AARMS TROPICAL GEOMETRY

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1. Introduction

These notes are my lecture notes from a four week graduate summer school on Tropical Geometry held at the University of New Brunswick in July/August 2008 under the auspices of the Atlantic Association for Research in the Mathematical Sciences (AARMS). The course was aimed at graduate students finishing their first year of graduate studies, though in practice a quarter of the class was about to start graduate school, and a few were more experienced. The only official prerequisites are a solid course in abstract algebra, and linear algebra. Recommended background reading was *Ideals, Varieties, and Algorithms* [CLO07], by Cox, Little, and O'Shea.

These are lecture notes, so are not attempting to be complete, both in content and in references. In particular, these notes only cover one aspect of this exciting emerging field - search for "tropical geometry" in mathscinet or on the arXiv to see much much more.

There are almost certainly errors, both typographical, and mathematical (both minor and major) in these notes. Please let me know (D.Maclagan at warwick.ac.uk) any mistakes you find. Thanks are due to AARMS 2008 students for the typos already corrected!

2. Lecture 1

What is tropical geometry?
First answer: (Algebraic) geometry where instead of working over the complex numbers (or some other field) we work over the *tropical semiring* $(\mathbb{R}, \oplus, \otimes)$. Here $\oplus$ is the usual minimum, and $\otimes$ is the usual addition.
Example: $(5 \oplus 6) \otimes 7 = 12$.

When necessary we consider $(\mathbb{R} \cup \infty, \oplus, \otimes)$, so ∞ becomes the additive identity. Then $(\mathbb{R} \cup \infty, \oplus, \otimes)$ is a semiring (ie associative, distributative etc - just no additive inverse).

In algebraic geometry we often work with polynomials. In tropical geometry we "tropicalize" these polynomials, which turns them into piecewise linear functions.
Example: $f(x,y) = x^2 + y^2 - 1$. This tropicalizes to $\operatorname{trop}(f) = x^2 \oplus y^2 \oplus 0 = \min(2x, 2y, 0)$. This is a piecewise linear function. (See below for why the -1 turns into 0). See Figure 1.

In algebraic geometry we study the common zeros of polynomial equations (Warning: Oversimplification!). These are called varieties. Tropically this corresponds to taking the nonlinear locus of the polynomial $\operatorname{trop}(f)$.
Example: Let $f(x,y) = x + y + 1$. The variety of f is the set $\{(x,y) \in \mathbb{C}^2 : x+y+1 = 0\}$, which is a line in $\mathbb{C}^2$. Then $\operatorname{trop}(f) = \min(x, y, 0)$. This is a piecewise

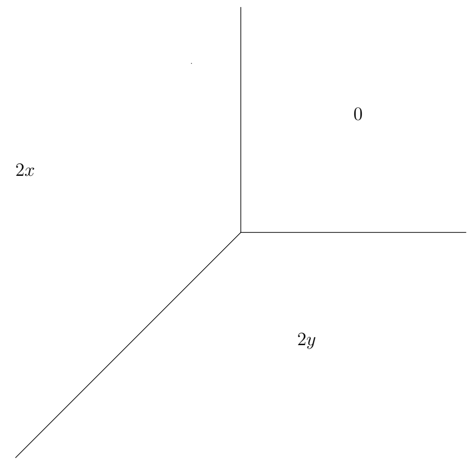

FIGURE 1.

linear function with graph shown in Figure 2. The nonlinear locus is the three line segments $x = y \leq 0$, $x = 0 \leq y$, and $y = 0 \leq x$. This is also shown in Figure 2.

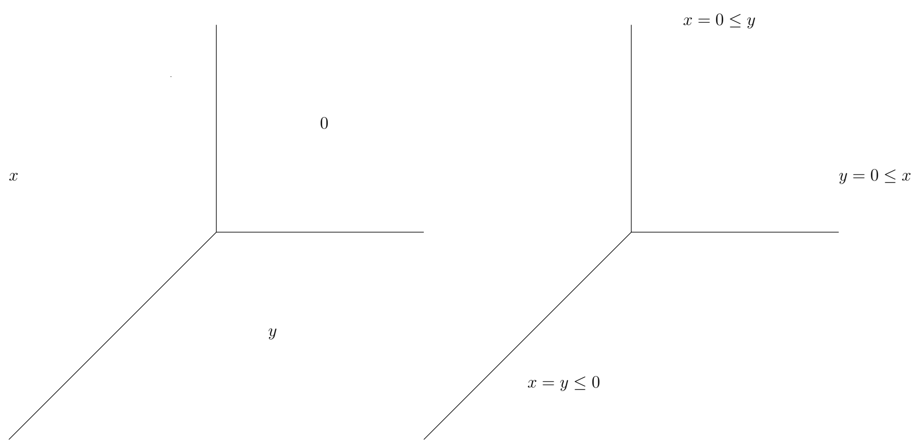

FIGURE 2.

Thus varieties turn into polyhedral complexes under the tropicalization map.

Warning: An issue with this first answer is that not everything tropicalizes well. In particular, maps between varieties do not tropicalize precisely as expected. ("Tropicalization is not functorial"). For this reason we will be careful over the next two weeks to define things formally.

Motivation

Why tropicalize? A first reason is that polyhedral geometry is (often) easier than algebraic geometry. Many invariants of the variety become invariants of the resulting polyhedral complex.

Example: Let $f = x + y + 1$. Then the set $f = 0$ is the line $\{(t, -1-t) : t \in \mathbb{C}\}$ in $\mathbb{C}^2$, so is one-dimensional. The tropical variety, shown on the right in Figure 2, is also one-dimensional.

It is true in general (we will see later) that dimension is preserved under tropicalization. Other (primarily intersection theoretic) invariants are also preserved.

