MATH 244 SAMPLE FINAL

SOLUTIONS

(1) $y(t) = 2(t-1)e^{2t} + 3e^t$.
(2) $y = -\sqrt{2x - 2x^2 + 4}$.
(3) Let $S(t)$ be the amount of salt in the tank in pounds after t minutes. Then $dS/dt = 1 - S/50$, and $S(0) = 0$, so after 10 minutes we have $50(1-e^{-1/5}) = 9.06$ pounds of salt in the tank. We then have the salt begin to leave the tank according to the differential equation $dS/dt = -S/50$. We then get a final amount of $50(1-e^{-1/5})e^{-1/5} = 7.42$ pounds of salt.
(4) The backwards Euler method gives $y_n = (y_{n-1} - 3t_n h)/(1-2h)$, so $y(0.1)$ is approximately $(1 - 3(0.1)(0.1))/(1-0.2) = 0.97/0.8 = 1.2125$.
(5) $y_1 = e^t \sin(\sqrt{5}t)$, $y_2 = e^t \cos(\sqrt{5}t)$.
(6) The general solution is $y(t) = c_1 e^{2t} + c_2 t e^{2t} + 1/2t^2 e^{2t}$, from which we get $(y) = c_1 = 1$, and $y'(0) = 2c_1 + c_2 = 0$, so $c_1 = 1, c_2 = -2$. So $y(t) = (1/2t^2 - 2t + 1)e^{2t}$.
(7) $y(t) = c_1 + c_2 e^t + c_3 e^{-t} + \sin(t)$.
(8) (a) $y(x) = c_1|x| + c_2|x|\ln|x|$.
(b) The function x^r approaches zero if and only if $r > 0$. From the equation $r^2 + (\alpha-1)r + 5/2 = 0$ we see $r = 1/2(1-\alpha \pm \sqrt{((\alpha-1)^2 - 10)})$. If the part under the squareroot is positive it suffices to have $1-\alpha > 0$, so $\alpha < 1$. If the part under the squareroot is negative then we again need $1 - \alpha > 0$. Finally, if the part under the squareroot is zero, we would have $1 - \alpha = \sqrt{10}$, and $y(x) = x^{\sqrt{10}/2}\ln(x)$ a solution, which does not approach zero. So the solution is $\alpha < 1$.
(9) The general solution is $\mathbf{x}(t) = c_1 e^{3t}(1,5)^t + c_2 e^{-t}(1,1)^t$, so $\mathbf{x}(0) = c_1(1,5)^t + c_2(1,1)^t = (1,3)^t$, and thus $c_1 = 1/2 = c_2$. Thus the solution is $x(t) = 1/2e^{3t} + 1/2e^{-t}$, $y(t) = 5/2e^{3t} + 1/2e^{-t}$.
(10) $\mathbf{x}(t) = c_1 e^{-t}\begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin(2t) \\ -\cos(2t) \end{pmatrix}$.
(11) $\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2(\begin{pmatrix} 4t+1 \\ 8t \end{pmatrix}$.
(12) (a) This has complex eigenvalues with a positive real part, so is an outward directed spiral.
(b) This has one eigenvalue -3 and two independent eigenvalues, so all trajectories end towards the origin on straight lines.
(c) This has eigenvalues ± 1, with eigenvectors $(1,1)$ for -1 and $(3,2)$ for 1. The trajectories thus head in on the straight line through the origin containing $(1,1)$ and out on the straight line through the origin containing $(3,2)$. Other trajectories sweep between them for an unstable saddle point at the origin.
(13) (a) The (real) critical points are $(1,1), (-1,-1)$.

(b) The relevant matrix is $\begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}$, so for $(1,1)$ we have repeated eigenvalues of -2, so this is asymptotically stable. For $(-1,-1)$ we have one positive and one negative eigenvalue, so this is an unstable saddle.

(14) The recurrence is $a_{n+2} = 1/(n+2)a_n$, so $y_1(x) = 1 + x^2/2 + x^4/8 + x^6/48 + x^8/384 + \cdots = \sum_{n=0}^{\infty} x^{2n}/(2^n n!)$ and $y_2(x) = x + x^3/3 + x^5/15 + x^7/105 + \cdots = \sum_{n=0}^{\infty} 2^n n! x^{2n+1}/(2n+1)!$.