MA 243: HYPERBOLIC DISTANCE

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This note is to provide a contained proof of the fact that the distance in the hyperbolic plane $\mathbb{H}^2$ is well-defined. Recall that the Lorentz inner product of vectors $\mathbf{v} = (t_1, x_1, y_1), \mathbf{w} = (t_2, x_2, y_2) \in \mathbb{R}^3$ is

$$\mathbf{v} \cdot_L \mathbf{w} = -t_1 t_2 + x_1 x_2 + y_1 y_2.$$

We define the *distance* between $\mathbf{v}, \mathbf{w} \in \mathbb{H}^2 = \{(t,x,y) \in \mathbb{R}^3 : -t^2 + x^2 + y^2 = -1, t > 0\}$ to be

$$d(\mathbf{v}, \mathbf{w}) = \cosh^{-1}(-\mathbf{v} \cdot_L \mathbf{w}).$$

For this definition to be well-defined we need to know that $\mathbf{v} \cdot_L \mathbf{w} \leq -1$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{H}^2$.

Our proof will use the following lemma. Define $q_L(\mathbf{v}) = \mathbf{v} \cdot_L \mathbf{v}$.

Lemma 1. *If $\mathbf{v} = (t_1, x_1, y_1), \mathbf{w} = (t_2, x_2, y_2) \in \mathbb{R}^3$ with $q_L(\mathbf{v}) < 0$ and $\mathbf{v} \cdot_L \mathbf{w} = 0$ then $q_L(\mathbf{w}) > 0$.*

Proof. We have $-t_1^2 + x_1^2 + y_1^2 < 0$ and $-t_1 t_2 + x_1 x_2 + y_1 y_2 = 0$. Thus

$$\begin{aligned}
(-t_2^2 + x_2^2 + y_2^2)t_1^2 &= -t_1^2 t_2^2 + t_1^2(x_2^2 + y_2^2) \\
&= -(x_1 x_2 + y_1 y_2)^2 + t_1^2(x_2^2 + y_2^2) \\
&> -(x_1 x_2 + y_1 y_2)^2 + (x_1^2 + y_1^2)(x_2^2 + y_2^2) \\
&= ((x_1, y_1) \cdot (x_2, y_2))^2 + |(x_1, y_1)|^2 |(x_2, y_2)|^2 \\
&\geq 0,
\end{aligned}$$

where the last equality is the Cauchy-Schwartz inequality. Since $q_L(\mathbf{v}) < 0$ we must have $t_1 > 0$, so $-t_2^2 + x_2^2 + y_2^2 = q_L(\mathbf{w}) \geq 0$. □

We use this to show that we can choose coordinates well so that $P = (1,0,0)$ and $Q = (\cosh(s), \sinh(s), 0)$ for any $P, Q \in \mathbb{H}^2$. This will use the analogue for the Lorentz inner product of the *Gram-Schmidt algorithm* to construct an orthonormal basis of a vector space. Google `Gram-Schmidt` for a list of good resources on this topic.

Lemma 2. *Fix $\mathbf{v} = (t_1, x_1, y_1) \neq \mathbf{w} = (t_2, x_2, y_2) \in \mathbb{R}^3$. Then there is a linear map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ that preserves the Lorentz inner product,* *so* $(\mathbf{x} \cdot_L \mathbf{y}) = \phi(\mathbf{x} \cdot_L \phi(\mathbf{y}))$*, and such that* $\phi(\mathbf{v}) = (1,0,0)$ *and* $\phi(\mathbf{w}) = (\cosh(r), \sinh(r), 0)$ *for some* $r > 0$*.*

Proof. Let $\mathbf{f}_0 = \mathbf{v}$. Note that $q_L(\mathbf{f}_0) = -1$, since $\mathbf{v} \in \mathbb{H}^2$. Set $\mathbf{w}' = \mathbf{w} + (\mathbf{w} \cdot_L \mathbf{f}_0)\mathbf{f}_0$. Then $\mathbf{w}' \cdot_L \mathbf{f}_0 = \mathbf{w} \cdot_L \mathbf{f}_0 + (\mathbf{w} \cdot_L \mathbf{f}_0)\mathbf{f}_0 \cdot_L \mathbf{f}_0 = 0$, so $q_L(\mathbf{w}') > 0$ by Lemma 1. Set

$$\mathbf{f}_1 = \frac{\mathbf{w}'}{\sqrt{q(\mathbf{w}')}}.$$

Then $\mathbf{w} = c\mathbf{f}_0 + s\mathbf{f}_1$, for $c = -(\mathbf{w} \cdot_L \mathbf{f}_0)$ and $s = \sqrt{(}\mathbf{w}') > 0$.

Choose $\mathbf{u} \in \mathbb{R}^3$ not in the span of $\mathbf{v}$ and $\mathbf{w}$. Set

$$\mathbf{w}'' = \mathbf{u} + (\mathbf{u} \cdot_L \mathbf{f}_0)\mathbf{f}_0 - (\mathbf{u} \cdot_L \mathbf{f}_1)\mathbf{f}_1.$$

Again $q(\mathbf{w}'') > 0$. Set $\mathbf{f}_2 = \mathbf{w}''/\sqrt{q(\mathbf{w}'')}$.

Now let A be the 3 matrix with columns $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$. Note that A is invertible. Let $T(\mathbf{x}) = A\mathbf{x}$. Let J be the matrix

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} T(\mathbf{x}) \cdot T(\mathbf{y}) &= (A\mathbf{x}) \cdot_L (A\mathbf{y}) \\ &= \mathbf{x}^T A^T J A \mathbf{y} \\ &= \mathbf{x}^T J \mathbf{y} \\ &= \mathbf{x} \cdot_L \mathbf{y}, \end{aligned}$$

since $A^T J A = J$. Thus T preserves $\cdot_L$. Let $S = T^{-1}$. Then S also preserves $\cdot_L$, and $S(\mathbf{v}) = (1,0,0)$, and $S(\mathbf{v}) = (c, s, 0)$, with $s > 0$. Now S is a continuous function preserving q, so it preserves the branches of the hyperboloid $q(\mathbf{x}) = -1$. Since $t_2 > 0$ we have $S(Q)_1 = c > 0$. Since $c, s > 0$ and $-c^2 + s^2 = -1$, there is $r > 0$ for which $c = \cosh(r)$, $s = \sinh(r)$. □

Corollary 3. *We have* $-\mathbf{v} \cdot_L \mathbf{w} \geq 1$ *for all* $\mathbf{v}, \mathbf{w} \in \mathbb{H}^2$*.*