MATH 559 HOMEWORK 1

DUE: MONDAY, FEBRUARY 5

All rings R are commutative with 1.

(1) Recall that a ring R is Noetherian if it satisfies the *ascending chain condition*: There is no infinite properly ascending chain of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$ in R. Show that R is Noetherian if and only if every ideal I of R is finitely generated. (This is a standard result, and in the textbook - the point of this exercise is just to remind you of this result, and if you look it up it is optional to actually write down the proof.)

(2) Let A be an abelian group. A ring R is graded by A if we can write $R = \oplus_{a \in A} R_a$ where if $f \in R_a$, $g \in R_b$ then $fg \in R_{a+b}$. The most standard case is $A = \mathbb{Z}$, where we also often assume that $R_i = 0$ for $i < 0$. A homogeneous element of R is an element of some R_i, and a homogeneous ideal is one generated by homogeneous elements.
   (a) Show that if R is graded by A and $f \in R$ then we can write f uniquely as a sum of homogeneous elements (called the homogeneous components of f).
   (b) If I is a homogeneous ideal in ring graded by an abelian group A, then $f \in I$ if and only if each homogeneous compponenent of f lies in I.

(3) The reverse lexicographic term-order was defined as: $x^u \prec x^v$ if $\deg(x^u) \prec \deg(x^v)$ or $\deg(x^u) = \deg(x^v)$ and the last nonzero entry of $(v-u)$ is negative. Why is the degree condition necessary? In other words, why can we not define an order by $x^u \prec x^v$ if the last nonzero entry of $(v-u)$ is negative?

(4) Show that if $\prec$ is a term order then there are no infinite descending chains $x^{u_1} \succ x^{u_2} \succ x^{u_3} \succ \dots$.

(5) **Proof of Hilbert basis theorem** The goal of this question is to prove the Hilbert basis theorem, which states that the polynomial ring is Noetherian. Let $S = k[x_1, \dots, x_n]$.
   (a) Let M be an ideal of S generated by monomials. Show that a monomial x^u lies in M if and only if it is divisible by some monomial generator.

   (b) Show that a monomial ideal in M must be finitely generated. (Hint: induction on n). This result is usually called "Dickson's Lemma".
   (c) Conclude that every ideal $I \subset S$ is finitely generated.

(6) Let $S = \Bbbk[x_1, \ldots, x_n]$, and let $I \subseteq S$ be an ideal.
   (a) Show that the monomials not in $\mathrm{in}_\prec(I)$ form a $\Bbbk$-basis for S/I, where $\prec$ is any term order.
   (b) If R is a graded ring, then an R-module M is graded if we can write $M = \oplus_{a \in A} M_a$ as an abelian group, with in addition if $r \in R_a$ and $m \in M_b$ then $rm \in M_{a+b}$. Show that if I is a homogeneous ideal with respect to any abelian group grading of S for which the variables are homogeneous then the S-module S/I is graded.
   (c) Show that in this case the monomials not in $\mathrm{in}_\prec(I)$ of degree a form a $\Bbbk$-basis for $(S/I)_a$.

(7) Compute a Gröbner basis for the ideal $\langle x^2y - 1, xy^2 - 1 \rangle$ with respect to the reverse lexicographic term order with $x \succ y$.