Motivating Example: Counting Curves

One of the first successful applications of tropical geometry has been to enumerative geometry, primarily in the work of Mikhalkin. This allows a simple answer to the classical question of counting the number of rational curves in $\mathbb{P}^2$ of a given degree d passing through a set of fixed points in general position. A curve C in $\mathbb{P}^2$ is given by a homogeneous polynomial $f(x, y, z) = 0$. The curve C is rational if it is isomorphic to $\mathbb{P}^1$ (informally, if there is a parameterization $\phi : \mathbb{C} \to C$ so C is the Zariski closure of the image of ϕ). The degree of the curve is the degree of the polynomial. In order for this number to be finite, we ask that the points be in general position, and that there be $3d-1$ of them. Here "general position" means that there is a (Zariski) open set in $(\mathbb{P}^2)^{3d-1}/S_{3d-1}$ for which this number is constant.

Definition 2.1. Let $N_{0,d}$ be the number of rational curves of degree d passing through a collection of $3d-1$ points in $\mathbb{P}^2$ in general position.

Example:

- $d=1$ A curve of degree one is a straight line, which is rational. Thus $N_{0,1}$ is one, as there is a unique line joining any two distinct points in $\mathbb{P}^2$.
- $d=2$ All curves of degree two are rational, and there is a unique curve through any five points in general position in $\mathbb{P}^2$ (see exercises). Thus $N_{0,2} = 1$.
- $d=3$ $N_{0,3} = 12$. This was computed by Steiner in 1848, and was possibly known earlier.
- $d=4$ $N_{0,4} = 620$. This was computed by Zeuthen in 1873.
- $d=5$ $N_{0,5} = 87304$. This (and all later ones) were unknown until the early 90s.
- $d=6$ $N_{0,6} = 26312976$.

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In 1994 Kontsevich gave a recursive formula that determines all of these numbers from $N_{0,1} = 1$. This involved developing the moduli space of stable maps, which is at the foundations of Gromov-Witten theory. Giving a self-contained proof of this would take more than this entire course. However in the last week we will (hopefully) give a self-contained proof of the Kontsevich recursion using tropical methods.

We now return to the question of tropicalizing polynomials. Earlier we said $\operatorname{trop}(x^2 + y^2 - 1) = \min(2x, 2y, 0)$. The -1 turned mysteriously into a 0. We will now partially explain this (though a full explanation will be later in the week).

Let

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n})),$$

where by $\mathbb{C}((t^{1/n}))$ we mean the ring of Laurent series in the variable $t^{1/n}$. This ring K is the ring of *Puiseux series*. An element $a \in K$ has the form

$$a = \sum_{q \in \mathbb{Q}} a_q t^q,$$

where $\{q \in \mathbb{Q} : a_q \neq 0\}$ is bounded below and has a common denominator.

Write $K^* = K \setminus \{0\}$. Let $\operatorname{val} : K^* \to \mathbb{R}$ be given by $\operatorname{val}(a) = \min\{q : a_q \neq 0\}$. This lets us define the tropicalization of a polynomial formally.

Definition 2.2. Let $S := \mathbb{C}[x_1, \ldots, x_n]$, and write

$$f = \sum_{u \in \mathbb{N}^n} c_u x^u,$$

where $x^u := \prod_{i=1}^n x_i^{u_i}$. Then

$$\operatorname{trop}(f) = \min_{u \in \mathbb{N}^n : c_u \neq 0} \left(\operatorname{val}(c_u) + \sum_{i=1}^n u_i x_i \right).$$

The tropical hypersurface of f is

$\operatorname{trop}(V(f)) = \{w \in \mathbb{R}^n :$ the minimum in the definition of $\operatorname{trop}(f)(w)$ is achieved at least twice$\}$.

Note that we could also define $\operatorname{val}\colon K \to \mathbb{R}C \cup \infty$ by setting $\operatorname{val}(0) = \infty$. Then $\operatorname{trop}(f) = \min_{u \in \mathbb{N}^n}(\operatorname{val}(c_u) + \sum_{i=1}^n u_i x_i)$. Note also that $\operatorname{val}(-1) = 0$, so this explains the earlier zero.

Example: Let $f = tx^2 + 2xy + 3ty^2 + 5x + 7y - (t^2 + t^5)$. Then $\operatorname{trop}(f) = \min(2x+1, x+y, 2y+1, x, y, 2)$. This function is illustrated in Figure 3.

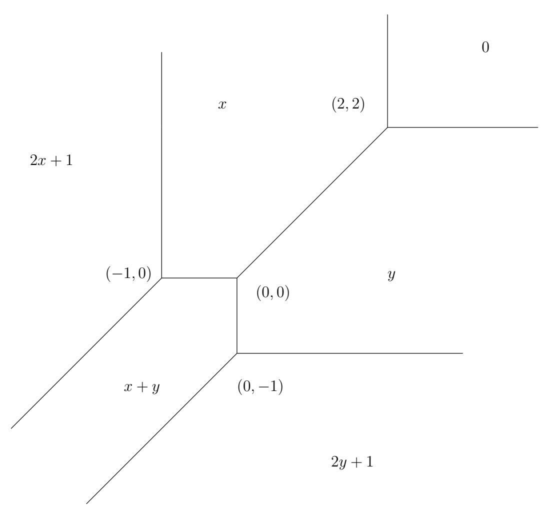

FIGURE 3.

Example: Let $f = (t^2 - t^{5/2})y^2 + 5x^2 - 7xy + 8y - tx + t^3$. Then $\operatorname{trop}(f) = \min(2+2y, 2x, x+y, y, x+1, 3)$. This is illustrated in Figure 4.

Example: Let $f = tx^2 + 3xy - 7(t^3 + t^5)y^2 + ty - 7x + 5$. Then $\operatorname{trop}(f) = \min(2x+1, x+y, 2y+3, y+1, x, 0)$. This is illustrated in Figure 5.

Example: Let $f = t^2x - 7(t + t^3)y + t^5$. Then $\operatorname{trop}(f) = \min(x+2, y+1, 5)$. This is illustrated in Figure 6.

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Example: Let $f = t^3x^2 - 7tx + 8xy - 7y^2 + 6$. Then $\operatorname{trop}(f) = \min(2x+3, x+1, x+y, 2y, 0)$. This is illustrated in Figure 7.

