3G6 COMMUTATIVE ALGEBRA - HOMEWORK 6

NOT ASSESSED - BUT EXAMINABLE.

(1) Give two different minimal primary decompositions of the ideal $I = \langle x^2y^3, x^3y^2, x^4y\rangle \subset K[x,y]$.
(2) Show that if $I \subset R$ satisfies $P = \sqrt{I}$ is prime, then R/I has only one minimal prime.
(3) Write down an example of an ideal $I \subset K[x,y,z]$ with $|\operatorname{Ass}(R/I)| > 1$ but $\sqrt{I}$ prime.
(4) Show that if $\operatorname{val}\colon K \setminus \{0\} \to \mathbb{R}$ is a valuation, then $\operatorname{val}(1) = 0$, $\operatorname{val}(-a) = \operatorname{val}(a)$, and $\operatorname{val}(1/a) = -\operatorname{val}(a)$.
(5) Show that if $\operatorname{val}\colon K \setminus \{0\} \to \mathbb{R}$ is a valuation, and $a, b \in K \setminus \{0\}$ with $\operatorname{val}(a) \neq \operatorname{val}(b)$, then $\operatorname{val}(a+b) = \min(\operatorname{val}(a), \operatorname{val}(b))$.
(6) Let $K = k(t)$ be the ring of rational functions in t, with the valuation given in lectures. Show that the residue field of K is isomorphic to k.
(7) Show that the residue field $R/\mathfrak{m}$ of $\mathbb{Q}$ with the p-adic valuation is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.
(8) Let $K = \mathbb{C}(\!(t)\!) = \{\sum_{i=N}^{\infty} a_i t^i : a_i \in \mathbb{C}, N \in \mathbb{Z}\}$.
   (a) Check that the natural addition and multiplication make this into a field. Here by "natural" I mean the extension of the addition and multiplication from the ring of power series. This is the field of Laurent series.
   (b) Check (as claimed in lectures) that the function given by $\operatorname{val}(\sum_{i=N}^{\infty} a_i t^i = N)$ when $a_N \neq 0$ obeys the valuation axioms.
   (c) What is the valuation ring of this valuation? What is the residue field?
(9) The field of Puiseux series is $\mathbb{C}\{\{t\}\} = \cup_{n\geq 1}\mathbb{C}(\!(t^{1/n})\!)$.
   (a) Check that $\mathbb{C}\{\{t\}\}$ is a field.
   (b) We define $\operatorname{val}\colon \mathbb{C}\{\{t\}\}\setminus\{0\} \to \mathbb{R}$ by $\operatorname{val}(\sum a_q t^q) = \min_{a_q \neq 0}(q)$. Show that this is a valuation.
   (c) What is the valuation ring of this valuation? What is the residue field?