Mathematics 244: Lab 5

Trajectories in the Phase Plane

Spring 2006

0. Introduction and Setup

In this lab, we shall use Maple to study the **qualitative properties** of **autonomous systems** of two differential equations.

The first step is to obtain the **seed file** from the web page and save it in your directory on eden. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own. Please turn in only the printout of your main Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet. Remove any extraneous material from the worksheet. There is also a **supplementary worksheet** that suggests additional experiments that may lead to better answers to the questions considered in this project.

As in Lab4, we require the `DEtools`, `plots`, and `LinearAlgebra` packages. There are also some constants that will be used in both of the remaining sections. To assure that these definitions are available, both the seed file and the supplementary worksheet contain the following instructions.

```
with(plots): with(DEtools): with(LinearAlgebra):
window:=x=-6..6,y=-6..6;
trange:=-6..6;
inits:=[[x(0)=3,y(0)=3],[x(0)=-1,y(0)=4],
        [x(0)=2,y(0)=-2],[x(0)=-1,y(0)=-4]];
```

Each numbered section of this lab deals with a system of two **autonomous** differential equations, i.e., a system of the form

$$x'(t) = F(x, y), \qquad y'(t) = G(x, y).$$

Note that the **distinguishing feature** of an autonomous system is that the expressions defining the functions F and G do not contain the independent variable t. This allows many properties of the solutions to be studied using the curves, called **trajectories**, that show the path in the xy plane followed by the solutions (It is an easy exercise, mentioned in Lab 4, to show that if an initial condition is on a trajectory, then the whole solution follows that trajectory). The Maple command `DEplot` may be used to draw trajectories and direction fields for such systems.

1. A linear system

Consider the linear system:

$$x' = x + 2y,$$
$$y' - 5x - 2y.$$

You teach Maple to recognize this equation using the instructions

```
L1:=[x+2*y,-5*x-2*y];
de1x:=diff(x(t),t)=eval(L1[1],{x=x(t),y=y(t)});
de1y:=diff(y(t),t)=eval(L1[2],{x=x(t),y=y(t)});
```

that appear in the seed file and supplementary worksheet for lab5. The right side of the equation is entered initially as `L1` using variables `x` and `y` that are converted to the expressions `x(t)` and `y(t)` when building the equations. The `eval` instructions are useful since the functions are required in some places, but simple variables are easier to use and clearer when their use is accepted.

This system has (0, 0) as an equilibrium point. We will study the stability of that point.

a. The Direction field

The following instruction can be used to recall previous definitions and construct a plot of the direction field of this system. The plot consists of small arrows pointing the way of the trajectories in the square $-6 \le x \le 6$, $-6 \le y \le 6$.

```
df1:=DEplot([de1x,de1y],[x(t),y(t)], trange, window,color=GREEN):
```

(note the colon at the end to suppress output of the plot structure). The `color` options is used to give a better view of the **nullclines** and solution curves to be added later. You should also test different values of `dirgrid` as in Lab 2 and customize the value for the plots in this project. A value should be chosen that allows individual arrows to be identified while including a sufficient number of arrows. To show this plot, you need only enter its name `df1;` after defining it. This **should not be done in the main worksheet**, since we will include this as part of a `display` command later.

b. Nullclines

The points where the direction field is **horizontal** (characterized by $dy/dt = 0$) or **vertical** (characterized by $dx/dt = 0$) form curves called **nullclines**. In many cases, these curves provide useful information about the behavior of trajectories without the excessive detail of a direction field. They can be plotted by the following instructions (the color is intended to distinguish these curves on the screen — it is not necessary to print your report in color).

```
dh1:=plot(-5*x/2,window,color=coral):
dv1:=plot(-x/2,window,color=violet):
display({df1,dh1,dv1},title="Slope field and nullclines");
```

Note that the nullclines usually **cut across the arrows** in the slope field since they are **not** solutions of the equation (except in rare cases). The line `dh1`, colored `coral` shows where the direction field is horizontal, and the line `dv1`, colored `violet` shows where the direction field is vertical.