Example: Let $f = t^4x^3 + t^2x^2y + t^2xy^2 + t^4y^3 + tx^2 + xy + ty^2 + x + y + t$. Then $\operatorname{trop}(f) = \min(3x+4, 2x+y+2, x+2y+2, 3y+4, 2x+1, x+y, 2y+1, x, y, 1)$. This is illustrated in Figure 8.

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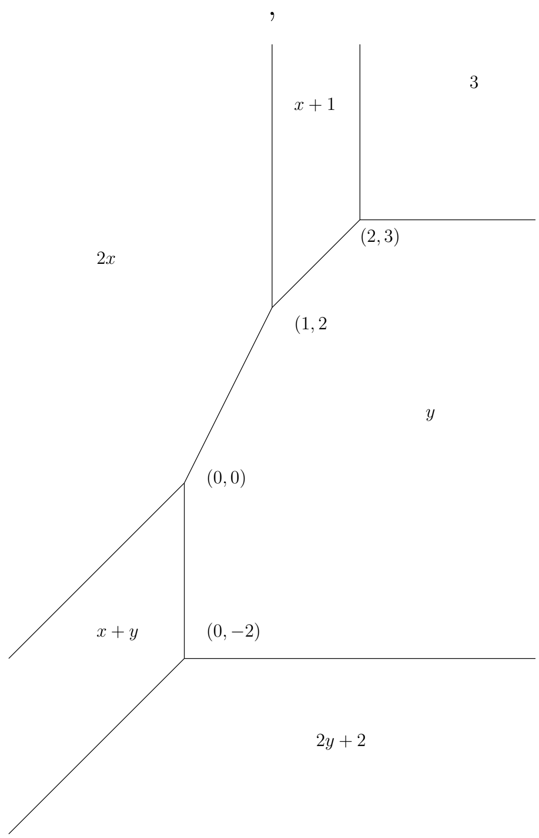

FIGURE 4.

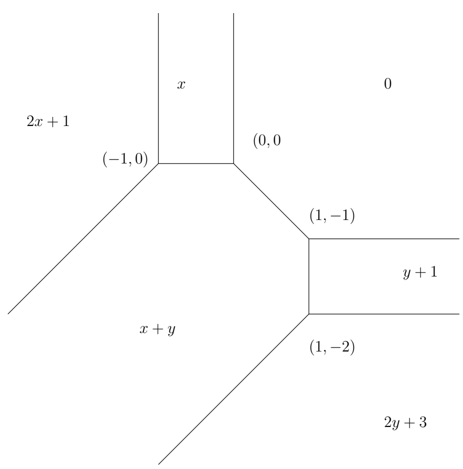

FIGURE 5.

Example: Let $f = x^2+2xy+3y^2+4x+5y+6$. Then $\mathrm{trop}(f) = \min(2x, 2y, x, y, 0)$. This is illustrated in Figure 9.

Note two important aspects of these pictures: Firstly, in almost all cases, there are the same number of “tentacles” in each of three directions, and that number is the degree of the polynomial. Secondly, in almost all cases, at each vertex of the graph,

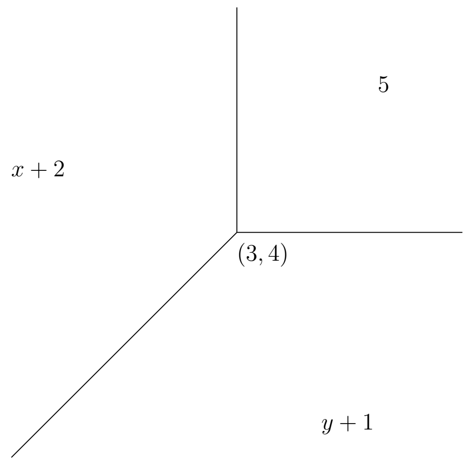

FIGURE 6.

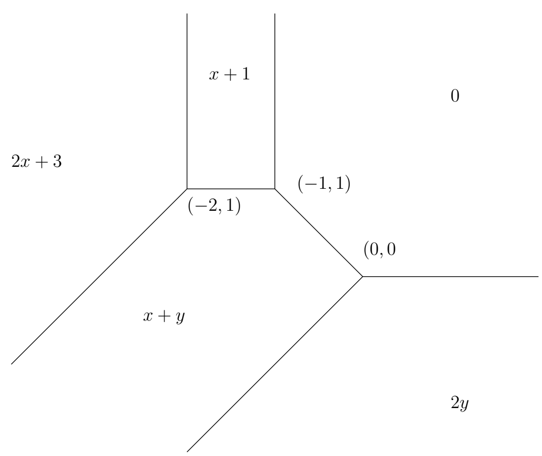

FIGURE 7.

the sum of the three vectors emanating from that vertex add to zero. In fact, with the appropriate notion of multiplicity (see next week), both of these are true in all cases.

Outline of Course:

Week 1: Introduction to varieties. Background tools, such as valuations and Gröbner bases.

Week 2: Fundamental theorems on tropical varieties. Some basic examples.

Week 3: More examples. Connections to toric varieties.

Week 4: Enumerative geometry. Presentations.

Presentations: During this month you will read a research paper or two in small groups (3–5) and do a presentation on the material during the last few class periods. There is a list of possible papers on the webpage. Check that out today!!

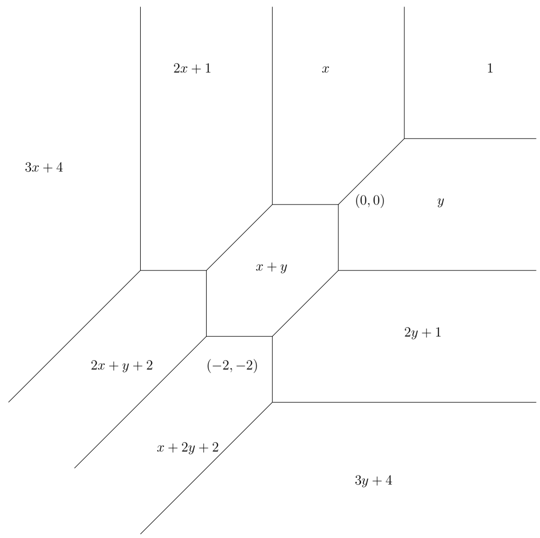

FIGURE 8.

