TCC - HILBERT SCHEMES AND MODULI SPACES - LECTURE 1

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1.1. **Introduction to the module.** If you plan to attend the lectures for this module, email me (`D.Maclagan@warwick.ac.uk`) to get on the mailing list for this module. Further information about the module will be posted on the webpage http://homepages.warwick.ac.uk/staff/D.Maclagan/Classes/TCCModuli/TCCModuli.html.

The plan for the term is:

(1) Today, we cover what a moduli space is, some examples, and the definition of a fine moduli space.
(2) Lectures 2 to 5 or 6 will cover the Hilbert scheme. We will go into details of its construction, and properties.
(3) Lectures 6 or 7 to 8 will cover some other moduli spaces: the moduli space of curves, and the moduli space of abelian varieties.

The focus will be on explicit constructions and examples. We will not cover everything about moduli spaces (it's only eight lectures!); in particular, we will not go into any detail about stacks.

Assumed background will be as follows:

(1) An undergraduate algebraic geometry module at the level of [Rei88], [Has07], or [SKKT00].
(2) Very beginnings of scheme theory ([Har77] Chapter II, sections 1 and 2 - the definition of an affine scheme).
(3) Comfort with basic commutative algebra of ideals in a polynomial ring.

For those taking the module for credit, the default option is to do complete one homework question per week. The target deadline is two weeks after the lecture. I will aim to have questions for a variety of backgrounds, and you are expected to choose a question at the right level for you. Some homeworks can be replaced by a report on a research paper on the topic of this module, with permission (this option is mostly intended for more advanced PhD students).

1.2. **First examples. Question:** What is a moduli space?

First answer: A moduli space is a space (variety/manifold/scheme/stack/...) where each point corresponds to an (equivalence class of) object(s) being considered.

Example 1. Projective space $\mathbb{P}^n_{\mathbb{C}} = (\mathbb{C}^{n+1} \setminus \{\mathbf{0}\})/\mathbf{v} \sim \lambda\mathbf{v}$ parameterises lines through the origin in $\mathbb{C}^{n+1}$. This gives a notion of lines being close.

Example 2. The Grassmannian $\mathrm{Gr}(r,n)$ parameterises r-dimensional planes in $\mathbb{C}^n$. This is a smooth variety (manifold) of dim $r(n-r)$ with affine charts $\mathbb{A}^{r(n-r)}$.

To see this, choose a basis $\mathbf{v}_1, \dots, \mathbf{v}_r$ for a subspace V, and write it as the rows of an $r \times n$ matrix. For a concrete example, consider $r = 2$, $n = 4$, and $\mathbf{v}_1 = (1,1,1,1)$ and $\mathbf{v}_2 = (1,2,3,4)$, so the matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Compute the row-reduced form of this matrix:

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

The rows of the row reduced form give a unique representative for the subspace V. If the first $r \times r$ submatrix is the identity matrix (so was invertible in the original matrix) then the remaining $r \times (n-r)$ submatrix can be anything, an all choices are different subspaces, so this gives a copy of $\mathbb{A}^{r(n-r)}$. All choices of $r \times r$ submatrices give the same result; there is nothing special about the order of the columns in Gaussian elimination. This means that $\mathrm{Gr}(r,n)$ is a union of affine cells. Check: the overlap maps are regular functions. This means that $\mathrm{Gr}(r,n)$ is a smooth variety of dimension $r(n-r)$.

To see projectivity, consider the map that takes the matrix to the vector of its $r \times r$ minors (the determinant of $r \times r$ submatrices), viewed as a point in $\mathbb{P}^{\binom{n}{r}-1}$ In our running example, this produces the vector

$$[1:2:3:1:2:1] \in \mathbb{P}^5,$$

where the coordinates are ordered $12, 13, 14, 23, 24, 34$. For the affine chart consisting of matrices of the form

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \tag{1}$$

we have the coordinates

$$[1 : c : d : -a : -b : ad - bc].$$

The image of the map $\mathrm{Gr}(r,n) \to \mathbb{P}^{\binom{n}{r}-1}$ is closed, and cut out by the *Plücker relations*. Writing p_I for the coordinates of $\mathbb{P}^{\binom{n}{r}-1}$, where $I \subseteq \{1,\ldots,n\}$ with $|I| = r$, the Plücker relations are

$$\sum_{i \in J_2} (-1)^{\text{sign}} p_{J_1 \cup i} p_{J_2 \setminus i}$$

for all $J_1, J_2 \subseteq \{1,\ldots,n\}$ with $|J_1| = r-1$, and $|J_2| = r+1$, and sign is the number of switches needed to move i from J_2 to J, keeping both in ascending order. For example, when $n = 4$, $J_1 = \{1\}$, and $J_2 = \{2,3,4\}$, we have

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

We can check on the affine chart (1) that

$$1(ad - bc) - (c)(-b) + (d)(-a) = 0.$$

Check: If $[p_I] \in \mathbb{P}^{\binom{n}{r}-1}$ satisfies the Plücker relations there is $V \subseteq \mathbb{C}^n$ with $[p_I]$ as the vector of minors. The Plücker embedding can also be described as follows. For $V \subseteq \mathbb{C}^n$ of dimension r, we have the inclusion

$$\wedge_{i=1}^r V \subseteq \wedge_{i=1}^r \mathbb{C}^n.$$

Note that $\wedge_{i=1}^r V$ is one-dimensional, so V determines a point in $\mathbb{P}(\wedge_{i=1}^r \mathbb{C}^n) = \mathbb{P}^{\binom{n}{r}-1}$.

