MA5 ALGEBRAIC GEOMETRY - HOMEWORK 5

DUE THURSDAY, 4TH DECEMBER 2PM

You are encouraged to work together on the homework, but please acknowledge all collaboration. You are also free to consult any texts you choose, but again please acknowledge references cited. Let me know if you find any (suspected) mistakes in these questions.

A: Warm-up problems

(1) Consider $X = V(3x_1+4x_2-x_3, x_1^2-5x_2^2-3x_3^2) \subset \mathbb{A}^3$. Compute the dimension of X.
(2) Show that if $X \subset \mathbb{A}^n$ is a d-dimensional subspace, then $\dim(X) = d$.
(3) Let $X = V(6xy + 3x + 2y + 1) \subseteq \mathbb{A}^2$. Compute all singular points of X.
(4) Let $X = V(5x^2+7xy-8x+y-1) \subseteq \mathbb{A}^2$. Compute all singular points of X.
(5) Let $X = \mathbb{V}(x_0x_2 - x_1^2) \subset \mathbb{P}^2$. Let $X_i = X \cap U_i \subseteq \mathbb{A}^2$. Show that $\dim(X_i)$ is the same for $i = 0, 1, 2$. What is it?

B: Exercises

(1) Let $X = V(y^2+5yz+z^2, x^2-2xy-5yz, 15xyz+4xz^2+50yz^2+11z^3) \subset \mathbb{A}^3_{\mathbb{C}}$. What is the dimension of X?
(2) Compute the dimension of the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ into $\mathbb{P}^5$.
(3) Find all singular points of $X = V(f) \subseteq \mathbb{A}^3$, where $f = (x-1)^2(x-2)^2 + (y-3)^2 + z^2$.
(4) Let $X \subseteq \mathbb{A}^n$ be an affine variety. Show that the set $\{a \in \mathbb{A}^n : X \text{ is singular at } a\}$ is a Zariski closed set. This is called the *singular locus* of X.
(5) Show that the ideal $I_{2,4}$ of the Grassmannian $G(2,4)$ is principal.
(6) Let $S = \Bbbk[x_I : I \subset \{1, \dots, n\}, |I| = d]$, and let $I_{d,n} = \langle p_{J_1J_2} : |J_1| = d-1, |J_2| = d+1\rangle$, where

$$p_{J_1J_2} = \sum_{j \in J_2} (-1)^{\mathrm{sign}(j)} x_{J_1 \cup j} x_{J_2 \setminus j}.$$

We will now show that $\mathbb{V}(I_{d,n}) \subseteq \mathbb{P}^{\binom{n}{d}-1}$ equals $G(d,n)$ in its Plücker embedding, as defined in class.

(a) Show that if $V \subseteq \Bbbk^n$ is a d-dimensional subspace, then $\phi(V)$ satisfies all equations $p_{J_1 J_2}$ if $J_1 \cap J_2 = \emptyset$. Hint: Think about the submatrix of A_V indexed by the columns $J_1 \cup J_2$. What can we assume about the columns indexed by J_1?

(b) Show that the same is true if $J_1 \cap J_2 \neq \emptyset$. Hint: One method here would be induction on d.

(c) Now consider $x \in \mathbb{V}(I_{d,n})$. Assume that $x_I \neq 0$ for $I = \{1, \dots, d\}$. Show that there is a unique d-dimensional V with $\phi(V) = [x]$.

C: Extensions

(1) Show that the Grassmannian $G(d,n)$ is smooth for all d, n.

(2) Smooth quadric surfaces.

(a) Let $X = \mathbb{V}(x_0x_3 - x_1x_2)$ be the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$. Show that for every point $p \in X$, there are exactly two lines L_1, L_2 in $\mathbb{P}^3$ with $p \in L_i \subset X$ for $i = 1, 2$.

(b) Fix a homogeneous polynomial $f \in K[x_0, x_1, x_2, x_3]$ of degree two, and let $X = \mathbb{V}(f) \subset \mathbb{P}^3$. Write $f = \mathbf{x}^T A \mathbf{x}$ where A is a 4×4 matrix and $\mathbf{x}$ is the vector $(x_1, x_2, x_3, x_4)^T$. What conditions on A mean that X is smooth?

(c) When the surface X of the previous part is smooth, show that for every point $p \in X$ there are exactly two lines in $\mathbb{P}^3$ containing p and contained in X (as in the first part). (Hint: There are several ways to do this. One method uses the first part. Another considers the intersection of X with the tangent plane to X at p.).