$R = \{a \in K : \mathrm{val}(a) \geq 0\}$

$m = \{a \in K : \mathrm{val}(a) > 0\}$

$k = R/m$

For $w \in \Gamma^n$ $\quad \mathrm{in}_w(f) = \sum_{\mathrm{val}(c_u) + w \cdot u = \mathrm{trop}(f)(w)} \overline{t^{-\mathrm{val}(c_u)} c_u} x^u \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$

<u>Fundamental Theorem</u>

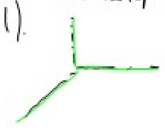
$X \subseteq (K^*)^n \quad K = \overline{K}$, nontrivial val

$X = V(I)$ $\quad \mathrm{trop}(X) = \mathrm{cl}(\overline{\mathrm{val}(X(\overline{K}))})$

$= \mathrm{cl}(\{w \in \Gamma^n : \mathrm{in}_w(I) \neq \langle 1 \rangle\})$

Euclidean top.

eg $V(x+y+1)$.



Γ = im val is a divisible group if $K = \overline{K}$.

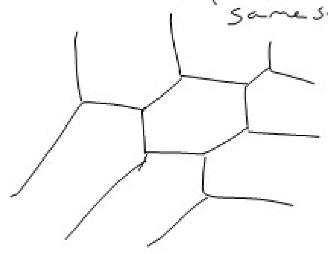
$X(K) = \{(y_1, \ldots, y_n) : y_i \in K^*, f(y) = 0 \ \forall f \in I\}$

<u>Structure Theorem</u> $\quad K = \overline{K}$

Let $X \subseteq (K^*)^n$ be an irred d-dim variety. Then $\mathrm{trop}(X)$ is the support of a pure d-dim weighted balanced polyhedral complex that is connected through codim 1.

same support

[The polyhedral complex structure is important, but not canonical.



<u>Fundamental theorem</u> $\quad \mathrm{val}(K^*) \neq \{0\}$

Easy case:

1 $\mathrm{trop}(X) = \mathrm{cl}(\{w \in \Gamma^n : \mathrm{in}_w(I) \neq \langle 1 \rangle\})$

Pf Let $w \in \mathrm{trop}(X) \cap \Gamma^n$

Then $w \in \mathrm{trop}(V(f)) \ \forall f \in I$, $\quad f = \sum c_u x^u$

so the minimum in $\mathrm{trop}(f)(w) = \min(\mathrm{val}(c_u) + w \cdot u)$ is achieved at least twice

This means that

$\mathrm{in}_w(f) = \sum_{\mathrm{val}(c_u) + w \cdot u = \mathrm{trop}(f)(w)} \overline{c_u t^{-\mathrm{val}(c_u)}} x^u$ is not a monomial

If $\mathrm{in}_w(I) = \langle 1 \rangle$, then there exists $f \in I$ with $\mathrm{in}_w(f) = 1$

Thus $\mathrm{in}_w(I) \neq \langle 1 \rangle$. This shows $\subseteq$.

Suppose $w \in \Gamma^n$ has $w \notin \mathrm{trop}(X)$.

Then $\exists f \in I$ with $w \notin \mathrm{trop}(V(f))$

so the minimum in $\mathrm{trop}(f)(w)$ is achieved once, so $\mathrm{in}_w(f)$ is a monomial

So $\mathrm{in}_w(I) = \langle 1 \rangle$.

② $\mathrm{cl}\,\mathrm{val}(X(K)) \subseteq \mathrm{trop}(X)$.

Pf Fix $y \in X$ & $w = (\mathrm{val}(y_1), \ \mathrm{val}(y_n))$

Since $y \in X$, $f(y) = 0 \ \forall f \in I$.

So $\sum c_u y^u = 0$ for $f = \sum c_u x^u$.

