3G6 COMMUTATIVE ALGEBRA - HOMEWORK 5

DUE THURSDAY 16TH MARCH, 2PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

A : Warm-up problems

(1) Verify that the ideal $I = \langle x^3, y^5, z^2\rangle \subset \mathbb{C}[x, y, z]$ is irreducible.
(2) Verify that the ideal $I = \langle x^3, x^2y, xy^3, y^5\rangle \subset \mathbb{C}[x, y, z]$ is primary. Show that it is not irreducible by writing it as the intersection of two larger ideals.
(3) Let M be the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. What is $\mathrm{Ass}(M)$?

B: Exercises

(1) Let G be a finitely generated abelian group, viewed as a $\mathbb{Z}$-module. What is $\mathrm{Ass}(G)$? (Hint: You secretly learned this in Algebra 1).

For the remaining questions, let $S = K[x_1, \ldots, x_n]$ be the ring of polynomials over a field K. Recall from the start of term that monomial ideals have a unique minimal generating set consisting of monomials.

(2) Show that a monomial ideal I is prime if and only if it is generated by some of the variables of S.
(3) Show that if $I = \langle x^{\mathbf{u}_1}, \ldots, x^{\mathbf{u}_r}\rangle$ is a monomial ideal in S, then

$$I = \langle x_1^{(\mathbf{u}_1)_1}, x^{\mathbf{u}_2}, \ldots, x^{\mathbf{u}_r}\rangle \cap \langle x^{\mathbf{u}_1}/x_1^{(\mathbf{u}_1)_1}, x^{\mathbf{u}_2}, \ldots, x^{\mathbf{u}_r}\rangle.$$

Here we have the convention that $x_1^0 = 1$.
(4) Show that a monomial ideal $I \subset S$ cannot be written as the intersection of two strictly larger monomial ideals if only if I has a generating set consisting of powers of variables. (In fact, this criterion characterizes when I is irreducible, but you do not need to prove this here).
(5) Show that a monomial ideal $I \subset S$ is primary if and only if whenever x_i divides a monomial minimal generator of I, some power x_i^m of x_i is in I.

(6) Describe an algorithm to compute an irreducible decomposition of a monomial ideal in S. You may assume that the criterion of Q4 characterizes irreducibility. Carry this out for the monomial ideal $I = \langle y^3, x^3, y^2z^3, xyz^3, x^2z^3 \rangle \subseteq K[x, y, z]$.

(7) Describe an algorithm to compute a minimal primary decomposition of a monomial ideal in S. Carry this out for the monomial ideal $I = \langle y^3, x^3, y^2z^3, xyz^3, x^2z^3 \rangle \subseteq K[x, y, z]$.

C: Extensions

For extra exercises see the non-assessed HW6. Other extra exercises can be found the recommended textbooks, in the sections indicated on the schedule.