3G6 COMMUTATIVE ALGEBRA - HOMEWORK 1

DUE TUESDAY 24 JANUARY, 2PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

Recall that all rings in this module are commutative with multiplicative identity, unless otherwise stated.

Let me know of any (suspected) typos in these problems as soon as possible.

A : Warm-up problems

(1) Why is $0a = 0$ true in any ring?

(2) Compare the definition of an ideal given in lectures to the one you were given in the Algebra II lecture notes. Why are these the same?

(3) Remind yourself of the Euclidean algorithm and long division of polynomials in one variable. Write down a polynomial f in x of degree 4 and a polynomial g of degree 2 and compute the remainder on dividing f by g.

(4) Remind yourself about Gaussian elimination (row operations on matrices). Write down a 3×4 matrix, and do row operations until it is in "row reduced form" (also called "reduced row echelon form").

(5) The reverse lexicographic term-order was defined as: $x^u \prec x^v$ if $\deg(x^u) < \deg(x^v)$ or $\deg(x^u) = \deg(x^v)$ and the last nonzero entry of $(v - u)$ is negative. Why is the degree condition necessary? In other words, why can we not define an order by $x^u \prec x^v$ if the last nonzero entry of $(v - u)$ is negative?

B: Exercises

(1) Let $S = K[x_1, \dots, x_n]$, and let $I \subset S$ be an ideal. Fix a term order $\prec$.
   (a) Show that S/I is a vector space over K.
   (b) Show that the monomials not in $\mathrm{in}_\prec(I)$ form a K-basis for S/I.

(2) Let $I \subset S := K[x_1, \dots, x_n]$ be an ideal, and fix a term order $\prec$ on S. Show that for every monomial $x^u \in \mathrm{in}_{\prec}(I)$, there is $f \in I$ with $\mathrm{in}_{\prec}(f) = x^u$. Hint: What is $\mathrm{in}_{\prec}(fg)$ for $f, g \in S$?

(3) Let $f = x^2y + xy^2 + y^2$, and $\prec$ be the lexicographic order with $x \succ y$. Apply the division algorithm to find the remainder on dividing f by $\{xy - 1, y^2 - 1\}$. Repeat with the polynomials in the order $\{y^2 - 1, xy - 1\}$. Do these two polynomials form a Gröbner basis for the ideal they generate?

(4) Use Buchberger's algorithm to compute a Gröbner basis for the ideal $\langle x^2, y^2, xy + yz\rangle \subset \mathbb{Q}[x, y, z]$ with respect to the reverse lexicographic term order with $x \succ y \succ z$.

(5) Learn to use a computer algebra system that computes Gröbner bases. I highly recommend `Macaulay2`, which is freely available at: http://www.math.uiuc.edu/Macaulay2/. `Macaulay2` is installed on the Linux machines on the ground floor. There is a short introduction in the Announcements section of the main module webpage. Another option is `Sage`, freely available from http://www.sagemath.org/. In either case there are tutorials available from the main program webpages.

Attach a printout of the Gröbner bases for $I = \langle x^5 + y^4 + z^3, x^3 + y^2 + z^2 - 1\rangle$ with respect to the reverse lex and lexicographic orders.

(6) In this question you will use Gröbner bases to solve a recreational mathematics problem that doesn't mention ideals or even polynomials. Consider a standard magic square, such as:

$$\begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}.$$

Recall that this means that the three rows, the three columns, and the two diagonals all sum to the same number (15 in our example). Note that if we read the rows as numbers forwards or backwards, we get the same sum of squares:

$$816^2 + 357^2 + 492^2 = 618^2 + 753^2 + 294^2.$$

Show this holds for an arbitrary magic square as follows: Write the magic square as

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

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(a) We know $a+b+c-d-e-f=0$ if the array is a magic square. Write down six more equations of the form $F=0$ for some linear polynomial F in $a, b, \ldots, i$ for which the array is a magic square if and only if these equations hold. Justify your answer.

(b) Let $I \subset \mathbb{Q}[a, \ldots, i]$ be the ideal generated by your seven polynomials. Show that if $F \in I$, then F is zero on any magic square.

(c) Show that $(100a+10b+c)^2+(100d+10e+f)^2+(100g+10h+i)^2-(100c+10b+a)^2-(100f+10e+d)^2-(100i+10h+g)^2 \in I$. Hint: Use Gröbner bases and your new Macaulay2 skills! (attach a printout if necessary).

C: Extensions

(1) Fix $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ and a term order $<$. Define an order $\prec$ on the monomials in $K[x_1, \ldots, x_n]$ by $x^{\mathbf{u}} \prec x^{\mathbf{v}}$ if $\mathbf{w} \cdot \mathbf{u} < \mathbf{w} \cdot \mathbf{v}$ or $\mathbf{w} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{v}$ and $x^{\mathbf{u}} < x^{\mathbf{v}}$. Show that $\prec$ is a term order. What goes wrong if we take $\mathbf{w} \in \mathbb{R}^n$ instead of $\mathbb{R}^n_{\geq 0}$? Deduce that there are an uncountable number of different term orders on $K[x_1, \ldots, x_n]$ for $n \geq 2$.

(2) (Much harder) Fix an ideal $I \subset K[x_1, \ldots, x_n]$. Show that there are only a finite number of different initial ideals $\mathrm{in}_{\prec}(I)$ as $\prec$ varies over different term orders. Hint: One approach uses Question 1 and Dickson's lemma. Why does this not contradict the previous question? (Easier) Give an example of two different term orders $\prec$, $\prec'$ with $\mathrm{in}_{\prec}(I) = \mathrm{in}_{\prec'}(I)$.

(3) Read more about Gröbner bases and their algorithms. An excellent reference is Chapter 2 of *Ideals, Varieties, and Algorithms* by Cox, Little, and O'Shea, and also second volume *Using Algebraic Geometry* by the same authors. What is the Gröbner fan?