Aside:
$\mathrm{val}: K \to \mathbb{R} \cup \infty$
$\mathrm{val}(0) = \infty$
$\mathrm{val}(0) = \mathrm{val}(a + -a)$
$\geq \min(\mathrm{val}(a), \mathrm{val}(-a))$

$\mathrm{val}(a+b) = \min(\mathrm{val}(a), \mathrm{val}(b))$
if $\mathrm{val}(a) \neq \mathrm{val}(b)$.

$$\mathrm{val}\left(\sum c_u y^u\right) = \infty$$
$$\vee\!/$$
$$\min(\mathrm{val}(c_u y^u))$$
$$= \min(\mathrm{val}(c_u) + w \cdot u)$$
$$= \mathrm{trop}(f)(w) < \infty$$

Thus if the minimum were achieved once we would have = & thus a contradiction

So the minimum in $\mathrm{trop}(f)(w)$ is achieved at least twice, so $w \in \mathrm{trop}(V(f))$

So $w \in \bigcap_{f \in I} \mathrm{trop}(V(f)) = \mathrm{trop}(X)$. □

This leaves the hard direction:
$$\mathrm{trop}(X) \subseteq \mathrm{cl}(\mathrm{val}(X(K)))$$
We will actually show.

Prop Let $X = V(I)$ be an irred d-dim variety in $(K^*)^n$. Fix $w \in \Gamma^n$ with $\mathrm{in}_w(I) \neq \langle 1 \rangle$ & $\alpha \in (k^*)^n$ with $\alpha \in V(\mathrm{in}_w(I))$. Then $\exists\, y \in X$ with $\mathrm{val}(y) = w$, &
$$\overline{t^{-w} y} = \alpha$$
ie if $K = \mathbb{C}\{\{t\}\}$ $y = \alpha t^w + \text{h.o.t.}$

Gröbner complex

Defn The homogenization of a polynomial $f \in K[x_1, \ x_n]$ $= \sum c_u x^u$ is $\tilde{f} = \sum c_u x^u x_0^{d-|u|}$ for $d = \max |u|$ with $c_u \neq 0$

eg $f = x_1 x_2 + x_3^3 + 5$
$\tilde{f} = x_0 x_1 x_2 + x_3^3 + 5x_0^3$

For an ideal $I \subseteq K[x_1^{\pm 1}, \ x_n^{\pm 1}]$ we denote by I_{aff} the ideal $I \cap K[x_1, \ x_n]$ & by $I_{\mathrm{proj}} = \langle \tilde{f} : f \in I_{\mathrm{aff}} \rangle$

Note: $V(I_{\mathrm{aff}}) \subseteq \mathbb{A}^n$ is the closure of $V(I) \subseteq (K^*)^n \subseteq \mathbb{A}^n$ in the Zariski topology

$V(I_{\mathrm{proj}}) \subseteq \mathbb{P}^n$ is the projective closure of $V(I) \subseteq (K^*)^n \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$

Lemma Let $I \subseteq K[x_1^{\pm 1}, \ x_n^{\pm 1}]$.
Then $\mathrm{in}_w(I) = \underbrace{\mathrm{in}_{(0,w)}(I_{\mathrm{proj}})}_{\subseteq\, k[x_0, \ x_n]}\Big|_{x_0 = 1} \ k[x_1^{\pm 1}, \ x_n^{\pm 1}]$

Pf $\mathrm{in}_w(f) = \mathrm{in}_{(0,w)}(\tilde{f})\big|_{x_0 = 1}$

$f = \sum c_u x^u$

<u>Point:</u> $\mathrm{trop}(V(I)) = cl(\{w \in \Gamma^n :$ $in_{(0,w)}(I_{proj})$ does not contain a monomial$\})$

Q: How does $in_w(I)$ change as w varies?

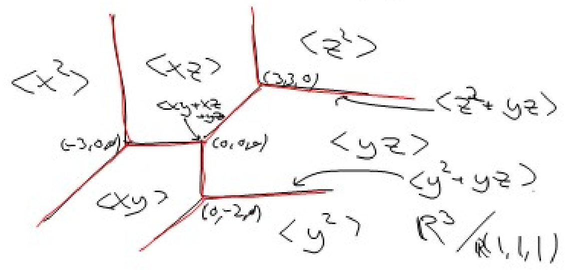
eg $f = 8x^2 + 5xy + 12y^2 + 7x - y + 2 \in \overline{\mathbb{Q}}[x^{\pm 1}, y^{\pm 1}]$
2-adic val.

