MA 243 HOMEWORK 5

SOLUTIONS

B: Exercises

1. **Use the main formula of spherical trig to calculate the distance from London to Christchurch, NZ on the surface of the earth, using that London is approximately $51°$ North, and Christchurch is approximately $43°$ South, $172°$ East. Recall that latitude is measured from the equator $0°$ north to the North Pole $= 90°$ N, and longitude is measured from the Greenwich observatory, which is in London. The circumference of the earth is 40,000 km by the definition of kilometer.**

   Consider the triangle with vertices the North Pole (N), London (L), and Christchurch (C). The angle between the lines NL and NC is $172°$. The distance NL is $40000(90-51)/360 = 4333$ kilometres, while the distance NC is $40000(90+43)/360 = 14777$ kilometres. Then from the main formula of spherical trig (the spherical cosine law) we have

$$\begin{aligned} d(C, L) = \quad & (40000/2\pi)\cos^{-1}((\cos(2\pi(90-51)/360)\cos(2\pi(90+43)/360) \\ & + \sin(2\pi(90-51)/360)\sin(2\pi(90+43)/360)\cos(172))) = 18925 \text{ kilometres.} \end{aligned}$$

   Note when you compare this to reality that this calculation used several approximations (for example, rounding longitudes/latitudes to the nearest degree). Anyone who gave an answer more than 40000 kilometres is advised to pause and think!

2. **Prove that $P, Q, R \in S^2$ are collinear if and only if either $d(P,Q) + d(Q,R) = d(P,R)$ after relabelling, or $d(P,Q) + d(Q,R) + d(P,R) = 2\pi$.**

   Suppose that P, Q, R are collinear. Let $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be the corresponding points in $\mathbb{R}^3$. Then after relabelling the angle PQR is equal to π, so by the spherical cosine law we have $\cos(d(P,R)) = \cos(d(P,Q))\cos(d(Q,R)) - \sin(d(P,Q))\sin(d(Q,R)) = \cos(d(P,Q) + d(Q,R))$, so either $d(P,R) = d(P,Q) + d(Q,R)$, or $d(P,R) = 2\pi - (d(P,Q) + d(Q,R))$, and thus $d(P,Q) + d(P,R) + d(Q,R) = 2\pi$.

3. **Show that if $T(x) = Ax$ is a linear map from $\mathbb{R}^2$ to itself (so A is a 2×2 matrix) with the property that T maps $\mathbb{H}^1$ to itself and preserves distance, then A has one of the following two forms:**

$$A = \begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}, \quad A = \begin{pmatrix} \cosh(s) & -\sinh(s) \\ \sinh(s) & -\cosh(s) \end{pmatrix}$$

Since T maps $\mathbb{H}^1$ to itself, we must have $T(1,0) \in \mathbb{H}^1$, and thus $T(1,0) = (\cosh(s), \sinh(s))$ for some s. Since T preserves distance we must have $\mathbf{v} \cdot_L \mathbf{w} = \mathbf{v}^T J \mathbf{w} = (A\mathbf{v}) \cdot_L (A\mathbf{w}) = \mathbf{v} A^T J A \mathbf{v}$, for all $\mathbf{v}, \mathbf{w} \in \mathbb{H}^1$, where

$$J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, if $\mathbf{v} = (1,0)$ we have $\mathbf{v} \cdot_L \mathbf{w} = -w_1$, and $T(\mathbf{v}) \cdot_L T(\mathbf{w}) = -\cosh(s)(A_{11}w_1 + A_{12}w_2) + \sinh(s)(A_{21}w_1 + A_{22}w_2)$. Since $T(1,0) = (\cosh(s), \sinh(s))$, $A_{11} = \cosh(s)$, and $A_{21} = \sinh(s)$, so this expression is $(-\cosh(s)^2 + \sinh(s)^2)w_1 + (-A_{12}\cosh(s) + A_{22}\sinh(s))w_2$, so assuming that $w_2 > 0$ we can conclude that $-A_{12}\cosh(s) + A_{22}\sinh(s) = 0$, so $(A_{12}, A_{22}) = \lambda(\sinh(s), \cosh(s))$ for some λ. Finally, we note that this means that $T(\mathbf{v}) \cdot_L T(\mathbf{w}) = -v_1 w_1 + \lambda^2 v_2 w_2$, so $\lambda = \pm 1$. Thus

$$A = \begin{pmatrix} \cosh(s) & \sinh(s) \\ \sinh(s) & \cosh(s) \end{pmatrix}, \quad \text{or} \quad A = \begin{pmatrix} \cosh(s) & -\sinh(s) \\ \sinh(s) & -\cosh(s) \end{pmatrix}.$$

4. **Show that if L is a hyperbolic line then there is a distance preserving bijection from L to $\mathbb{H}^1$.**

In the notes on the webpage it is shonw that given any two points $P, Q \in \mathbb{H}^2$ there is a map $T(\mathbf{x}) = A\mathbf{x}$ with $T(P) = (1,0,0)$ and $T(Q) = (\cosh(s), \sinh(s), 0)$, and the map T preserves distance. Thus any line can be taking by a distance preserving bijection to the line $L' = \{y = 0\} \cap \mathbb{H}^2$. The distance on $\mathbb{H}^1$ is then the same as the distance on L' in $\mathbb{H}^2$, so the result follows.