Tropical semiring

$\mathbb{R} \cup \{\infty\}, \oplus, \otimes$

$\oplus$ = min, $\otimes$ = $+$

A tropical polynomial F is a piecewise linear fcn.

Tropical hypersurface $V(F) = \{w \in \mathbb{R}^n : F \text{ is not diff'al at } w\}$

$= \{w : \text{min in } F \text{ is achieved twice}\}$

If $f = \sum c_u x^u \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

where K is a field with a val.

then $\operatorname{trop}(f)(w) = \min(\operatorname{val}(c_u) + w \cdot u)$

$\operatorname{trop}(V(f)) := V(\operatorname{trop}(f))$

If $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,

then $\operatorname{trop}(V(I)) = \bigcap_{f \in I} \operatorname{trop}(V(f))$.

$V(I) \subseteq (K^*)^n$

Note: $\operatorname{trop}(V(f)) = \operatorname{trop}(V(x^u f))$ for any x^u.

(since $\operatorname{trop}(x^u f)(w) = \operatorname{trop}(f)(w) + \underbrace{w \cdot u}_{\text{diff'able everywhere}}$)

Today: Fundamental theorem & structure theorem.

First: Gröbner theory & and an alternative characterization of tropical varieties

Recall $\Gamma = \operatorname{im} \operatorname{val}$.

Fix a splitting $\varphi : \Gamma \to K^*$ of the valuation on K.

ie $\varphi : \Gamma \to K^*$ group hom

$\operatorname{val}(\varphi(w)) = w$

eg $K = \mathbb{C}\{\{t\}\}$ $\quad w \mapsto t^w$ $\quad \Gamma = \mathbb{Q}$

$K = \mathbb{Q}$ p-adic val $\quad w \mapsto p^w$ $\quad \Gamma = \mathbb{Z}$

$K = \mathbb{C}$ trivial val. $\quad 0 \mapsto 1$ $\quad \Gamma = \{0\}$

Notation: $w \mapsto t^w$

ie t^w means $\varphi(w)$

<u>Lemma</u> If K is algebraically closed, then a splitting always exists.

Pf. Lemma 2.1.13.

aside: $\operatorname{val}_2(\sqrt{3}) = \frac{1}{2} \operatorname{val}_2(3) = 0$

<u>Defn</u> Let $R = \{a \in K : \mathrm{val}(a) \geq 0\}$

This is a local ring, with maximal ideal $m = \{a \in K : \mathrm{val}(a) > 0\}$

Let $k = R/m$ — the residue field.

eg $K = \mathbb{C}\{\{t\}\}$ $R =$ power series with rational exp. with a common denom

$m =$ " " & constant term 0

$k = \mathbb{C}$

eg $K = \mathbb{Q}$ p-adic val.

$R = \mathbb{Z}_{(p)}$ $k = \mathbb{Z}/p\mathbb{Z}$

$m = p\mathbb{Z}_{(p)}$.

<u>Defn</u> Let $f = \sum c_u x^u \in K[x_1^{\pm 1}, x_n^{\pm 1}]$

Fix $w \in \Gamma^n$ (or in $\mathbb{Q}^n$ if $\Gamma = (0)$)

The initial term of $f = \sum c_u x^u$ is

$$in_w(f) = \overline{t^{-\mathrm{trop}(f)(w)} f(t^{w_1}x_1, \ t^{w_n}x_n)}$$

(for $a \in R$, $\overline{a}$ is the image in k)

$$= \overline{t^{\underbrace{-\mathrm{trop}(f)(w)}_{\min(\mathrm{val}(c_u) + w \cdot u)}} \sum \underbrace{c_u t^{w \cdot u}} x^u}$$

$$= \sum_{\substack{\mathrm{val}(c_u) + w \cdot u \\ = \mathrm{trop}(f)(w)}} \overline{t^{-\mathrm{val}(c_u)} c_u} \, x^u \in k[x_1^{\pm 1}, x_n^{\pm 1}]$$

eg $f = 6x^2 + 5xy + 3y^2 + 4x + 7y + 9 \in \mathbb{Q}[x_1^{\pm 1} y^{\pm}]$ 3-adic val.

$w = (2,1)$

$\mathrm{trop}(f) = \min(2x+1, x+y, 2y+1, x, y, 2)$

$\mathrm{Trop}(f)(w) = \min(5, 3, 3, 2, 1, 2) = 1$

$in_w(f) = \overline{3^{-0} \cdot 7}\, y = y \in \mathbb{Z}/3\mathbb{Z}[x^{\pm}, y^{\pm}]$

$w = (1,1)$ $in_w(f) = \overline{4}x + \overline{7}y = x + y$

$w = (0,-1)$ $\mathrm{trop}(f)(w) = \min(1, -1, -1, 0, -1, 2)$

$in_{(0,-1)}(f) = \overline{3^{-1} 3}\, y^2 + \overline{5}\, xy + \overline{7}\, y = y^2 + 2xy + y \in \mathbb{Z}/3[x^{\pm}, y^{\pm}]$

<u>Defn</u> Fix an ideal $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$w \in \Gamma^n$ (or $\mathbb{Q}^n$ if $\Gamma = (0)$)

The initial ideal of I wrt w is

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$$

Warning: If I is $\langle f_1, \ldots, f_r \rangle$,

$\langle \mathrm{in}_w(f_1), \ldots, \mathrm{in}_w(f_r) \rangle \subseteq \mathrm{in}_w(I)$

but this inclusion can be proper.

A generating set $f_1, \ldots, f_r$ for I with equality is called a Gröbner basis for I.

Notes: Gröbner bases can be computed. If $K = \mathbb{C}$, or $K = \mathbb{C}(t)$ this can be done with usual Gröbner bases. For more general fields there are also alg.