This describes $\mathrm{Gr}(r,n)$ as a projective variety.

One advantage of putting a projective variety description on the set of all r-dimensional subspaces of $\mathbb{C}^n$ is that it puts a topology on this set, so a way to decide if two subspaces are close, and gives a notion to limits of families.

However a natural question is:

Question: (Why) is this the only option?

A priori there might be another parameter space for r-dimensional subspaces of $\mathbb{C}^n$ that places a different topology on this set. Luckily we will see that there is a reasonable notion of uniqueness in this case.

Another question is:

Question: The Grassmannian makes sense, with the same description, over any field. What about in more generality? Is there a unifying way to do this?

1.3. **A category theory interlude.** Recall that a functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of functions $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F\colon \mathrm{Hom}(C,C') \to \mathrm{Hom}(F(C),F(C'))$ satisfying certain compatibility conditions. A contravariant functor has instead $F\colon \mathrm{Hom}(C,C') \to \mathrm{Hom}(F(C'),F(C))$.

The functor of points h_X of a scheme X is the contravariant functor

$$\text{Schemes} \to \text{Sets}$$

that takes a scheme Y to the set of morphisms from Y to X:

$$Y \mapsto h_X(Y) = \text{Mor}_{Sch}(Y, X),$$

and a morphism $\phi \colon Y \to Y'$ to the map of sets $h_X(Y') \to h_X(Y)$ given by taking $g \in h_X(Y') = \text{Mor}(Y', X)$ to $g \circ \phi$.

A reference for this topic is Chapters I.4 and VI of [EH00]. Two key facts about functors of points are:

(1) h_X determines X. To start to see that this is reasonable, note that if $Y = \text{Spec}(K)$ for a field K, then as a topological space Y is a point, so the set of morphisms is the set of K-valued points of X. This follows from Yoneda's lemma below.
(2) To know h_X, it suffices to know its restriction to affine schemes, so to know $h_X(\text{Spec}(R))$ for R a commutative ring.

A functor h : Schemes $\to$ Sets is *representable* if $h \cong h_X$ for some scheme X.

There is at most one scheme X for which $h \cong h_X$. This is a consequence of Yoneda's lemma.

Recall that a natural transformation ϕ between two functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{C} \to \mathcal{D}$ is the data of, for all objects C in $\mathcal{C}$ an element $\phi(C) \in \text{Hom}(F(C), G(C))$ with the property that for all objects C, C' in $\mathcal{C}$ and $f \in \text{Hom}(C, C')$, the diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{\phi(C)} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\phi(C')]{} & G(C') \end{array}$$

commutes. The map ϕ is an isomorphism if each $\phi(C)$ is an isomorphism.

Lemma 3 (Yoneda's Lemma)**.** *Let $\mathcal{C}$ be a category and let X, X' be objects of $\mathcal{C}$.*

(1) *If F is any contravariant functor from $\mathcal{C}$ to the category of sets, then there is a one-to-one correspondence between the natural transformations from* $\text{Hom}(-, X)$ *to F and the elements of $F(X)$. This correspondence is natural when both are regarded as functors from $\mathcal{C} \times$* $\text{Func}(\mathcal{C}, \text{Sets})$ *to* Sets*.*
(2) *If the functors* $\text{Hom}(-, X)$ *and* $\text{Hom}(-, X')$ *from $\mathcal{C}$ to the category of sets are isomorphic, then $X \cong X'$.*

A natural transformation α from $\operatorname{Hom}(-, X)$ to F consists of the data of, for each object C in $\mathcal{C}$ a function $\alpha_C\colon \operatorname{Hom}(C, X) \to F(C)$ satisfying the compatibility requirements. In particular, for $C = X$, we get a function $\alpha_X\colon \operatorname{Hom}(X, X) \to F(X)$. The set $\operatorname{Hom}(X, X)$ always contains the identity map $\mathrm{id} : X \to X$. The correspondence of part 1 takes α to $\alpha_X(\mathrm{id} : X \to X)$.

To see that this is a bijection, note that given $p \in F(X)$, we can construct for all $C \in \mathcal{C}$ a function $\alpha_{p,C}\colon \operatorname{Hom}(C, X) \to F(C)$ by

$$\alpha_{p,C}(f) = F(f)(p) \in F(C)$$

for $f \in \operatorname{Hom}(C, X)$. We now check that this is a natural transformation. Given $g \in \operatorname{Hom}(C, C')$, we have the diagram

$$\begin{array}{ccc} \operatorname{Hom}(C', X) & \xrightarrow{\alpha_{p,C'}} & F(C') \\ {\scriptstyle \operatorname{Hom}(g,X)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ \operatorname{Hom}(C, X) & \xrightarrow[\alpha_{p,C}]{} & F(C) \end{array}$$

For $f \in \operatorname{Hom}(C', X)$, $\alpha_{p,C}(f) = F(f)(p)$, so going across and down we get $F(g)(F(f)(p)) = F(g \circ f)(p)$, while $\operatorname{Hom}(g, X)(f) = g\ circf$, so going down and then across we get $F(g \circ f)(p)$ as well. Thus the diagram commutes, so α is a natural transformation.

