MA5Q6 GRADUATE ALGEBRA - HOMEWORK 7

DUE MONDAY 3/12, 12PM

Hand in all questions to Carole Fisher's office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

(1) Consider the morphisms $\psi\colon \mathbb{Z}^3 \xrightarrow{A} Z^5$ and $\phi\colon \mathbb{Z}^5 \xrightarrow{B} \mathbb{Z}$ given by the matrices

$$A = \begin{pmatrix} -6 & -26 & -82 \\ 0 & 4 & 3 \\ 1 & 0 & 7 \\ 0 & 2 & 5 \\ 2 & 10 & 30 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -2 & -4 & -2 & -4 \end{pmatrix}.$$

(a) Verify that $0 \to Z^3 \xrightarrow{\psi} Z^5 \xrightarrow{\phi} Z \to 0$ is a complex.

(b) (Not to be handed in). "Recall" that the *Smith normal form* of an integer $d \times n$ matrix A is a $d \times n$ matrix D with $D_{ij} = 0$ if $i \neq j$, and $D_{ii}|D_{i+1,i+1}$ for which there are matrices $U \in \mathrm{GL}(d, \mathbb{Z})$, $V \in \mathrm{GL}(n, V)$ with $D = UAV$. If you have not seen the Smith normal form before, look it up. Learn how to compute it in your favourite computer algebra system. For example, in maple the relevant commands are `ismith` or `SmithForm`.

(c) Given a complex

$$0 \to Z^a \xrightarrow{\psi} \mathbb{Z}^b \xrightarrow{\phi} \mathbb{Z}^c \to 0.$$

Give an algorithm to compute the homology $\ker(\phi)/\operatorname{im}(\psi)$. Hint: Use the Smith normal form of the matrix of ϕ to change coordinates on $\mathbb{Z}^b$ so $\ker(\phi)$ has a nice form. Then use the Smith normal form on the matrix of ψ to compute the cokernel of the map $\mathbb{Z}^a \to \ker(\phi)$.

(d) Apply this algorithm to the complex at the start of the questions to compute the homology $\ker(\phi)/\operatorname{im}(\psi)$.

(2) Show that $\mathrm{Tor}_0^R(M, N) = M \otimes_R N$

(3) Recall that $\mathrm{Ext}^i(A, D)$ is the ith homology of the complex obtained by applying $\mathrm{Hom}(-, D)$ to a projective resolution for A. Show that if P is a projective module then $\mathrm{Ext}^i(P, A) = 0$ for $i > 0$.

(4) The Wedderburn-Artin theorem implies that $\mathbb{C}[\mathbb{Z}/5\mathbb{Z}]$ is a product of matrix rings over division rings. Write down this isomorphism explicitly (ie write down an explicit ring homomorphism from $\mathbb{C}[\mathbb{Z}/5\mathbb{Z}]$ to a product of matrix rings, and check that this is an isomorphism).

(5) Repeat the previous question with $\mathbb{Z}/5\mathbb{Z}$ replaced by S_3.

(6) Regard $\mathbb{C}^n$ as a $\mathbb{C}[S_n]$ module where permutations π act by permuting coordinates (eg $(123)\mathbf{e}_1 = \mathbf{e}_2$ where $\mathbf{e}_i$ is the ith standard basis vector). Write $\mathbb{C}^n$ as a direct sum of simple $\mathbb{C}[S_n]$-modules.