg 0$.

Definition 1. The Hilbert function of a homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$ is the function $h_I : \mathbb{N} \to \mathbb{N}$ given by $h_I(d) = \dim_K (S/I)_d$ for all d.

Example 3.

(1) If $I = \langle 0 \rangle$, $h_I(d) = \dim_K S_d = \binom{n+d}{d} = \binom{n+d}{n}$. This equals $1/n!(d+n)\dots(d+1)$, so is a polynomial of degree n in d.

(2) If $I = \langle f \rangle$, where f is homogeneous of degree m, then

$$\begin{aligned} h_I(d) &= \begin{cases} \dim_K S_d & d < m \\ \dim_K S_d - \dim_K S_{d-m} & d \geq m \end{cases} \\ &= \begin{cases} \binom{n+d}{n} & d < m \\ \binom{n+d}{n} - \binom{n+d-m}{n} & \end{cases} \end{aligned}$$

Note that $h_I(d)$ is a polynomial in d of degree $n-1$ for $d \gg 0$.

For any homogeneous ideal $h_I(d)$ agrees $p_I(d)$ for $d \gg 0$, where $p_I \in \mathbb{Q}[t]$ is a polynomial. This polynomial is called the *Hilbert polynomial* of I.

When Z is a subscheme of $\mathbb{P}^n$ with associated ideal I, we have $\dim(Z) = \deg(p_I)$. The degree of Z is $\dim(Z)!$ times the leading coefficient of p_I.

Geometrically, if Z is defined over a field K, we have

$$p_I(d) = \chi(\mathcal{O}_Z(d)) = \sum_{i=0}^{\dim Z} (-1)^i \dim_K H^i(\mathbb{P}^n, \mathcal{O}_Z(d)),$$

which equals $\dim_K H^0(\mathbb{P}^n, \mathcal{O}_Z(d))$ for $d \gg 0$ by Serre vanishing.

Example 4. (1) When Z is a hypersurface of degree m in $\mathbb{P}^n$, $p_I = \binom{n+t}{n} - \binom{n+t-m}{n}$.

(2) When Z is a line in $\mathbb{P}^n$, $p_I(t) = t+1$.

(3) When Z is a smooth curve of degree d and genus g, $p_I(t) = dt+1-g$. If Z is embedded by a complete linear series $L(D)$, then this follows from the Riemann-Roch theorem. The Riemann-Roch theorem states that

$$l(D) - l(K-d) = \deg(D) + 1 - g.$$

Apply this to tD, using that $h(t) = l(tD)$. We have $l(K-tD) = 0$ for $t \gg 0$, so $p(tD) = \deg(tD) + 1 - g = t\deg(D) + 1 - g$.

Our new Hilbert functor is:

$$\mathrm{Hilb}_{P,\mathbb{P}^n} : \mathrm{Schemes} \to \mathrm{Sets}$$

given by

$$\mathrm{Hilb}_{P,\mathbb{P}^n}(B) = \{ \text{ subschemes } Z \subseteq \mathbb{P}^n_B \text{ such that } Z \to B \text{ is a flat family, with every fibre having Hilbert polynomial } P\}.$$

We will show that this representable, so there is a (projective) scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ with

$$\mathrm{Hilb}_{P,\mathbb{P}^n} \cong \mathrm{Mor}(-, \mathrm{Hilb}_P(\mathbb{P}^n)).$$

The original Hilbert functor is then represented by the disjoint union of all Hilb_P as P varies over all (countably infinite) Hilbert polynomials.

Example 5. (1) When $P(t) = r$ is constant, Hilb_P is the Hilbert scheme of points in $\mathbb{P}^n$. This has an irreducible component with an open set consisting of subschemes of $\mathbb{P}^n$ consisting of r distinct reduced points. However, as we will see later it can have many more components.

(2) When $P(t) = t+1$, then Z is a line in $\mathbb{P}^n$, and $\mathrm{Hilb}_P = \mathrm{Gr}(2, n+1)$.

(3) When $P(t) = 2t+1$, $\mathrm{Hilb}_P(\mathbb{P}^2)$ parameterises all conics in $\mathbb{P}^2$. When we work over a field K, these can be described by an equation

$$ax_0^2 + bx_0x_1 + cx_0x_1 + dx_1^2 + ex_1x_2 + fx_2^2, \tag{2}$$

so we associate the point $[a:b:c:d:e:f] \in \mathbb{P}^5$ to the conic. Thus $\mathrm{Hilb}_{2t+1}(\mathbb{P}^2) \cong \mathbb{P}^5$. The equation (2) defines the universal family of $\mathrm{Hilb}_{2t+1}(\mathbb{P}^5)$ as a subscheme of $\mathbb{P}^2 \times \mathbb{P}^5$.

In general, when $P = \binom{n+t}{n} - \binom{n+t-r}{n}$ the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ parameterises hypersurfaces of degree r in $\mathbb{P}^n$, so $\mathrm{Hilb}_P(\mathbb{P}^n) \cong \mathbb{P}^{\binom{n+r}{r}-1}$.

There is an implicit exercise here: if a subscheme Z has this Hilbert polynomial, it must be a hypersurface of this degree.

(4) When $P(t) = \binom{t+r}{r}$, then (again, an exercise!) if a subscheme Z of $\mathbb{P}^n$ has Hilbert polynomial P, then it is a linear subspace of dimension r, so $\mathrm{Hilb}_P(\mathbb{P}^n) = \mathrm{Gr}(r+1, n+1)$.

The situation of the last two examples, where every subscheme Z with the given Hilbert polynomial has a particular form, does not generalise well. We will revisit this with the Hilbert scheme of points later. Another example is given by the "twisted cubic" (the image of the Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$. This has Hilbert polynomial $3t+1$. However $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$ has two irreducible components [PS85]. One component generically parameterises twisted cubics, and the other generically parameterises a plane cubic plus a point. This phenomenon happens routinely in the study of moduli spaces: we try to describe a geometric object by specifying invariants, but often other objects appear in the space as well.

We next show that the Hilbert scheme actually exists (i.e., that the Hilbert functor is representable).

References

[Eis95] David Eisenbud, *Commutative algebra: with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995.

[PS85] Ragni Piene and Michael Schlessinger, *On the Hilbert scheme compactification of the space of twisted cubics*, Amer. J. Math. **107** (1985), no. 4, 761–774.