We then need to check that for all natural transformations α we have $\alpha_{\alpha_X(\mathrm{id}:X\to X)} = \alpha$, and for all $p \in F(X)$ we have $\alpha_{p,X}(\mathrm{id} : X \to X) = p$. (Exercise!)

For part 2, if $\alpha\colon \operatorname{Hom}(-, X) \to \operatorname{Hom}(-, X')$ is an isomorphism with inverse α', then $\alpha_X(\mathrm{id} : X \to X) \in \operatorname{Hom}(X, X')$ has inverse $\alpha'_{X'}(\mathrm{id} : X' \to X') \in \operatorname{Hom}(X', X)$, so X is isomorphic to X'. (Check details!)

1.4. **Fine moduli spaces.** A *moduli problem* asks to classify/parameterise all families of objects of a particular type, up to some notion of equivalence. For example, one might ask to classify lines through the origin in $\mathbb{A}^n$, with equivalence being identity, or one might ask to classify smooth curves of genus g up to isomorphism.

A *moduli functor* is a functor Schemes $\to$ Sets of the form:

$$B \mapsto \{\text{ a family of objects being parameterised over a base } B\}/\sim$$

where $\sim$ is a notion of equivalence. A scheme X is a *fine moduli space* for a moduli functor F if X represents F: $F = h_X$. Part 2 of Yoneda's lemma says that fine moduli spaces, when they exist, are unique. In this case we have an answer to the question "Why is this space the only option for a moduli space?".

Example 4. For projective space we have the moduli functor Schemes $\to$ Sets given by

$$B \mapsto \{\text{subsheaves } \mathcal{F} \subseteq \mathcal{O}_B^{n+1} \text{ that are locally summands of rank } n\}.$$

Restricted to affine schemes, so $B = \operatorname{Spec}(R)$, this becomes

$$R \mapsto \{ \text{ submodules } M \subset R^{n+1} \text{ that are locally free direct summands of rank } n\}.$$

Example 5. For the Grassmannian we have the moduli functor Schemes $\to$ Sets given by

$$B \mapsto \{\text{subsheaves } \mathcal{F} \subseteq \mathcal{O}_B^{n} \text{ that are locally free of rank } r\}.$$

Restricted to affine schemes, so $B = \operatorname{Spec}(R)$, this becomes

$$R \mapsto \{ \text{ submodules } M \subseteq R^{n} \text{ that are locally free direct summands of rank } r\}.$$

The property that a scheme X is a fine moduli space for a moduli functor F is equivalent to the existence of a *universal family* $\pi\colon U \to X$ with the property that whenever $\psi\colon Y \to B$ is a family of the required form (so an element of $F(B)$), there is a unique morphism $\phi\colon B \to X$ such that

$$\begin{array}{ccc} Y = U \times_X B & \longrightarrow & U \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[\phi]{} & X \end{array}$$

If we know that X represents F (so $F \cong \operatorname{Mor}(-, X)$, we can take $\pi\colon U \to X$ to be the element of $F(X)$ corresponding to $\mathrm{id}\colon X \to X \in \operatorname{Hom}(X, X)$. Given a universal family $\pi\colon U \to X$, for each B we get a function $\alpha_B\colon F(B) \to \operatorname{Mor}(B, X)$, which in the cases we will consider is a natural isomorphism.

Example 6. The universal family of $\mathbb{P}^2$ is

$$U = \{([x_0 : x_1 : x_2], (y_0, y_1, y_2) : \operatorname{rk}\left(\begin{smallmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{smallmatrix}\right) = 1\} \subseteq \mathbb{P}^2 \times \mathbb{A}^3.$$

The map $\pi\colon U \to \mathbb{P}^2$ is projection onto the first factor, and the fibre over a point $[x] \in \mathbb{P}^2$ is the line through the origin spanned by x in $\mathbb{A}^3$

References

[EH00] David Eisenbud and Joe Harris, *The geometry of schemes*, Graduate Texts in Mathematics, vol. 197, Springer-Verlag, New York, 2000.

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Has07] Brendan Hassett, *Introduction to algebraic geometry*, Cambridge University Press, Cambridge, 2007.

[Rei88] Miles Reid, *Undergraduate algebraic geometry*, London Mathematical Society Student Texts, vol. 12, Cambridge University Press, Cambridge, 1988.

[SKKT00] Karen E. Smith, Lauri Kahanpää, Pekka Kekäläinen, and William Traves, *An invitation to algebraic geometry*, Universitext, Springer-Verlag, New York, 2000.