MA 243 HOMEWORK 7

DUE: THURSDAY, 22, 2007, BY 12PM

Hand in the problems in Section B *only* to the boxes outside the undergraduate office. You are encouraged to work together on the problems, but your written work should be your own.

A : Warm-up problems

(1) Prove the hyperbolic double-angle formula:

$$\cosh(\beta + \gamma) = \cosh(\beta)\cosh(\gamma) + \sinh(\beta)\sinh(\gamma).$$

(2) Calculate the intersection (if any) of the following lines in $\mathbb{H}^2$, and any common perpendiculars:
   (a) $L = \{x = y\} \cap \mathbb{H}^2, L' = \{t = 2x\} \cap \mathbb{H}^2$
   (b) $L = \{x = 2t\} \cap \mathbb{H}^2, L' = \{y = 3t\} \cap \mathbb{H}^2$
   (c) $L = \{5x = 4t\} \cap \mathbb{H}^2, L' = \{5y = 3t\} \cap \mathbb{H}^2$

B: Exercises

(1) Show that if $L = \Pi \cap \mathbb{H}^2$ is a line in $\mathbb{H}^2$ then there are an infinite number of vectors $\mathbf{v} \in \Pi$ with $q_L(\mathbf{v}) = 1$. Deduce that given a line L and a point P not on L there are an infinite number of lines L' passing through P and not intersecting L. Compare with $\mathbb{E}^2$ and S^2.

(2) Show that if A is a 3×3 matrix with columns $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2$ with $\mathbf{f}_0 \cdot_L \mathbf{f}_0 = -1$, $\mathbf{f}_1 \cdot_L \mathbf{f}_1 = \mathbf{f}_2 \cdot_L \mathbf{f}_2 = 1$, and $\mathbf{f}_i \cdot_L \mathbf{f}_j = 0$ for $i \neq j$, then

$$A^T J A = J,$$

for

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(3) (a) Show that if $T(\mathbf{x}) = A\mathbf{x}$ is a hyperbolic motion then $\det(A) = \pm 1$ (thus hyperbolic motions are either orientation preserving or reversing!)
   (b) Conclude that either 1 or -1 is an eigenvalue of A. (See below for a continuation of this question).

C: Extensions

(1) (Hyperbolic motions continued). Show that if $\mathbf{v}$ is an eigenvector of A with eigenvalue ± 1, then we can find an appropriate change of basis so that A is in block form with one block of size one and one block of size two. What must the block of size two look like? Why does this imply the classification of Section 3.11 of the notes?

(2) In class we used the hyperbolic cosine law to prove the triangle inequality, and then argued from the triangle inequality that collinearity is preserved by distance, so motions must take lines to lines. However the first step in the proof of the hyperbolic cosine law was to choose a good coordinate system in which $P = (1, 0, 0)$ and $Q = (\cosh(\beta), \sinh(\beta), 0)$. How do we know that that is possible in such a way that the line joining P to Q is taken to the line joining $(1, 0, 0)$ to $(\cosh(\beta), \sinh(\beta), 0)$? In other words, why does the map taking P to $(1, 0, 0)$ and Q to $(\cosh(\beta), \sinh(\beta), 0$ take lines to lines?