TCC - HILBERT SCHEMES AND MODULI SPACES - EXERCISE 3

DIANE MACLAGAN

If you are taking this module for credit write up your solution to one of these questions and email them to me. Aim to do this within two weeks of the lecture. Choose an exercise at the appropriate level for your background; let me know if you are having trouble finding such an exercise.

(1) The syntax to compute the regularity of S/I in the computer algebra system `Macaulay2` is:

```
R=QQ[x_0,x_1,x_2,x_3]
I=ideal(x_0*x_3-x_1*x_2,x_0*x_2-x_1^2,x_1*x_3-x_2^2)
regularity(R^1/I)
```

You can use `Macaulay2` online at `https://www.unimelb-macaulay2.cloud.edu.au/#home`.

(a) Check that the results above are consistent with Gotzmann's regularity theorem, by checking that the ideal I is saturated, and computing its Hilbert polynomial and the resulting decomposition. You may find the commands `hilbertPolynomial` and `saturate` useful. What I have called the Hilbert polynomial of I, `Macaulay2` calls the Hilbert polynomial of R/I.

(b) Repeat this calculation with another ideal of your choice, preferably one that has geometric meaning for you. For this ideal, also compute a saturated lexicographic ideal with this Hilbert polynomial, and compute its regularity.

(2) Is there a value of n for which $\mathrm{Hilb}_{t^3}(\mathbb{P}^n)$ is nonempty? If so, what range can you give for n?

(3) Continue the last example from the lecture, which discussed $\mathrm{Hilb}_2(\mathbb{P}^1)$.

(a) Instead of restricting to an affine chart, we should really work with homogeneous coordinates on the Grassmannian, getting an ideal with four generators, as in Question 3 from Week 1 (make sure that you are looking at the corrected version!). What equations do you get in this case? Can you describe the subscheme of the Grassmannian $\mathrm{Gr}(2, 4)$?

(b) Repeat this exercise using the true Gotzmann number $D = 2$. Can you describe the subscheme of the Grassmannian $\mathrm{Gr}(1, 3) = \mathbb{P}^2$?

   (c) What is the relationship between your answers to these two questions? Is this what you expect?

(4) For the Hilbert polynomial $P(t) = 2$, the Gotzmann number is 2, as $2 = \binom{t+0}{0} + \binom{t-1+0}{0}$. This means that the Hilbert scheme of 2 points in $\mathbb{P}^2$, $\mathrm{Hilb}_2(\mathbb{P}^2)$ embeds into $\mathrm{Gr}(4,6)$, which in turn embeds into $\mathbb{P}^{14}$. Describe explicitly the equations defining this embedding. Optional bonus question: We will see that this Hilbert scheme is irreducible. Can you see that from this description? One way to approach this question would be to consider the ideals defining two distinct points, and compare the closure of the locus of such ideals in $\mathrm{Gr}(4,6)$ with the subscheme you have constructed, but this is a large computation.

(5) This lecture had several omitted proofs. Investigate one of them, and write a summary (at a level that fits in the equivalent of two typed pages).