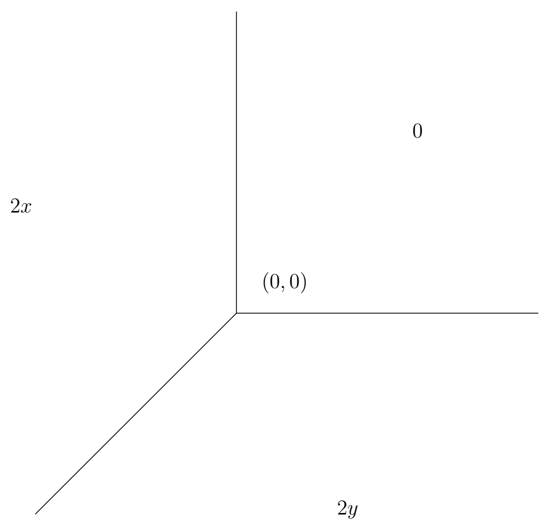

FIGURE 9.

3. LECTURE 2

The goal of today is to introduce affine and projective varieties. We adopt the simplistic motto that "Algebraic geometry is the study of solutions of polynomial equations". We are interested in the *geometry* of these solution spaces.

For example, consider the three equations $x^2+y^2-1=0$, $xy=0$, and $x^2+y^2=-1$. The first of these describes a circle of radius one, while the second is the union of two lines. The third has no solutions over the real numbers, but has solutions if work over the complex numbers (or at least an algebraically closed field), as we always will.

Definition 3.1. Let $\Bbbk$ be an algebraically closed field (such as $\mathbb{C}$). Then affine space is

$$\mathbb{A}^n = \{(a_1, \dots, a_n) : a_i \in \Bbbk\} = \Bbbk^n.$$

Philosophically $\mathbb{A}^n$ should be thought of as $\Bbbk^n$ without a distinguished origin.

Definition 3.2. Let $S = \Bbbk[x_1, \dots, x_n]$ be the polynomial ring in n variables with coefficients in $\Bbbk$. Given $f_1, \dots, f_s \in S$ the (affine) variety defined by the f_i is

$$V(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in \mathbb{A}^n : f_i(a_1, \dots, a_n) = 0 \text{ for } 1 \leq i \leq s\}.$$

Example: $V(x + y - 1)$ is the line $y = x - 1$.
Example: $V(x^2 - y, x^3 - z, y^3 - z^2) = \{(t, t^2, t^3) : t \in \mathbb{C})\}$. This is the affine "twisted cubic" curve.
Note: $V(f_1, f_2) = V(f_1 + f_2, f_1 - f_2) = V(f_1, f_2, f_1 + f_2, xf_1 + y^2 f_2)$.

Recall that an ideal $I \subseteq S$ is a set closed under addition and multiplication by elements of S. The ideal I generated by $f_1, \dots, f_s \in S$ is

$$I = \langle f_1, \dots, f_s \rangle = \{\sum_{i=1}^{s} g_i f_i : g_i \in S\}.$$

Lemma 3.3. *The variety $V(f_1, \dots, f_s)$ only depends on the ideal $\langle f_1, \dots, f_s \rangle$, so if $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_r \rangle$ then $V(f_1, \dots, f_s) = V(g_1, \dots, g_r)$.*

Thus we will talk about varieties as defined by ideals. Note that if the ideal is principal (generated by one element) then we call the variety a hypersurface. All the examples we saw yesterday were hypersurfaces.

Operations on varieties (The Ideal/Variety dictionary).

(1) $V(I) \cap V(J) = V(I + J)$. Here $I + J = \{f + g : f \in I, g \in J\}$. If $I = \langle f_1, \dots, f_s \rangle$, $J = \langle g_1, \dots, g_r \rangle$, then $I + J = \langle f_1, \dots, f_s, g_1, \dots, g_r \rangle$.
(2) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$, where $IJ = \{fg : f \in I, g \in J\}$. If $I = \langle f_1, \dots, f_s \rangle, J = \langle g_1, \dots, g_r \rangle$ then $IJ = \langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq r \rangle$. A description of $I \cap J$ in terms of the f_i and g_j is not as simple (though there are algorithms to compute it).

Warning: We see here that we can have $V(I) = V(J)$ if $I \neq J$ (for example $IJ \neq I \cap J$ in general). For example,

$$V((x - y)^2) = V(x - y) = \{(a, a) : a \in \Bbbk\}.$$

Solution: Hilbert's Nullstellensatz.

Theorem 3.4. *Let $\Bbbk$ be an algebraically closed field. Then $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$, where*

$$\sqrt{I} = \{f \in S : f^r \in I \textit{ for some } r\}.$$

For a proof, see any book on commutative algebra (for example, [Eis95], or [CLO07]).
Example: If $I = \langle x^2 \rangle$, then $\sqrt{I} = \langle x \rangle$.

A subvariety of a variety $V(I)$ is a variety $V(J)$ with $V(J) \subset V(I)$. Note that if $V(J)$ is a subvariety of $V(I)$ then $\sqrt{I} \subseteq \sqrt{J}$.

We place a topology on $\mathbb{A}^n$ by setting the closed sets to be $\{V(I) : I \text{ is an ideal of } S\}$. This is the *Zariski topology.* To check that $\emptyset$ and $\mathbb{A}^n$ are closed, note that $\emptyset = V(1)$, and $\mathbb{A}^n = V(0)$. Exercise: Check that the finite union of closed sets and the arbitrary intersection of closed sets are closed. We denote by $\overline{U}$ the closure in the Zariski topology of a set U. This is the smallest set of the form $V(I)$ for some I that contains U.

