MA 243 HOMEWORK 7

SOLUTIONS

B: Exercises

1. **Show that if $L = \Pi \cap \mathbb{H}^2$ is a line in $\mathbb{H}^2$ then there are an infinite number of vectors $\mathbf{v} \in \Pi$ with $q_L(\mathbf{v}) = 1$. Deduce that given a line L and a point P not on L there are an infinite number of lines L' passing through P and not intersecting L. Compare with $\mathbb{E}^2$ and S^2.**

   Let $\mathbf{f}_0 \in L$, so $q_L(\mathbf{f}_0) = -1$. We can find a Lorentz basis $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2$ with $\mathbf{f}_1 \in \Pi$. Since $\mathbf{f}_0 \cdot_L \mathbf{f}_1 = 0$, we have $q_L(\mathbf{f}_1) = \lambda > 0$. Then for $a > \sqrt{1/\lambda}$ the vector $\mathbf{w}_a = 1/(\lambda a^2 - 1)(\mathbf{f}_0 + a\mathbf{f}_1) \in \Pi$ satisfies $q_L(\mathbf{w}_a) = 1$, so there are an infinite number of vectors $\mathbf{v}$ in Π with $q_L(\mathbf{v}) = 1$.

   Let P have position vector $\mathbf{w}$, and let $\Pi_a = \text{span}(\mathbf{w}, \mathbf{w}_a)$. Then $P \in L_a = \Pi_a \cap \mathbb{H}^2$, and $\Pi_a \cap \Pi = \text{span}(\mathbf{w}_a)$, so $L \cap L_a = \emptyset$, and $L_a \neq L_{a'}$ if $a \neq a'$. This contrasts with $\mathbb{E}^2$, where there is a unique line through a point P not intersecting a given line L, and S^2, where there are no lines not intersecting a given line L.

2. (a) **Show that if $T(\mathbf{x}) = A\mathbf{x}$ is a hyperbolic motion then $\det(A) = \pm 1$ (thus hyperbolic motions are either orientation preserving or reversing!)**

   Since $A^T JA = J$, $\det(A^T JA) = -\det(A)^2 = -1$, so $\det(A)^2 = 1$, and thus $\det(A) = \pm 1$.

   (b) **Conclude that either 1 or -1 is an eigenvalue of A. (See below for a continuation of this question).**

   We will make repeated use of the following fact. Let $\mathbf{x}, \mathbf{y}$ be two vectors in $\mathbb{R}^3$ or $\mathbb{C}^3$, with $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$, where again λ, μ can be real or complex. Then either $\lambda\mu = 1$, or $\mathbf{x}^T J\mathbf{y} = 0$. This follows from the fact that $\mathbf{x}^T J\mathbf{y} = (A\mathbf{x})^T J(A\mathbf{y}) = \lambda\mu\mathbf{x}^T J\mathbf{y}$.

   In particular if $A\mathbf{v} = \lambda\mathbf{v}$, where $\lambda, \mathbf{v}$ may be complex, then either $\lambda^2 = 1$, so $\lambda = \pm 1$ as required, or $\mathbf{v}^T J\mathbf{v} = 0$. So we may assume from now that if $\mathbf{v}$ is an eigenvector, then $\mathbf{v}^T J\mathbf{v} = 0$.

   In fact the same is true for generalized eigenvectors: if $(A - \lambda I)^k\mathbf{x} = 0$ for some k, and $A\mathbf{y} = \mu\mathbf{y}$ with $\lambda\mu \neq 1$ then

$\mathbf{x}^T J\mathbf{y} = 0$. This follows by induction on k, with the base case $k = 1$ above, since if $(A - \lambda I)^k\mathbf{x} = 0$, then $\mathbf{x}' = A\mathbf{x} - \lambda\mathbf{x}$ satisfies $(A-\lambda I)^{k-1}\mathbf{x}' = 0$, so $\mathbf{x}^T J\mathbf{y} = (A\mathbf{x})^T J(A\mathbf{y}) = \mu(\lambda\mathbf{x}+\mathbf{x}')^T J\mathbf{y} = \mu\lambda\mathbf{x}^T J\mathbf{y}$.

If there is only one eigenvalue λ for A, then we have $\lambda^3 = \pm 1$, and λ real (since A is a 3×3 matrix), so $\lambda = \pm 1$ as required. Otherwise let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of A, where we may have $\lambda_2 = \lambda_3$, but we may assume $\lambda_1 \neq \lambda_i$ for $i = 2, 3$. We assume that none of the λ_i is ± 1, so no product $\lambda_i\lambda_j$ is equal to ± 1 either. Let $\mathbf{x}_i$ be a basis of (generalized) eigenvectors for $\mathbb{R}^3$ (or $\mathbb{C}^3$ if $\lambda_i, \mathbf{x}_i$ are complex), so $(A - \lambda_i I)^3\mathbf{x}_i = 0$ for $i = 1, 2, 3$. Then by above we have $\mathbf{x}_1^T J\mathbf{x}_i = 0$ for $i = 1, 2, 3$, so $\mathbf{x_1}^T J\mathbf{w} = 0$ for all $\mathbf{w} \in \mathbb{R}^3$. But this is only possible if $\mathbf{x}_1 = \mathbf{0}$, contradicting $\mathbf{x}_1$ being an eigenvector. We thus conclude that some λ_i is ± 1 as required.

3. **Consider the line $L = \{t = 2x\} \cap \mathbb{H}^2$ in $\mathbb{H}^2$. Show that in the Poincaré disk model of $\mathbb{H}^2$, L is taken to an arc of the circle of radius $\sqrt(3)$ centred at the point $(2, 0)$ (using the identification of projecting from $(-1, 0, 0)$ as in the diagram on p70 of Cannon, Floyd, Kenyon and Parry, which is linked on the main webpage under announcements).**

The line L consists of the points $(2x, x, \sqrt{3x^2 - 1})$ for $x > 0$. The line through $(2x, x, \sqrt{3x^2 - 1})$ and $(-1, 0, 0)$ passes through $(0, x/(2x + 1), \sqrt{3x^2 - 1}/(2x + 1))$, so the line L is equal to the points $\{(x/(2x + 1), \sqrt{3x^2 - 1}/(2x + 1)) : x > 0\}$ in the Poincaré disk model. The result now follows from the calculation

$$(x/(2x + 1) - 2)^2 + (\sqrt{3x^2 - 1}/(2x + 1))^2 = 3.$$