MA5 ALGEBRAIC GEOMETRY - HOMEWORK 2

DUE THURSDAY, 30/10/14, AT 2PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the homework, but please acknowledge all collaboration. You are also free to consult any texts you choose, but again please acknowledge references cited. Let me know if you find any (suspected) mistakes in these questions.

A : Warm-up problems

(1) Let $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ be given by $\phi(t) = (t^4, t^2, t+1)$. Compute $\phi^*(f)$ for the following $f \in \mathbb{k}[x,y,z]$:
  (a) $f = x$;
  (b) $f = z$;
  (c) $f = x^2 - y^2 + 3xz$

(2) Repeat the previous question for $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ given by $\phi(x,y) = (x+y, x^2-y^2)$ and $f = z_1^2 - z_2^2$, $f = z_1$, and $f = z_2$. Here z_1, z_2 are the coordinates on the second copy of $\mathbb{A}^2$.

(3) Check that the map $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ given by $\phi(x,y) = (x+y, x-y)$ is an isomorphism. Determine whether $\phi(x,y) = (2x+y, x+3y)$ and $\phi(x,y) = (2x+y, 4x+2y)$ are isomorphisms.

(4) Let $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ be given by $\phi(t) = (t, t^3, t^4)$, and let $Y = V(xz - y^2) \subseteq \mathbb{A}^3$. Does ϕ determine a morphism from $\mathbb{A}^1$ to Y?

(5) Check that the polynomial ring is the *free commutative k-algebra*; this means that to define a k-algebra homomorphism f from the polynomial ring $\mathbb{k}[x_1, \dots, x_n]$, it suffices to give $f(x_i)$, and these can be chosen abitrarily. .

B: Exercises

(1) Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ and $Z \subseteq \mathbb{A}^p$ be varieties, and let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms. Let $\alpha = \psi \circ \phi$. Show that $\alpha^* = \phi^* \circ \psi^*$.

(2) Let $\phi : \mathbb{A}^n \to \mathbb{A}^n$ be the morphism defined by $\phi^* : \mathbb{k}[z_1, \dots, z_n] \to \mathbb{k}[x_1, \dots, x_n]$ where $\phi^*(z_i) = \sum_{j=1}^n a_{ij} x_j$ for $a_{ij} \in \mathbb{k}$. Give necessary and sufficient conditions for ϕ to be an isomorphism.

(3) Let $Y = V(x^2 - y, xy^2 - z^2)$, and consider $\phi : \mathbb{A}^1 \to Y$ given by $\phi(t) = (t^2, t^4, t^5)$. Show that ϕ is a bijection, but not an isomorphism. (Recall that a morphism can have several descriptions, so it does not suffice to give one description of the inverse, and note that that is not a morphism.)

(4) Let $X = V(xy - z) \subseteq \mathbb{A}^3$, and let $\phi : \mathbb{A}^3 \to \mathbb{A}^2$ be given by $\phi(x, y, z) = (x, z)$. What is the image $\phi(X) \subseteq \mathbb{A}^2$? What is the closure $\overline{\phi(X)}$?

(5) Let $\phi : \mathbb{A}^1 \to \mathbb{A}^4$ be the morphism given by $\phi(t) = (t^2, t^3, t^5, t^6)$. Find equations for $\overline{\operatorname{im}(\phi)}$. Did we need to take the closure here?

(6)
 (a) Let $\alpha = \sqrt[3]{3} + \sqrt{7}\sqrt[4]{2}$. Show that there is a polynomial p in the ideal $\langle a^3 - 3, b^2 - 7, c^4 - 2, \tilde{\alpha} - (a + bc)\rangle \subseteq \mathbb{Q}[a, b, c, \tilde{\alpha}]$ depending only on $\tilde{\alpha}$ with $p(\alpha) = 0$ (ie substituing the real number α for the variable $\tilde{\alpha}$ gives zero). If we can choose this polynomial to be irreducible, it is the minimal polynomial of α. You may assume the Nullstellensatz: $I(V(I)) = \sqrt{I}$ when k is algebraic closed.
 (b) Use the previous part to compute the minimal polynomial of α. (Optional: compare with answers obtained from other computer algebra systems).
 (c) Let α be a root of the polynomial $x^2 + a_1x + a_0 \in \mathbb{Q}[x]$, and β be a root of the polynomial $x^2 + b_1x + b_0 \in \mathbb{Q}[x]$ (so $a_i, b_j \in \mathbb{Q}$). Write down polynomials with $\alpha + \beta$ and $\alpha\beta$ as roots.

C: Extensions

(1) Understand the statement given in class about the equality of categories for varieties and k-algebras (only for those who plan to learn more algebraic geometry or related fields).