TCC - HILBERT SCHEMES AND MODULI SPACES - EXERCISE 1

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If you are taking this module for credit write up your solution to one of these questions and email them to me. Aim to do this within two weeks of the lecture. Choose an exercise at the appropriate level for your background; let me know if you are having trouble finding such an exercise.

(1) (For people new to the Grassmannian) The purpose of this exercise is to check that Plücker relations define the Grassmannian as a projective variety.

Let $S = K[p_I : I \subset \{1,\dots,n\}, |I| = r]$, and let $I_{r,n} = \langle P_{J_1J_2} : |J_1| = r-1, |J_2| = r+1\rangle$, where

$$P_{J_1J_2} = \sum_{j \in J_2} (-1)^{\mathrm{sign}(j)} p_{J_1 \cup j} p_{J_2 \setminus j}.$$

We will now show that $\mathbb{V}(I_{r,n}) \subseteq \mathbb{P}^{\binom{n}{r}-1}$ equals $G(r,n)$ in its Plücker embedding, as defined in lecture.

(a) Show that if $V \subseteq K^n$ is a r-dimensional subspace, then the Plücker coordinates of V (the vector of $r \times r$ minors of any matrix whose rows are a basis for V) satisfies all equations $P_{J_1J_2}$ if $J_1 \cap J_2 = \emptyset$. Hint: Think about the submatrix of A_V indexed by the columns $J_1 \cup J_2$. What can we assume about the columns indexed by J_1?

(b) Show that the same is true if $J_1 \cap J_2 \neq \emptyset$. Hint: One method here would be induction on r.

(c) Now consider $[p] \in \mathbb{V}(I_{r,n})$. Assume that $p_I \neq 0$ for $I = \{1,\dots,r\}$. Show that there is a unique r-dimensional V with Plücker coordinates $[p]$.

(2) (Yoneda's lemma)

(a) Finish checking that the correspondence in the first part of the Lemma is a bijection.

(b) Check that the correspondence given in the first part of the Lemma is natural, in the sense that there is a natural transformation between the two functors from $\mathcal{C} \times \mathrm{Func}(\mathcal{C}, \mathrm{Sets})$ to Sets.

(c) Check that the elements of $\mathrm{Hom}(X, X')$ and $\mathrm{Hom}(X', X)$ given in the second part of the Lemma are in fact inverses.

(3) (Universal families) Let U be the variety in $\mathbb{P}^5 \times \mathbb{A}^4$ defined by

$$\begin{aligned} U =& V(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}, p_{23}x_1 - p_{12}x_2 + p_{12}x_3, p_{24}x_1 - p_{14}x_2 + p_{12}x_4, \\ & p_{34}x_1 - p_{14}x_3 + p_{13}x_4, p_{34}x_2 - p_{24}x_3 + p_{23}x_4). \end{aligned}$$

(a) Let $\pi \colon U \to \mathbb{P}^5$ be projection onto the $\mathbb{P}^5$ factor. Show that $\pi \colon U \to \mathrm{Gr}(2,4)$ is the universal family of the Grassmannian $\mathrm{Gr}(2,4)$ of planes in 4-space.

(b) A plane in 4-space can be defined by *two* linear equations. Can we remove some of the equations given above?

(4) Generalise the previous example to $\mathrm{Gr}(r,n)$.

(5) Check that the Grassmannian represents the moduli functor given. See Eisenbud-Harris for hints and further technical background needed.