3G6 COMMUTATIVE ALGEBRA - HOMEWORK 3

DUE TUESDAY 21 FEBRUARY, 2PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

A : Warm-up problems

(1) (If you didn't do this when working on HW2) Let I be an ideal of a ring R. Show that there is a bijection between ideals in R/I and ideals in R containing I.
(2) Let $R = \mathbb{Z}/10\mathbb{Z}$, and let $U = \{1, 5\}$. Describe $R[U^{-1}]$.
(3) Show that $\langle x^2 + y^2, xy \rangle \subset \mathbb{C}[x, y]$ is not a free module.
(4) Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, where the entries are in $\mathbb{Z}/5\mathbb{Z}$. Compute the characteristic polynomial of A.

B: Exercises

(1) Show that if R is a local ring with maximal ideal $\mathfrak{m}$, then $R_\mathfrak{m} \cong R$.
(2) Let R be a domain, and let $f \in R$ be a nonzero non-unit. Let $U = \{1, f, f^2, f^3, \dots\}$. We write $R[1/f]$ for $R[U^{-1}]$. Prove that $R[1/f]$ is not finitely-generated as an R-module. Give an example to show that this might not be true if R is not a domain (ie give an example of R that is not a domain, and $f \in R$ with $R[1/f]$ finitely generated as an R-module. Be sure to carefully justify your answer!
(3) Let $R = \mathbb{C}[x, y]$, and let $\phi\colon R^2 \to R^2$ be the R-module homomorphism given by the matrix

$$A = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where $f_{ij} \in \mathbb{C}[x, y]$. Give necessary and sufficient conditions for ϕ to be an isomorphism. You must completely justify your answer.

(4) Let $\mathfrak{m}$ be the unique maximal ideal in $\mathbb{C}[x,y]_{\langle x,y\rangle}$. Show that $\mathfrak{m} = \langle x+7y+8x^2+9y^2, 2x-y-4x^3+8x^5y^5\rangle$. (Hint: Nakayama's lemma)

(5) (Eisenbud Exercise 2.6: Generalized Chinese Remainder Theorem). Let R be a ring, and let $Q_1, \dots, Q_n$ be ideals of R such that $Q_i + Q_j = R$ for all $i \neq j$. Show that $R/(\cap_i Q_i) \cong \prod_i R/Q_i$ as follows:
  - (a) Consider the map of rings $\phi \colon R \to \prod_i R/Q_i$ obtained from the n projection maps $R \to R/Q_i$. Show that $\ker \phi = \cap_i Q_i$.
  - (b) Show that if $\mathfrak{m}$ is a maximal ideal of a ring R, Q is an ideal of R, and $\psi \colon R \to R_{\mathfrak{m}}$ is the homomorphism sending r to $r/1$, then $\psi(Q)R_{\mathfrak{m}} = R_{\mathfrak{m}}$ if $Q \not\subseteq \mathfrak{m}$.
  - (c) Let $\mathfrak{m}$ be a maximal ideal of R. Show that the hypothesis that $Q_i + Q_j = R$ for all $i \neq j$ means that at most one of the Q_i is contained in $\mathfrak{m}$. Use this to show that the homomorphism ϕ of part (a) is surjective.

C: Extensions

(1) Let $R = K[x,y]/\langle xy\rangle$, and let $P = \langle x, y-1\rangle$, and $P' = \langle x-1, y\rangle$ be two ideals in R. Show that P and P' are prime. Describe $\mathrm{Spec}(R_P)$ and $\mathrm{Spec}(R_{P'})$ (i.e., classify all prime ideals in these two rings).

(2) Show that localization has the following universal property: If $\phi \colon R \to S$ is a ring homomorphism which takes all elements of $U \subset R$ to units of S, then there is a unique induced ring homomorphism from $R[U^1]$ to S. Show that this property defines the localization up to unique isomorphism: if T is a ring for which every ring homomorphism from R to a ring S that takes elements of U to units of S factors uniquely through T, then T is uniquely isomorphic to $R[U^1]$.

(3) Let R be a ring. Show that an ideal that is maximal with respect to not being finitely generated (ie any larger ideal is finitely generated) is prime.

(4) Let R be a ring. Show that an ideal that is maximal among those that are not principal is prime.

(5) Generalize your answer to Question B3.