TCC - HILBERT SCHEMES AND MODULI SPACES - EXERCISE 5

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If you are taking this module for credit write up your solution to one of these questions and email them to me. Aim to do this within two weeks of the lecture. Choose an exercise at the appropriate level for your background; let me know if you are having trouble finding such an exercise.

(1) Write out a proof of Theorem 3 from the notes. Add any details that you need to fully understand it to the version found in the references.

(2) Prove Lemma 7 from the notes, which characterizes the generators of $\mathrm{Syz}(M)$ for a monomial ideal $M \subseteq K[x, y]$.

(3) Recall that the Grassmannian $\mathrm{Gr}(r, n)$ is the Hilbert scheme $\mathrm{Hilb}_{\binom{n+r-1}{n-1}}(\mathbb{P}^{n-1})$. Use the approach of this week to verify that the Grassmanian is smooth. (We already know this from our description of the affine charts of the Grassmannian!)

(4) Show that the twisted cubic (the image of the Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$) is a smooth point of $\mathrm{Hilb}_{3t+1}(\mathbb{P}^3)$. Hint: You can compute the dimension of the tangent space using Macaulay2 using the commands:

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S = QQ[a..d];
I = ideal(b^2-a*c, b*c-a*d, c^2-b*d)
N = sheaf prune Hom(I,S^1/I)
HH^0(N)
HH^1(N)
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Why do you know that the Hilbert scheme has at least that dimension? (See Mike Stillman's Arizona Winter School lectures for more on explicit methods here).

(5) Compute the dimension of the tangent space to the ideal $\langle x^2, xy, xz, y^2, yz, z^2\rangle \subseteq K[x, y, z]$ in $\mathrm{Hilb}^4(\mathbb{A}^3)$.