Another operation on varieties

(3) $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$, where

$$I : J^\infty) = \{f \in S : \text{ for all } g \in J \text{ there exists } N > 0 \text{ with } fg^N \in I\}$$

is the *saturation* of the ideal I by the ideal J.

Important (for us) example: Let $J = \langle \prod_{i=1}^n x_i \rangle$. Then $V(J) = V(\prod_{i=1}^n x_i) = \cup_{i=1}^n V(x_i)$. For example, when $n = 2$, so $S = \mathbb{C}[x_1, x_2]$, then $J = \langle x_1x_2 \rangle$, and $V(J)$ is the union of the two coordinate axes. The complement $\mathbb{A}^n \setminus V(J) = T^n = (\mathbb{C}^*)^n$, and for $I \in S$, $\overline{V(I) \setminus V(J)} = \overline{V(I) \cap T^n}$.

Example: $I = \langle x_1^2 + 3x_1x_2 \rangle$, $J = \langle x_1x_2 \rangle$. Then $(I : J^\infty) = \{f \in S : \exists N \text{ such that } f x_1^N x_2^N \in I \rangle = \langle x_1 + 3x_2 \rangle$.

Definition 3.5. A variety X is *irreducible* if it cannot be written as the union of two proper subvarieties. This is a topological notion.

Proposition 3.6. *Let $X \subset \mathbb{A}^n$ be a variety. Then X can be written uniquely (up to order) as an irredundant union of irreducible varieties. These are called the irreducible components of X.*

Proof. We first show that such a decomposition exists. If X is irreducible then we are done. Otherwise we can write $X = X_1 \cup X_2$ where the X_i are proper subvarieties. Given a variety Y we write $I(Y)$ for the radical ideal defining Y. We must have $I(X) \subsetneq I(X_i)$ for $i = 1, 2$. Suppose now that we have a decomposition $I = \cup_{i=1}^s X_i^s$. If all of the X_i are irreducible, we are done. Otherwise there is some X_j that can be written in the form $X_j = X_j' \cup X_j''$ where X_j', X_j'' are proper subvarieties of X_j, so we replace X_j by X_j' and X_j'' in the decomposition and renumber to have $X_1^{s+1}, \dots, X_{s+1}^{s+1}$. In this fashion we can get a decreasing sequence $X_{i_1}^1 \supsetneq X_{i_2}^2 \supsetneq X_{i_3}^3 \supsetneq \dots$ s with corresponding increasing sequence $I(X_{i_1}^1) \subsetneq I(X_{i_2}^2) \subsetneq I(X_{i_3}^3) \subsetneq \dots$. Since S is Noetherian this sequence must terminate at some stage s, at which point each X_i^s is irreducible, and $X = X_1^s \cup \dots \cup X_s^s$ is an irreducible decomposition.

Now suppose that $X = X_1 \cup \dots \cup X_s = Y_1 \cup \dots Y_r$ are two irredundant irreducible decompositions of X. Since $Y_1 \subset X$, $\cup_{i=1}^s (Y_1 \cap X_i) = Y_1$, so since Y_1 is irreducible there must be $1 \leq j_i \leq s$ with $Y_1 \subset X_{j_i}$. Similarly for all other Y_k there is j_k with $Y_k \subset X_{j_k}$. Reversing the roles of X_i and Y_i, we also get for each $1 \leq i \leq s$ there is l_i with $X_i \subseteq Y_{l_i}$. But this means that $Y_i \subseteq X_{j_i} \subseteq Y_k$ for $k = l_{j_i}$, so we must have $k = i$, so for each $1 \leq i \leq r$ there is k with $X_k = Y_i$. Since the decomposition in the X_i is irredundant, the X_i must be equal to the Y_j up to order. □

Example: Let $I = \langle x_1^2 + 3x_1x_2 \rangle$. Then $I = \langle x_1 \rangle \cap \langle x_1 + 3x_2 \rangle$, so $V(I) = V(x_1) \cup V(x_1 + 3x_2)$. The varieties $V(x_1), V(x_1 + 3x_2)$ are both irreducible, so these are the irreducible components. The saturation of I by $J = \langle x_1x_2 \rangle$ removes the component $V(x_1)$ that does not intersect the torus.

Definition 3.7. The *coordinate ring* of X is $S/I(X)$. This is the ring of polynomial functions on X.

Projective Varieties.

Definition 3.8. Projective space $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \mathbf{0})/\sim$ where $\mathbf{v} \sim \lambda\mathbf{v}$ for all $\lambda \neq 0$. The points of $\mathbb{P}^n$ are the equivalence classes of lines through the origin $\mathbf{0}$. We write $[x_0 : x_1 : \cdots : x_n]$ for the equivalence class of $\mathbf{x} = (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$. A line in $\mathbb{P}^n$ is the equivalence class of a two-dimensional subspace in $\mathbb{C}^{n+1}$.

We can think of $\mathbb{P}^n$ as $\mathbb{A}^n$ with some points "at infinity" added. For example, there is a bijection between $\mathbb{A}^n$ and the points of $\mathbb{P}^n$ of the form $[1 : x_1 : \cdots : x_n]$. The "points at infinity" are then those with first coordinate 0.
Example: $\mathbb{P}^2 = (\mathbb{C}^3 \setminus (0,0,0))/\sim$. Then $\mathbb{P}^3 = \{[1 : x_1 : x_2] : (x_1, x_2) \in \mathbb{A}^2\} \cup \{[0 : x_1 : x_2] : x_1, x_2 \in \mathbb{C}^2\} = \{[1 : x_1 : x_2] : (x_1, x_2) \in \mathbb{A}^2\} \cup \{[0 : 1 : x_2] : x_2 \in \mathbb{C}\} \cup \{[0 : 0 : 1]\}$. So we see that the points at infinity are a copy of $\mathbb{P}^1$.
Note: Polynomials don't make sense as functions on $\mathbb{P}^n$. For example, $[1 : 2 : 3] = [2 : 4 : 6] \in \mathbb{P}^2$, but the function $x_1 + x_2$ has different values (5 or 10) on these two points. However if $f \in S$ is homogeneous, then $\{x \in \mathbb{P}^n : f(x) = 0\}$ is well-defined. This is because if $f(x) = 0$, then $f(\lambda x) = 0$ for all $\lambda \neq 0$, since if f is homogeneous of degree k, then $f(\lambda x) = \lambda^k f(x)$.

