3G6 COMMUTATIVE ALGEBRA - HOMEWORK 4

DUE TUESDAY 7 MARCH, 2PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

A : Warm-up problems

(1) Show that if $I \subset J \subset K[x_1, \ldots, x_n]$, then $V(J) \subset V(I)$. Show that if $X \subset Y \subset K^n$, then $I(Y) \subset I(X)$.
(2) Let $f = x^4 + 2x^3 + x^2$. What is $\sqrt{\langle f \rangle}$? In general, what can you say about the radical of a principal ideal in a UFD?
(3) Show that localization commutes with quotients: If $N \subset M$ are R-modules, and U is a multiplicatively closed subset of R, then $(M/N)[U^{-1}] \cong (M[U^{-1}])/(N[U^{-1}])$.

B: Exercises

(1) Show that $\mathbb{Z}[\sqrt{3}]$ is integrally closed.
(2) Let $R = \mathbb{Q}[t^2, t^5] \cong \mathbb{Q}[x, y]/\langle x^5 - y^2 \rangle$. Compute the integral closure of R in its field of fractions.
(3) We saw in lectures that if S is a finite extension of R (i.e., $R \subset S$ and S is a finitely generated R-module), then S is integral over R. Give a counterexample to the converse: Give an example $R \subset S$ for which S is integral over R, but S is not a finitely generated R-module.
(4) The squarefree part of a monomial $x^{\mathbf{u}}$ is $\prod_{i : u_i > 0} x_i$. Show that if $I = \langle x^{\mathbf{u}_1}, \ldots, x^{\mathbf{u}_r} \rangle$ is a monomial ideal, then the radical $\sqrt{I}$ is generated by the squarefree parts of $x^{\mathbf{u}_1}, \ldots, x^{\mathbf{u}_r}$. Use this to compute the radical of $\langle x^{10}, x^2y^5, x^3y^3z^3, x^2z^4, y^7, y^9z, z^8 \rangle$.
(5) Show that if $I \subset K[x_1, \ldots, x_n]$ is an ideal, and $\prec$ is a term order with $\mathrm{in}_\prec(I)$ a radical ideal, then I is radical. Give a counterexample to show that the converse is not true (i.e., there is a radical ideal I for which $\mathrm{in}_\prec(I)$ is not radical).
(6) Let I, J be ideals in $K[x_1, \ldots, x_n]$. Recall that $I + J = \{i + j : i \in I, j \in J\}$, and $IJ = \langle ij : i \in I, j \in J \rangle$. Show that $V(I + J) = V(I) \cap V(J)$, and $V(I \cap J) = V(IJ) = V(I) \cup V(J)$.

C: Extensions

(1) Let n be a squarefree integer (no divisible by m^2 for any integer m), and let $K = \mathbb{Q}(\sqrt{n})$. Let $\alpha = (1 + \sqrt{n})/2$ if $n \equiv 1 \mod 4$, and $\alpha = \sqrt{n}$ if $n \equiv 2$ or $3 \mod 4$ (the case $n \equiv 0 \mod 4$ is ruled out by the squarefree hypothesis). Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$ is $\mathbb{Z}[\alpha]$.

(2) Question B2 can be generalized as follows. A *submonoid of* $\mathbb{N}^n$ is a subset $A \subset \mathbb{N}^n$ that is closed under addition. The monoid algebra $K[A]$ with coefficients in a field K is the set of formal finite sums $\sum_{i=1}^{r} a_i m_i$ with $a_i \in K$, and $m_i \in A$ for which addition is coordinate-wise $((2(2,3)+3(3,7))+(3(2,3)-(3,7)) = (5(2,3)+2(3,7)))$ and multiplication by using the distributative property: $(2(2,3)+3(3,7))(3(2,3)-(3,7)) = (6(4,6)+7(5,10)-3(3,7))$. What can you say about when the domain $K[A]$ is integrally closed in its field of fractions? Start with the case that $n = 1$, so $A \subseteq \mathbb{N}$.

(3) Let I be an ideal in $K[x]$ where K is an arbitrary field (ie not necessarily algebraically closed). Give an algorithm to compute the radical $\sqrt{I}$. Hint: One can take a (formal) derivative f' of a polynomial f with coefficients in any field. Think about the gcd of f and f'. Your algorithm will show that if, for example, $K = \mathbb{Q}$, the radical of $\langle f \rangle$ is generated by a polynomial with coefficients in $\mathbb{Q}$ even if f has no rational roots.