TCC - HILBERT SCHEMES AND MODULI SPACES - EXERCISE 2

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If you are taking this module for credit write up your solution to one of these questions and email them to me. Aim to do this within two weeks of the lecture. Choose an exercise at the appropriate level for your background; let me know if you are having trouble finding such an exercise.

(1) Write a complete proof of the equivalence of X representing a functor F from the category of Schemes to the category of Sets, and the existence of a universal family. Hint: you may want to consider what the fact that X represents F means for for $F(\phi)$ where $\phi\colon B \to B'$ is a morphism.

(2) Show that if Z is a subscheme of $\mathbb{P}^n$ corresponding to homogeneous ideal I in $K[x_0, \dots, x_n]$ with Hilbert polynomial $p_I(t) = t + 1$, then Z is a line in $\mathbb{P}^n$.

(3) Show that if Z is a subscheme of $\mathbb{P}^n$ corresponding to a homogeneous ideal I in $K[x_0, \dots, x_n]$ with Hilbert polynomial $\binom{n+t}{n} - \binom{n+t-r}{n}$ then Z is a hypersurface of degree r.

(4) Show that if Z is a subscheme of $\mathbb{P}^n$ corresponding to a homogeneous ideal I in $K[x_0, \dots, x_n]$ with Hilbert polynomial $\binom{t+r}{r}$ then Z is a linear subspace of $\mathbb{P}^n$ of dimension r.

(5) Compute the Hilbert polynomial of a subscheme $Z \subseteq \mathbb{P}^3$ consisting of the union of a reduced plane cubic, plus a reduced point not in the plane. You may want to first compute this for a particular example, and then argue why it does not depend on the choice of such Z. In the computer algebra system `Macaulay2`, the Hilbert polynomial of a homogeneous ideal I in a polynomial ring R is computed with the command `hilbertPolynomial(R/I,Projective=>false)` .

(6) The last question from Exercise 1 can be carried over to this week.