MA 243 HOMEWORK 2

SOLUTIONS

B: Exercises

(1) **Let $T : \mathbb{E}^n \to \mathbb{E}^n$ be a motion. Fix a choice of coordinates $\phi : \mathbb{E}^n \to \mathbb{R}^n$.**

(a) **Show that $\phi \circ T : \mathbb{E}^n \to \mathbb{R}^n$ is another choice of coordinates.** Since ϕ and T are both bijections, their composition is as well. In addition, $|\phi \circ T(P) - \phi \circ T(Q)| = d(T(P), T(Q)) = d(P, Q)$, so $\phi \circ T$ is distance preserving, and is thus a choice of coordinates.

(b) **Define $T' : \mathbb{R}^n \to \mathbb{R}^n$ by $T' = \phi \circ T \circ \phi^{-1}$. Show that T' is a distance preserving map (so $|T'(\mathbf{x}) - T'(\mathbf{y})| = |\mathbf{x} - \mathbf{y}|$).**
We have $|T'(\mathbf{x}) - T'(\mathbf{y}| = |\phi \circ T \circ \phi^{-1}(\mathbf{x}) - \phi \circ T \circ \phi^{-1}(\mathbf{y})|$. Since ϕ is distance preserving, this equals $d(T \circ \phi^{-1}(\mathbf{x}), T \circ \phi^{-1}(\mathbf{y}))$. Since T is a motion, this is $d(\phi^{-1}(\mathbf{x}), \phi^{-1}(\mathbf{y}))$. Again, since ϕ is distance-preserving, this equals $|\mathbf{x} - \mathbf{y}|$ as required.

(c) **Show that if $\psi : \mathbb{E}^n \to \mathbb{R}^n$ is another choice of coordinates, then $T : \mathbb{E}^n \to \mathbb{E}^n$ defined by $T = \phi^{-1} \circ \psi$ is a motion.** We have $d(T(P), T(Q)) = d(\phi^{-1} \circ \psi(P), \phi^{-1} \circ \psi(Q)) = d(\psi(P), \psi(Q)) = d(P, Q)$, where the second equality is because ϕ is distance preserving, and the third is because ψ is.

(2) **Let T be motion obtained by rotating by θ anti-clockwise about a point P in $\mathbb{E}^2$, and let S be the motion obtained by rotating by ω anti-clockwise about the same point P. Write down the matrices A and B for T and S in some choice of coordinates.**

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad B = \begin{pmatrix} \cos(\omega) & -\sin(\omega) \\ \sin(\omega) & \cos(\omega) \end{pmatrix}.$$

Describe the motion $T \circ S$ geometrically, and write down its matrix in the same choice of coordinates. The motion $T \circ S$ is rotation by $\theta + \omega$ anti-clockwise. Its matrix is:

$$\begin{pmatrix} \cos(\theta+\omega) & -\sin(\theta+\omega) \\ \sin(\theta+\omega) & \cos(\theta+\omega) \end{pmatrix}.$$

Compare this to the matrix AB (unsimplified) and explain your answer.

$$AB = \begin{pmatrix} \cos(\theta)\cos(\omega) - \sin(\theta)\sin(\omega) & -\cos(\theta)\sin(\omega) - \sin(\theta)\cos(\omega) \\ \cos(\theta)\sin(\omega) + \sin(\theta)\cos(\omega) & \cos(\theta)\cos(\omega) - \sin(\theta)\sin(\omega) \end{pmatrix}.$$

The equality of the two previous matrices follows from the trigonometric angle-sum formula.

(3) **Let T be the motion of $\mathbb{E}^2$ of anti-clockwise rotation by $\pi/2$ about a point P, and let S be the motion of $\mathbb{E}^2$ of anti-clockwise rotation by $\pi/2$ about a point Q distance one from P. Fix a coordinate choice in which P is taken to $(0,0)$, and Q is taken to $(1,0)$.**

(a) **Write down the expression for T in these coordinates.**

$$T(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}.$$

(b) **Write down the expression for S in *these* coordinates.**

$$S(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(c) Write down the composition $S \circ T$ in these coordinates

$$S \circ T(\mathbf{x}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(d) **Describe $S \circ T$ geometrically.** Rotation by π about the point $(1/2, -1/2)$.