MA5 ALGEBRAIC GEOMETRY - HOMEWORK 1

DUE THURSDAY 16/10, 2PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

A : Warm-up problems

(1) Recall from Algebra II the definition of a quotient ring R/I where I is an ideal in a ring R. Verify that ring homomorphism $\phi : R/I \to S$ from R/I to a ring S are in bijection with ring homomorphisms $\tilde{\phi} : R \to S$ with $\tilde{\phi}(r) = 0$ for all $r \in I$.

(2) Recall from Algebra II the definition of an algebraically closed field. Why is $\mathbb{C}$ algebraically closed? Which other fields you know are algebraically closed?

(3) Recall from Introduction to Topology (or this class) the definition of a topological space. Check that the Zariski topology defined in class satisfies the axioms of a topological space.

(4) Recall from Algebra II the definition of a prime ideal. Check that an ideal I in a commutative ring R is prime if and only if the quotient ring R/I is a domain (has no zero-divisors). Find all prime ideals in the following rings:
   (a) $\mathbb{Z}$,
   (b) $\mathbb{C}[x, y]$,
   (c) $\mathbb{C}[x]/\langle x^2 \rangle$, and
   (d) $\mathbb{C}[x, y]/\langle xy \rangle$.

(5) Let $I \subset S := \Bbbk[x_1, \ldots, x_n]$ be an ideal, and fix a term order $\prec$ on S. Show that for every monomial $x^u \in \mathrm{in}_\prec(I)$, there is $f \in I$ with $\mathrm{in}_\prec(f) = x^u$. Hint: What is $\mathrm{in}_\prec(fg)$ for $f, g \in S$?

B: Exercises

(1) Let $f = x^2y + xy^2 + y^2$, and $\prec$ be the lexicographic order with $x \succ y$. Apply the division algorithm to find the remainder on dividing f by $\{xy - 1, y^2 - 1\}$. Repeat with the polynomials in the order $\{y^2 - 1, xy - 1\}$. Do these two polynomials form a Gröbner basis for the ideal they generate?

(2) Fix a term order $\prec$ on the polynomial ring $K[x_1, \dots, x_n]$. Recall that $\mathcal{G} = \{g_1, \dots, g_s\}$ is a *reduced* Gröbner basis for an ideal I if $\mathcal{G}$ is a Gröbner basis, $\{\mathrm{in}_\prec(g_1), \dots, \mathrm{in}_\prec(g_s)\}$ is an irredundant (no repeats) minimal generating set for $\mathrm{in}_\prec(I)$, and for each g_i, no term of g_i other than its initial term is divisible by $\mathrm{in}_\prec(g_j)$ for any $1 \leq j \leq s$. Show that every ideal has a unique reduced Gröbner basis.

(3) Show, by giving an example, that the reduced Gröbner basis and initial ideal for an ideal depend on the choice of term order. Characterize which homogeneous ideals have only one initial ideal. An ideal is homogeneous if it has a generating set consisting of homogeneous polynomials. A polynomial is homogeneous if every monomial occuring in the polynomial has the same degree.

(4) (Hassett #3.7) Recall that the closure $\overline{S}$ of a set S in a topological space is the smallest closed set containing S. Compute the Zariski closures $\overline{S} \subset \mathbb{A}^2$ for $K = \mathbb{Q}$ for the following sets:
   (a) $S = \{(n^2, n^3) : n \in \mathbb{N}\}$
   (b) $S = \{(x, y) : x^2 + y^2 < 1\}$
   (c) $S = \{(x, y) : x + y \in \mathbb{Z}\}$

(5) Learn to use a computer algebra system that computes Gröbner bases. I highly recommend `Macaulay2`, which is freely available at: http://www.math.uiuc.edu/Macaulay2/. An older version packaged for windows is at: http://www.commalg.org/m2win/. `Macaulay2` is installed on the Linux machines on the ground floor. Another option is `Sage`, freely available from http://www.sagemath.org/. In either case there are tutorials available from the main program webpage.

   Attach a printout of the Gröbner bases for $I = \langle x^5 + y^4 + z^3, x^3 + y^2 + z^2 - 1\rangle$ with respect to the revlex and lexicographic orders.

(6) (CLO p99 #11) Let $X = V(x + y + z - 3, x^2 + y^2 + z^2 - 5, x^3 + y^3 + z^3 - 7)$.
   (a) Show that $x^4 + y^4 + z^4 = 9$ for all $(x, y, z) \in X$.
   (b) What is the value of $x^5 + y^5 + z^5$ for $(x, y, z) \in X$? (Hint: Compute remainders)
   (c) (Harder) What is the size of X? (you may answer for either the field $K = \mathbb{R}$, or $K = \mathbb{C}$). (Hint: Compute a lexicographic Gröbner basis - you will probably want to use a computer).

C: Extensions

(1) The algorithm to compute Gröbner bases is called Buchberger's algorithm. It can be thought of as a generalization of Gaussian elimination. The key step is the formation of the S-polynomial of two polynomials $f, g \in I$:

$$S(f,g) = \operatorname{lcm}(\operatorname{in}_\prec(f), \operatorname{in}_\prec(g))/\operatorname{in}_\prec(f) f - \operatorname{lcm}(\operatorname{in}_\prec(f), \operatorname{in}_\prec(g))/\operatorname{in}_\prec(g) g.$$

The Buchberger algorithm then proceeds by computing the S-polynomial of any pair of generators for I, finding the remainder on dividing this polynomial by the generators, and adding this remainder to the generating set if nonzero. This procedure continues until the generating set stabilizes, at which point the generating set is a Gröbner basis.

(a) Read a proof that this algorithm works (for example, in Cox, Little, O'Shea, or Hassett).

(b) Use it to compute a Gröbner basis for the ideal $I = \langle x^2, y^2, xy + yz \rangle \subset k[x,y,z]$ with respect to the revlex term order. (Eisenbud *Commutative Algebra*, Ex 15.27).

(2) An ideal $I \subset \Bbbk[x_1, \ldots, x_n]$ is *radical* if $I = \{f \in \Bbbk[x_1, \ldots, x_n] : f^k \in I \text{ for some } k > 0\}$.

(a) Let $I \subset \Bbbk[x_1, \ldots, x_n]$ be an ideal. Show that if $\operatorname{in}_\prec(I)$ is radical, then I is radical.

(b) Given an example to show that the converse does not hold; that is, give an example of a radical ideal all of whose initial ideals are radical.

(c) Compute the radical of the ideal $\langle x_1^2, x_1x_2^3, x_2^2x_3 \rangle \subset \Bbbk[x_1, x_2, x_3]$.

(d) Characterize which monomial ideals are radical.