MA5Q6 ALGEBRAIC GEOMETRY - HOMEWORK 1

DUE TUESDAY 16/10, 12PM

Hand in all questions not marked as "(Not to be handed in)" to the box outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own. Write on your homework the names of the people you worked with, and any references you consult.

This first homework is largely warm-up, and so I can see the level of the class (so also let me know if you found this easy, or difficult).

(1) (Not to be handed in) Look up Russell's Paradox. Why is the set of all bijections $f : X \to X$ of a set X a set?

(2) How many semigroups are there on two elements, up to isomorphism? How many monoids? How many groups?

(3) Show the following:
   (a) In a monoid the identity element e is unique.
   (b) In a group the inverse of an element is unique.
   (c) If a is a unit in a ring R with identity, then its left and right inverses coincide.
   (d) The set of units of a ring with identity form a group under multiplication.

(4) Let G be the subgroup of the group $M_2(\mathbb{C})$ of 2×2 matrics with entries in the complex numbers generated by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and let H be the subgroup of $M_2(\mathbb{C})$ generated by

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

What is the cardinality (size) of G and H? Are they isomorphic? Which group you have hopefully met before is each of these?

(5) Let G be a group. The set of automorphisms $\mathrm{Aut}(G)$ of G is the set of all group isomorphism $\phi : G \to G$.
   (a) Show that $\mathrm{Aut}(G)$ is a group.
   (b) Show that $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Aut}(\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/(p-1)\mathbb{Z}$ when p is prime.

(6) (Not to be handed in) Look up the quarternion algebra. Convince yourself that this is a division ring.

(7) Recall that the characteristic of field k is the smallest n for which 1 added to itself n times is zero, or zero if there is no such n. Show that the characteristic of a field is either zero or a prime p.