We call an ideal $I \subset \Bbbk[x_0, \dots, x_n]$ *homogeneous* if it has a homogeneous generating set.

Definition 3.9. Let I be a homogeneous ideal in $S = \Bbbk[x_0, \dots, x_n]$. Then the variety of I is

$$V(I) = \{[x] \in \mathbb{P}^n : f(x) = 0 \ \ \forall [x] \in \mathbb{P}^n\}.$$

Example: $V(x_0 + x_1 + x_2) = \{[1 : t : -1 - t] : t \in \mathbb{C}\} \cup \{[0 : 1 : -1]\}$.
Example: $V(x_0, x_1, x_2) = \emptyset$.

The same rules apply for varieties in $\mathbb{P}^n$ as for $\mathbb{A}^n$:

(1) $V(I) \cap V(J) = V(I + J)$;
(2) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$;
(3) $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$.

As in the affine case, there is not a bijection between homogeneous ideals and projective varieties. Let $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$. We call $\mathfrak{m}$ the "irrelevant ideal", as it is the largest ideal not corresponding to a nonempty subvariety of $\mathbb{P}^n$.

Lemma 3.10. *Let $V(I), V(J)$ be subvarieties of $\mathbb{P}^n$. Then $V(I) = V(J) \neq \emptyset$ if and only if*

$$\sqrt{I} = \sqrt{J}.$$

Also, $V(I) = \emptyset$ if and only if $I = \langle 1 \rangle$ or $\sqrt{I} = \mathfrak{m}$.

Proof. We first consider the case $V(I) \neq \emptyset$. Let $\widetilde{V(I)}, \widetilde{V(J)}$ denote the subvarieties of $\mathbb{A}^{n+1}$ defined by I and J. Note that if $x \in \widetilde{V(I)}$ then $\lambda x \in \widetilde{V(J)}$ for all $\lambda \neq 0$ (and similarly for $V(J)$). Thus $V(I) = V(J)$ if and only if $(\widetilde{V(J)}) \setminus \{\mathbf{0}\} = (\widetilde{V(J)}) \setminus \{\mathbf{0}\}$. Now if $f \in S$ satisfies $f(\lambda x) = 0$ for all $x \in \widetilde{V(I)} \setminus \mathbf{0}$ then $f(\mathbf{0}) = 0$, so $\overline{\widetilde{V(I)} \setminus \mathbf{0}} = \widetilde{V(I)}$. Thus $\widetilde{V(I)} \setminus \mathbf{0} = \widetilde{V(J)} \setminus \mathbf{0}$ if and only if $\sqrt{I} = \sqrt{J}$ by the Nullstellensatz.

Also $V(I) = \emptyset$ if and only if $\widetilde{V(I)} = \emptyset$ or $\widetilde{V(I)} = \{\mathbf{0}\}$, so if and only if $I = \langle 1 \rangle$ or $\sqrt{I} = \mathfrak{m}$. $\square$

Definition 3.11. The homogeneous coordinate ring of a projective variety $X = V(I)$ is S/I, where $S = \Bbbk[x_0, \ldots, x_n]$.

Subvarieties of tori.

The last case of varieties that we will consider is that of subvarieties of tori. This is actually a special case of affine varieties. Let $S = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ring of Laurent polynomials.

Example: $f = 3x_1x_2^2 + 5x_1x_2^3 + 7x_1^{-5}x_2 \in S$.

The ring S is the coordinate ring of the *algebraic torus* $T^n \cong (\Bbbk^*)^n$. The name comes from the fact that $(\mathbb{C}^*)^n$ deformation retracts to the standard topological n-dimensional torus $(S^1)^n$. The ring S is the ring of all those rational functions (quotients of polynomials) that are defined everywhere on T^n.

An ideal $I \subset S$ determines a subvariety

$$V(I) = \{\mathbf{x} \in T^n : f(x) = 0 \text{ for all } f \in I\} \subseteq T^n.$$

Note that it makes sense to consider $f(x)$ for $x \in T^n$, since any $f \in S$ has the form $g/(\prod_{i=1}^n x_i)^N$ for some polynomial g and $N \geq 0$, so is defined at any $x \in T^n$.

Note that a subvariety of T^n is actually also an affine variety, which can be embedded into $\mathbb{A}^{n+1}$. If $X = V(I) \subset T^n$, choose a generating set for I consisting of polynomials $\{f_1, \ldots, f_s\}$. This can always be done, since every monomial is a unit in S. Let S' be the polynomial ring $\Bbbk[x_1, \ldots, x_n, y]$, and let J be the ideal $\langle f_1, \ldots, f_s, y\prod_{i=1}^n x_i - 1\rangle$, where we consider the f_i here as elements of S'. Then the affine variety of J in $\mathbb{A}^{n+1}$ consists of the points $\{(x, 1/\prod_{i=1}^n x_i) \in \mathbb{A}^{n+1} : x \in V(I) \subset T^n\}$.

Warning: You'll notice I'm using S for three different rings here: $S = \Bbbk[x_1, \ldots, x_n]$, the coordinate ring of $\mathbb{A}^n$; $S = \Bbbk[x_0, \ldots, x_n]$, the homogeneous coordinate ring of $\mathbb{P}^n$; and $S = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the coordinate ring of T^n. The meaning should always be clear from context, and this has the advantage that one can summarize the previous discussion in the following form:

Note also that in each case there is a largest ideal defining a variety X, which we denote by $I(X)$. For example if $X = V(I) \subseteq \mathbb{A}^n$, then $I(X) = \sqrt{I}$.