$\tilde{f} = 8x^2 + 5xy + 12y^2 + 7xz - yz + 2z^2 \in \overline{\mathbb{Q}}[x, y, z]$

$w = (0,0,0)$.
$in_w(\tilde{f}) = xy + xz + yz$

$w = (2,2,2)$ $in_w(\tilde{f}) =$ 〃

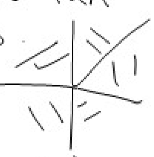
(Note: When $\tilde{f}$ is homogeneous,
$(1, \ldots, 1) \cdot \underline{u} = \deg(\tilde{f})$ for all x^u in f,
so $in_w(\tilde{f}) = in_{(w+\lambda(1,\ldots,1))}(\tilde{f})$ for all λ.



<u>Theorem</u>

Let I_{proj} be a homogeneous ideal in $K[x_0, \ldots, x_n]$. Then there is a finite polyhedral complex $\Sigma \subseteq \mathbb{R}^{n+1}$ for which if $\sigma \in \Sigma$ then $in_w(I)$ is constant for all w in the relative interior of σ (interior in its affine span).

The polyhedral complex Σ is called the Gröbner complex of I.
It generalizes the Gröbner fan of usual Gröbner bases.

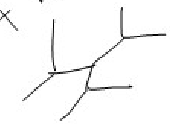


(A polyhedral cone is a polyhedron of the form $\{x \in \mathbb{R}^n : Ax \le 0\}$
A: $s \times n$ matrix

A polyhedral fan is a polyhedral complex where all polyhedra are cones).

Cor There are only finitely many initial ideals as w varies.

Cor Let $X \subseteq (K^*)^n$ be a variety.
Then $\mathrm{trop}(X)$ is the support of a polyhedral complex



Pf By the lemma, $w \in \operatorname{trop}(X) \cap \Gamma^n$ if and only if $\operatorname{in}_{(0,w)}(I_{proj})$ does not contain a monomial.

So $\operatorname{trop}(X) =$ subcomplex of Gröbner complex consisting of σ for which $\operatorname{in}_w(I_{proj})$ does not contain a monomial for $w \in \operatorname{relint}(\sigma)$.

Sketch of Pf [for details, see survey article on webpage]

1. Show that the Hilbert functions of I & $\operatorname{in}_w(I)$ agree
   ie $\dim_K \left(\frac{K[x_0 \dots x_n]}{I} \right)_d = \dim_k \left(\frac{k[x_0, \dots x_n]}{\operatorname{in}_w(I)} \right)_d$
2. Show $\forall w \in \Gamma^n$, $\forall v \in \mathbb{Q}^n$ $\exists \varepsilon > 0$ s.t. $\operatorname{in}_v(\operatorname{in}_w(I)) = \operatorname{in}_{w+\varepsilon v}(I)$
3. Every ideal has a monomial initial ideal.
4. There are only a finite number of monomial initial ideals of I.
   This gives a bound D on the degrees of generators of initial ideals.
5. Construct a polynomial $F = \sum a_u x^u$ with the Gröbner complex the regions of linearity of $\operatorname{trop}(F)$.
   $F = \prod_{i=1}^{D} f_i$ $\quad \operatorname{trop}(f_d)$ is linear on regions $\sigma \Rightarrow$ $w, w' \in \sigma$ $\operatorname{in}_w(I)_d = \operatorname{in}_{w'}(I)_d$

Aside:

Q. Given I, $w \in \Gamma^n$, what is the polyhedron in the Gröbner complex corresponding to w?

A. Choose $v \in \mathbb{Q}$ for which $\operatorname{in}_v(\operatorname{in}_w(I))$ is monomial, & let $g_1, \dots, g_s$ be a reduced GB for I wrt $w + \varepsilon v$

ie $g_i = x^{u_i} + \sum c_{iu} x^u$

$x^{u_i} \nmid x^u$ for $w + \varepsilon v$

The polyhedron is given by

$$w \cdot u_i \leq \operatorname{val}(c_{iu}) + w \cdot u$$

Γ-rational.

The polyhedron for w is a face of this.