TCC : TROPICAL GEOMETRY

HW 1

(1) Show that if $\mathrm{val} : K^* \to \mathbb{R}$ is a valuation, and $\mathrm{val}(a) \neq \mathrm{val}(b)$, then $\mathrm{val}(a+b) = \min(\mathrm{val}(a), \mathrm{val}(b))$.
(2) Show that if K is an algebraically closed field with a nontrivial valuations $\mathrm{val} : K^* \to \mathbb{R}$ (ie there is $a \in K^*$ with $\mathrm{val}(a) \neq 0$) then $\mathrm{im}\,\mathrm{val}$ is dense in $\mathbb{R}$.
(3) Give formulas to solve the tropical cubic.
(4) Draw the tropical varieties $\mathrm{trop}(V(f))$ for the following $f \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$.
   (a) $f = t^3x + (t + 3t^2 + 5t^4)y + t^{-2}$;
   (b) $f = (t^{-1} + 1)x + (t^2 - 3t^3)y + 5t^4$;
   (c) $f = t^3x^2 + xy + ty^2 + tx + y + 1$;
   (d) $f = 4t^4x^2 + (3t + t^3)xy + (5 + t)y^2 + 7x + (-1 + t^3)y + 4t$;
   (e) $f = tx^2 + 4xy - 7y^2 + 8$;
   (f) $f = t^6x^3 + x^2y + xy^2 + t^6y^3 + t^3x^2 + t^{-1}xy + t^3y^2 + tx + ty \ + 1$.
(5) Let $f = ax + by + c$, where $a, b, c \in \mathbb{C}\{\{t\}\}$. What are the possibilities for $\mathrm{trop}(V(f))$? How does this change if $\mathbb{C}\{\{t\}\}$ is changed to $\mathbb{Q}$ with the p-adic valuation? Such tropical varieties are called tropical lines. (Harder) Repeat this question for $f = ax^2 + bxy + cy^2 + dx + ey + f$ (tropical quadrics).
(6) Show that any two general tropical lines in $\mathbb{R}^2$ intersect in a unique point, and there is a unique tropical line containing any two general points in $\mathbb{R}^2$. What are the notions of genericity here?
(7) (Much much harder) Show that if f and g are two general polynomials in $K[x^{\pm 1}, y^{\pm 1}]$ of degrees d and e respectively then $\mathrm{trop}(V(f)) \cap \mathrm{trop}(V(g))$ consists of at most de points. This can be refined by adding multiplicities to make the intersection consist of exactly de points counted with multiplicity (and more generally to n general polynomials in n variables). This is the tropical Bézout theorem.