Summary:

Let X, Y be subvarieties of $\mathbb{A}^n$, $\mathbb{P}^n$ or $\mathbb{T}^n$, with ideals $I(X), I(Y)$ in the respective coordinate ring. Then

(1) $I(X \cup Y) = I(X) \cap I(Y) = I(X)I(Y)$;
(2) $I(X \cap Y) = I(X) + I(Y)$;
(3) $I(\overline{X \setminus Y}) = I(X) : I(Y)^\infty$;
(4) The coordinate ring of X is $S/I(X)$.

Dimension

We will study many invariants of a variety X. A basic one is the *dimension* of X. We first give an intuitive definition of dimension. Nice ("smooth" or "nonsingular") complex varieties are real manifolds of dimension $2d$ for some integer d. We say that the (complex) dimension of such an X is d.

Example: The projective variety $\mathbb{P}^1$ is equal to the two-dimensional sphere S^2 as a set. This has real dimension two as a manifold, so the dimension of $\mathbb{P}^1$ is one.

Saying that $\dim(X) = d$ is intuitively saying that near most points X looks like $\mathbb{C}^d$. (Intentionally vague sentence!)

Formally, the dimension of an irreducible variety X is the length d of the longest chain

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_d = X$$

of irreducible subvarieties. (Note that this definition works for subvarieties of $\mathbb{A}^n, \mathbb{P}^n, T^n$.)

Example: $\{V(x_1, x_2) = (0,0)\} \subsetneq V(x_1) \subsetneq \mathbb{A}^2$, so $\dim(\mathbb{A}^2) \geq 2$. In fact $\dim(\mathbb{A}^2) = 2$ (and $\dim(\mathbb{A}^n) = n$ for all n), but this is (surprisingly?) not trivial.

Example: $\{(1,1) = V(x_1 - 1, x_2 - 1) \subsetneq V(x_1^2 + x_2^2 - 1)$, so the dimension of $V(x_1^2 + x_2^2 - 1)$ is at least one. Again, in this case it is exactly one.

There is an equivalent algebraic definition of dimension. The *Krull dimension* of a ring R is the length d of the longest chain

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d = R$$

of prime ideals. If $X \subset \mathbb{A}^n$ or $X \subset T^n$ then $\dim(X)$ is the Krull dimension of the coordinate ring $S/I(X)$. If $X \subset \mathbb{P}^n$ then $\dim(X)$ is one less than the dimension of $S/I(X)$. For an overview of Krull dimension, see [Eis95, Chapter 8].

4. Exercises

These questions cover approximately Monday - Wednesday of week one. You do not need to do every question! This week some of you may have seen some of the content before, so concentrate on the new material. Do at least one question from each days material - ask me for advice on which questions are most appropriate for your background if you're not sure. You are *strongly* encouraged to work together. I will ask you to create a solution set as a group. This will involve typing up the answer to approximately one question each a week.

Tropical Questions

(1) Check that $(\mathbb{R}, \oplus, \otimes)$ is a semiring.

(2) Draw a picture of the tropical curve corresponding to the following polynomials in $K[x,y]$:
   (a) $f = t^3x + (t + 3t^2 + 5t^4)y + t^{-2}$;
   (b) $f = (t^{-1} + 1)x + (t^2 - 3t^3)y + 5t^4$;
   (c) $f = t^3x^2 + xy + ty^2 + tx + y + 1$;
   (d) $f = 4t^4x^2 + (3t + t^3)xy + (5 + t)y^2 + 7x + (-1 + t^3)y + 4t$;
   (e) $f = tx^2 + 4xy - 7y^2 + 8$;
   (f) $f = t^6x^3 + x^2y + xy^2 + t^6y^3 + t^3x^2 + t^{-1}xy + t^3y^2 + tx + ty + 1$.

(3) The goal of this exercise is to show the connection between tropical curves in the plane and triangulations of a certain point configuration. It requires some basic knowledge of polyhedral geometry (and is probably the hardest exercise in this problem set). Ask for hints/help once you've thought about it a little.

Fix $d > 0$. Let $\mathcal{A}_d = \{(a,b) : a+b \leq d, a,b \geq 0\}$. Fix a polynomial $f = \sum_{(a,b)\in\mathcal{A}_d} c_{ab}x^a y^b$ with $c_{ab} \in \mathbb{C}\{\{t\}\}$. The *regular triangulation* of $\mathcal{A}_d$ induced by f is obtained by taking the convex hull of the points $\{(a,b,\mathrm{val}(c_{ab}) : (a,b) \in \mathcal{A}\}$ and taking the (projections of the) set of *lower faces*. These are the faces that you can see if you look from $(0,0,-N)$ for $N \gg 0$.

Example: Let $d = 2$, so $\mathcal{A}_2 = \{(2,0),(1,1),(0,2),(1,0),(0,1),(0,0)\}$. Let $f = tx^2 + xy + ty^2 + x + y + t^6$. We form the convex hull of the points $\{(2,0,1),(1,1,0),(0,2,1),(1,0,0),(0,1,0),(0,0,6)\}$. The lower faces of this polytope are illustrated in Figure 10.

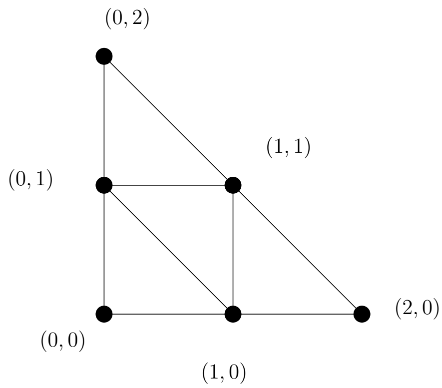

FIGURE 10.

(a) Draw the regular triangulation of $\mathcal{A}_2$ corresponding to the polynomial $f = tx^2 + xy + t^3y^2 + x + ty + 1$.
(b) Draw the regular triangulation of $\mathcal{A}_1$ corresponding to the polynomial $f = t^5x + t^3y + t^{10}$.
(c) Draw the regular triangulation of $\mathcal{A}_3$ corresponding to the polynomial $f = t^3x^3 + tx^2y + txy^2 + t^3y^3 + tx^2 + xy + ty^2 + tx + ty + t^3$.

