TCC - HILBERT SCHEMES AND MODULI SPACES - EXERCISE 4

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If you are taking this module for credit write up your solution to one of these questions and email them to me. Aim to do this within two weeks of the lecture. Choose an exercise at the appropriate level for your background; let me know if you are having trouble finding such an exercise.

(1) Write a careful exposition of the Gröbner flat family. One reference is section 15.8 of Eisenbud's commutative algebra. Your answer should be the equivalent of one typed page, and include a worked example.

(2) Show that a Borel-fixed ideal is a monomial ideal, and that, in characteristic zero, it is strongly-stable. Hint: Consider the effect of $g = I + aE_{ij}$ on monomials for $i < j$, where E_{ij} is the matrix with a one in the ijth position, and zeros elsewhere.

(3) (a) Show that the saturation of a Borel-fixed ideal is Borel-fixed.
    (b) Show that there are only finitely many saturated Borel-fixed ideals with a given Hilbert polynomial. Hint: There are in fact only finitely many monomial ideals with a given Hilbert polynomial. Use Gotzmann's bounds to restrict to one degree to prove this.

(4) Consider $\mathrm{Hilb}_{t+4}(\mathbb{P}^2)$. Show that there are exactly two Borel-fixed ideals saturated with respect to $\langle x_0, x_1, x_2\rangle$. Hint: First show that if a monomial ideal M is strongly stable and saturated with respect to $\langle x_0, x_1, x_2\rangle$ then no minimal generator of M is divisible by x_2. Give an explicit path in the Hilbert scheme between these two Borel-fixed points.

(5) Let $X = V(x_1x_2) \subseteq \mathbb{P}^2$. Compute $\mathrm{Hilb}_{t+1}(X)$ as a subscheme of $\mathbb{P}^2$ using the method we discussed, and show that that gives the expected answer that $\mathrm{Hilb}_{t+1}(X)$ is two reduced points.

(6) You may also choose to revisit a question from a previous week that you did not answer and hand it in for this week.