(sorry Bath)

Theorem [Fundamental theorem of tropical algebraic geometry]

Let $X = V(I)$ be a subvariety of $(K^*)^n$, where $K = \overline{K}$ val is nontrivial

The following subsets of $\mathbb{R}^n$ coincide:

1) $\operatorname{trop}(X) = \bigcap_{f \in I} \operatorname{trop}V(f)$

2) $\operatorname{cl}\{ w \in \mathbb{R}^n : \operatorname{in}_w(I) \neq \langle 1 \rangle \}$

closure in Euclidean top.

3) $\operatorname{cl}(\operatorname{val}(X(K)))$

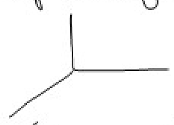
$\operatorname{val}(X(K)) = \{ (\operatorname{val}(y_1), \ldots, \operatorname{val}(y_n)) : y = (y_1, \ldots, y_n) \in X(K) \}$

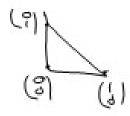
4) $\bigcup_{K \subseteq L} \operatorname{val}(X(L))$

field extension with val on L extending that on K.

eg $X = V(x+y+1) \subseteq (\mathbb{C}^*)^2$
$\subseteq (\mathbb{C}\{\{t\}\}^*)^2$

$\operatorname{Trop}(x+y+1) = \min(x, y, 0)$





$\langle \operatorname{in}_w(x+y+1) \rangle$

$= \langle 1 \rangle$ unless $\operatorname{in}_w(x+y+1)$

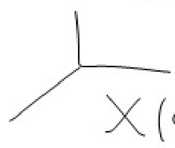
contains at least two terms

Options: $w = (\lambda, 0)$ $\lambda > 0$ $\operatorname{in}_w(x+y+1) = y+1$

$w = (0, \lambda)$ $\lambda > 0$ $\operatorname{in}_w(x+y+1) = x+1$

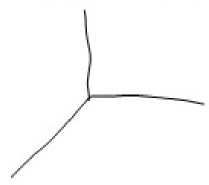
$w = (-\lambda, -\lambda)$ $\lambda > 0$ $\operatorname{in}_w(x+y+1) = x+y$

$w = (0,0)$ $\operatorname{in}_{(0,0)}(x+y+1) = x+y+1$



$X(\mathbb{C}\{\{t\}\}) = \{ (a, -1-a) : a \in \mathbb{C}\{\{t\}\} \setminus \{0, -1\} \}$

$(\operatorname{val}(a), \operatorname{val}(-1-a)) = \begin{cases} (\operatorname{val}(a), 0) & \text{if } \operatorname{val}(a) > 0 \\ (\operatorname{val}(a), \operatorname{val}(a)) & \text{if } \operatorname{val}(a) < 0 \\ (0, \operatorname{val}(b)) & \text{if } a = -1+b, \ \operatorname{val}(b) > 0 \\ (0,0) & \text{o/w} \end{cases}$



Theorem is due to many people.

Kapranov X a hypersurface $1 \leftrightarrow 3/4$

Sturmfels Gröbner connection $1 \leftrightarrow 2$

Speyer, Draisma, ...

Point: 1) Tropical varieties can be computed.

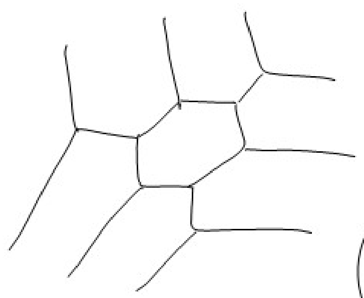
2) Trop(X) is a shadow of X. (Gröbner connection)

A goal of tropical geometry is to recover information about X from trop(X).

Structure theorem

Let $X \subseteq (K^*)^n$ be an irreducible (not union of two proper subvariet.) d-dimensional variety in $(K^*)^n$.

Then trop(X) is the support of a pure d-dim weighted balanced Γ-rational polyhedral complex connected through codim one.



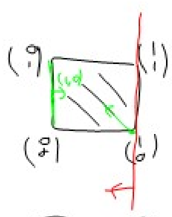
← union of rays & line segments with rational slope.

ie one-dimensional

Defn A polyhedron in $\mathbb{R}^n$ is a set of the form

$$\{x \in \mathbb{R}^n : Ax \leq b\}$$

A: $s \times n$ matrix; $\leq$: coordwise; $b \in \mathbb{R}^s$



$$\left\{x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$$

A face of P is a set of the form $\mathrm{face}_w(P) = \{y \in P : w \cdot y \leq w \cdot x \ \forall x \in P\}$

eg $\mathrm{face}_{(1,0)}(P)$ = left edge.

$\mathrm{face}_{(-1,1)}(P) = \{\binom{1}{0}\}$

P is Γ-rational if $A \in \mathbb{Q}^{s \times n}$ & $b \in \Gamma^s$.

This means P has rational facet (codim 1 face) normals & vertices in Γ^n.

A polyhedral complex Σ is a collection of polyhedra for which any nonempty intersection of two polyhedra is a face of each.

The support of Σ is the set of all $x \in \mathbb{R}^n$ in any polyhedron in Σ

The affine span of a polyhedron P is the affine subspace of $\mathbb{R}^n$ $\underline{u} + \mathrm{span}(\underline{v} - \underline{u} : \underline{v} \in P)$
$\underline{u} \in P$

$\dim(P) = \dim$ of its affine span.

Σ is pure of dim d if every maximal polyhedron in Σ wrt inclusion has dim d.

Σ is connected through codim one if we can walk between d-dim polyhedra, only crossing $(d-1)$-dim ones.
i.e. the graph with vertices d-dim polyhedra, edges common facets is connected