TCC: TROPICAL GEOMETRY

HW 2

(1) Verify (as much as possible) the fundamental theorem of tropical algebraic geometry and the structure theorem for $Y = V(f)$ for the following polynomials $f \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$:
  (a) $f = 3x + t^2 y + 2t$;
  (b) $f = tx^2 + xy + ty^2 + x + y + t$;
  (c) $f = x^3 + y^3 + 1$.

(2) Let $Y = \operatorname{trop}(V(x+2y+3z+4)) \subseteq (\mathbb{C}^*)^3$. Compute $\operatorname{trop}(Y)$, and a polyhedral complex Σ with support $\operatorname{trop}(Y)$. Show that Σ is balanced if we put the weight one on each top-dimensional cone.

(3) Let Σ be the pure one-dimensional polyhedral fan with cones $\operatorname{pos}((1,0))$, $\operatorname{pos}((0,1))$, $\operatorname{pos}((-1,1))$, $\operatorname{pos}((-1,-1))$. Find all weights $w \in \mathbb{N}^4$ for which Σ is balanced.

(4) Show that if K is an algebraically closed field with a valuation, then the residue field is algebraic closed.

(5) Show that the residue field of $\mathbb{C}\{\{t\}\}$ is $\mathbb{C}$. Show that the residue field of $\mathbb{Q}$ with the p-adic valuation is $\mathbb{Z}/p\mathbb{Z}$.

(6) Let $f = 8x^2 + xy + 12y^2 + 3 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. Compute all initial ideals $\operatorname{in}_w(I)$ of $I = \langle f \rangle$ as w varies when
  (a) $\mathbb{Q}$ has the trivial valuation,
  (b) $\mathbb{Q}$ has the 2-adic valuation, and
  (c) $\mathbb{Q}$ has the 3-adic valuation.