NATURAL TRANSFORMATIONS

MATH 551 NOTES - DIANE MACLAGAN

Definition 1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let S and T be covariant functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* α from S to T is a collection of morphisms $\{\alpha_C : C \in ob(\mathcal{C})\}$ in $\mathcal{D}$, where $\alpha_C \in \hom(S(C), T(C))$ (ie α_C is a morphism from $S(C)$ to $T(C)$) such that if $f : C \to C'$ is a morphism in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} S(C) & \xrightarrow{\alpha_C} & T(C) \\ {\scriptstyle S(f)}\downarrow & & \downarrow{\scriptstyle T(f)} \\ S(C') & \xrightarrow[\alpha_{C'}]{} & T(C') \end{array}$$

When all the maps α_C are equivalences (that is, $S(C)$ is equivalent to $T(C)$ for all $C \in \mathcal{C}$), then α is called a natural isomorphism, or natural equivalence.

To give an example, we need to recall some linear algebra. Let V be an n-dimensional vector space over $\mathbb{R}$ (so $V \cong \mathbb{R}^n$). The *dual* of V, denoted V^* is the set of linear maps from V to $\mathbb{R}$.

Lemma 2. *The set V^* is an n-dimensional vector space.*

Proof. We can add two linear maps $((f+g)(v) = f(v) + g(v))$, and multiply them by a scalar $((\lambda f)(v) = \lambda f(v))$, so V^* is a vector space.

To calculate the dimension, let $v_1, \dots, v_n$ be a basis for V, and let $f : V \to \mathbb{R}$ be a linear map. Then $f(\sum \lambda_i v_i) = \sum \lambda_i f(v_i)$, so the function f is determined by its values on $v_1, \dots, v_n$. Let f_i be the functional defined by setting $f_i(v_j) = 1$ if $i = j$, and 0 otherwise. Then $f_1, \dots, f_n$ are linearly independent, and a general $f = \sum f(v_i) f_i$, so they span V^*, and thus V^* is n-dimensional. $\square$

Note that while the above proof gives an explicit isomorphism of V and V^*, given by sending v_i to f_i, this isomorphism depends on the choice of basis. However if we consider the double dual V^{**}, we get a more canonical isomorphism. Let $\phi_v \in V^{**}$ be defined by $\phi_v(f) = f(v)$ for $f \in V^*$. Then the map $v \mapsto \phi_v$ is an isomorphism from V to V^{**}.

Let $\mathbf{Vect_n}$ be the category whose objects are all n-dimensional vector spaces, and whose morphisms are linear maps. Let 1 be the identity functor from $\mathbf{Vect_n}$ to $\mathbf{Vect_n}$ (so 1 takes V to V, and a linear map f to itself). Let F be the functor from $\mathbf{Vect_n}$ to itself that takes a vector space V to V^{**}. If $f\colon V \to W$ is a linear map, then the linear map $F(f)\colon V^{**} \to W^{**}$ is given by setting for $\beta \in V^{**}$, $F(f)(\beta)$ is the element of W^{**} that takes an element an element $\psi \in W^*$ to $\beta(\psi \circ f) \in \mathbb{R}$.

We claim that there is a natural transformation α that takes 1 to F. For $V \in \mathbf{Vect_n}$ let $\alpha_V : V \to V^{**}$ be the morphism in $\mathbf{Vect_n}$ defined by sending v to ϕ_v for all $v \in V$. Let $f : V \to W$ be a morphism in $\mathbf{Vect_n}$. Then $F(f) \circ \alpha_V : V \to W^{**}$ takes $v \in V$ to the element of W^{**} that takes $\psi \in W^*$ to $\phi_v(\psi \circ f) = (\psi \circ f)(v)$. Thus $(F(f) \circ \alpha_V)(v) = \phi_{f(v)}$. The morphism $\alpha_W \circ 1(f) : V \to W^{**}$ takes $v \in V$ to $\alpha_W(f(v)) = \phi_{f(v)}$, so the following square commutes:

$$\begin{array}{ccc} V & \xrightarrow{\alpha_V} & V^{**} \\ {\scriptstyle 1(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ W & \xrightarrow[\alpha_W]{} & W^{**} \end{array}$$

Since $V \cong V^{**}$, the map α is a natural isomorphism.

Exercise: Check that this doesn't work if we replace V^{**} by V^*.