c. Trajectories

To study the stability of an equilibrium point, it is also useful to draw the direction field together with several trajectories. In Lab 4, we did this by finding the **exact** solution to the equation. Since this will usually not be possible for the nonlinear equations to be studied later in this project, we use the **numerical methods** that are part of the `DEplot` command. This will allow us to experiment with those methods before they are really needed. These experiments will be done in the supplementary worksheet, with **only the conclusions** reported here. The initial points (3, 3), (−1, 4), (2, −2), and (−1, −4) were specified in the earlier definition of `inits`. To draw the direction field together with the trajectories through these points, execute the command `DEplot([de1x,de1y],[x(t),y(t)], trange, inits, window,linecolor=[RED, BLUE, BROWN, PLUM],thickness=2);` . This form of the command appears in the supplementary worksheet. **The result should not be satisfactory** since Maple chose a `stepsize` that was **much too large** (for our `trange`, the `stepsize` is .6). The trajectories drawn as coarse polygons instead of smooth curves, and these trajectories do not follow the arrows very closely. **Experiment** with different explicit values of a `stepsize` option until you find one that gives **smooth trajectories**. **Copy the completed command to your main worksheet and execute it there**. The plot in the main worksheet should have a `title`.

In a **Discussion** section, list the different stepsizes used in your experiments, and the reason for your choice. Your choice should be the **simplest value giving an acceptable graph** — a value that is too small uses more computer resources without improving the graph. **Simplicity** also calls for using values requiring only a few decimal places.

d. Stability

The plot reveals curves that spiral towards the origin, so the origin is a stable point of this system. The eigenvalues of the matrix can be shown to be complex numbers of negative real part, which is the algebraic characterization of stable spiral points.

It is also possible to recognize the nature of the equilibrium point using the **trace-determinant plane** without actually computing eigenvalues.

A more elaborate method of proving stability involves the construction of a Liapunov function. This is done in the supplementary worksheet.

Select **one** method for proving the origin stable and include the details in a **discussion** section.

2. An almost linear system.

Consider the almost linear system

$$x' = 2y - 2x + xy - x^2 = (2 + x)(y - x)$$
$$y' = 4y + 4x - xy - x^2 = (4 - x)(y + x).$$

The equilibrium solutions are $[x = 0, y = 0]$, $[x = -2, y = 2]$, and $[x = 4, y = 4]$. Maple can obtain these by using

```
F2:=2*y - 2*x + x*y - x^2;
G2:=4*y + 4*x - x*y - x^2;
eqpts:=solve({F2,G2},{x,y});
```

which are included in the seed file.

a. The Direction field.

The expressions `F2` and `G2` can be rewritten in the form needed to specify a differential equation using the commands `de2x:=diff(x(t),t)=eval(F2,x=x(t),y=y(t));`, and `de2y:=diff(y(t),t)=eval(G2,x=x(t),y=y(t));`. (a variant based on the `subs` command will also work). Modify the command that you used to plot the slope field in Section 1 to plot the slope field of this system. Test different options in the supplementary worksheet and copy your best `plot` command to the main worksheet. Use the `DEplot` command to obtain a plot of the direction field of this system using the values of `trange` and `window` defined earlier. You should **test** the plot in the supplementary worksheet, but not include the plot here until it is combined with other plots in a later display.

b. Nullclines.

In this example, the factored form of the expressions for dx/dt and dy/dt allows a simple parametric description of the nullclines. In particular, the slope field is horizontal along `dh2`, whose plot can be constructed using

```
dh2:=plot([[4,t,t=trange],[t,-t,t=trange]],window,color=coral):
```

Construct a similar instruction to produce a plot showing where the slope field is vertical (you should use `color=violet` as before), and obtain a `display` (including a title) combining the slope field and both sets of nullclines. The equilibrium points should be seen as points lying on one nullcline of each color.

c. Trajectories. Graph the solutions belonging to the **initial conditions** `inits` defined in part 1. Choose a suitable `stepsize` to produce graphs. As before, use different colors to allow the solutions belonging to different initial conditions to be easily identified.

Include a **discussion** in which you describe the properties of the **portions** of the trajectories shown in the graph. In **some cases** properties as $t \to \infty$ or as $t \to -\infty$ may be suggested by the graph you have. Include such observations in your discussion. Be sure to describe **all four solutions**.

d. Stability The type and stability of the critical points can be determined by examining the eigenvalues of the corresponding linear system. The entries of matrix of the linear system corresponding to each critical point can be obtained by Maple by executing the following sequence of commands, which calculate the partial derivatives of the right hand sides of the differential equations, assembles them into a matrix, and then substitutes the coordinates of the critical point for x and y. The `Jac` functions constructed earlier gives a function of x and y which gives the linearization at each **stationary point** when **evaluated at those values** of (x, y). The different stationary points may have different types.

In a **discussion section**:

(1) give the coordinates of each stationary point;

(2) **find the linearizations** at each stationary point;

(3) describe the type (node, spiral, or saddle) of each stationary point, and say whether it is stable of unstable.

End of Lab 5