The dual graph to a triangulation has a vertex for every triangle. There are two types of edges. The finite edges join two adjacent triangles, and have direction orthogonal to the common edge of the triangles. The infinite edges start at the triangles adjacent to the boundary of the large triangle $\mathrm{conv}((d,0),(0,d),(0,0))$, and have direction orthogonal to the external edge. This is defined up to the lengths of the finite edges.

Example: In the example above, a dual graph for the regular triangulation is shown in Figure 11.

(d) Draw a dual graph to the regular triangulation of $\mathcal{A}_2$ corresponding to $f = tx^2 + xy + t^3y^2 + x + ty + 1$.
(e) Draw a dual graph to the regular triangulation of $\mathcal{A}_1$ corresponding to $f = t^5x + t^3y + t^{10}$.
(f) Draw a dual graph to the regular triangulation of $\mathcal{A}_3$ corresponding to $f = t^3x^3 + tx^2y + txy^2 + t^3y^3 + tx^2 + xy + ty^2 + tx + ty + t^3$.
(g) Let $f = \sum_{(a,b)\in\mathcal{A}_d} c_{ab}x^a y^b$ with $c_{d0}, c_{0d}, c_{00} \neq 0$. Show that the tropical curve defined by f is the image under $x \mapsto -x$ of a dual graph to the regular triangulation defined by f.

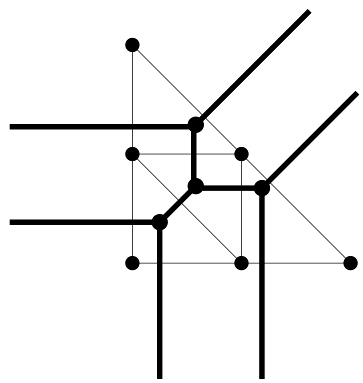

FIGURE 11.

(h) Check the previous claim for the examples of the first question.
(i) Conclude that for sufficiently general f there are d tentacles pointing in each direction. What can you say about the genericity condition? What happens in the other cases?

Varieties

(1) If you haven't already done so, read a proof of the Nullstellensatz. Suggested references: Cox, Little, O'Shea, or Eisenbud's commutative algebra course.
(2) Show that if $\langle f_1, \dots, f_s \rangle = \langle g_1, \dots, g_r \rangle$ then $V(f_1, \dots, f_s) = V(g_1, \dots, g_r)$.
(3) Show that $V(I) \cap V(J) = V(I + J)$.
(4) Show that $V(I) \cup V(J) = V(IJ) = V(I \cap J)$.
(5) Show that $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$.
(6) Let $I = \langle x^2, xy^3, y^2z, z^4 \rangle \subset \Bbbk[x, y, z]$. Compute $\sqrt{I}$. What are the irreducible components of $V(I)$?
(7) Show that the Zariski topology is a topology.
(8) Describe the subvariety of $\mathbb{P}^3$ defined by the ideal $I = \langle x_0x_2 - x_1x_3, x_0x_2 - x_1^2, x_1x_3 - x_2^2 \rangle$. Repeat for $I = \langle x_2^2 - x_1x_3, x_1^2 - x_0x_2, x_1x_2x_3 - x_0x_3^2, x_0x_1x_2 - x_0^2x_3 \rangle$. Explain what you notice. (You may find a computer algebra system helps here - ask around until you find a fellow student who knows how to use one if you don't).
(9) Show that if X is an affine or projective variety or a subvariety of a torus, then there is a largest ideal $I \subset S$ with $X = V(I)$ in the sense that if $X = V(J)$ then $J \subseteq I$.
(10) What is the dimension of the affine variety $V(I)$ for $I = \langle x_1, x_2 \rangle \subset \mathbb{A}^5$? What about the affine variety $V(x_1^2 - 3x_2x_3) \subset \mathbb{A}^3$? What is the dimension of the projective variety $V(\langle x_0x_2 - x_1x_3, x_0x_2 - x_1^2, x_1x_3 - x_2^2 \rangle)$?
(11) Let $I = \langle x_1^2 + x_2, x_2^2 + x_3 \rangle \subset \Bbbk[x_1, x_2, x_3]$. What is the multiplicity of the intersection of the affine varieties $V(I)$ and $V(x_i)$ for $i = 1, 2, 3$?
(12) Show that there is a unique curve of degree two through any five points in $\mathbb{P}^2$ in general position. What is the genericity condition?

5. LECTURE 4

Today we will discuss valuations and Puiseux series.

Let K be a field. We denote by K^* the nonzero elements of K. A *valuation* on K is a function $\mathrm{val}\colon K \to \mathbb{R} \cup \infty$ satisfying

(1) $\mathrm{val}(a) = \infty$ if and only if $a = 0$,
(2) $\mathrm{val}(ab) = \mathrm{val}(a) + \mathrm{val}(b)$ and
(3) $\mathrm{val}(a + b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\}$ for all $a, b \in K^*$.

We will always assume that $1 \in \mathrm{im}(\mathrm{val})$. Since $(\lambda\,\mathrm{val})\colon K \to \mathbb{R}$ is a valuation for any valuation val and $\lambda \in \mathbb{R}_{>0}$, this is not a serious restriction.

Example: $K = \Bbbk(x)$, the ring of rational functions. We can write any function $f/g \in K$ as a Laurent series $h = \sum h_i x^i$ where $h_i = 0$ for $i \ll 0$. Then $\mathrm{val}(f/g) = \min(i : h_i \neq 0)$. If i is the lowest exponent occuring in f and j is the lowest exponent occuring in g, then $\mathrm{val}(f/g) = i - j$.

Example: $K = \mathbb{Q}$, and $\mathrm{val}_p(q) = j$ when $q = p^j a/b$, where p does not divide a or b. For example $\mathrm{val}_2(12/5) = 2$, while $\mathrm{val}_2(1/10) = -1$. This the p-adic valuation.

Lemma 5.1. *If* $\mathrm{val}(a) \neq \mathrm{val}(b)$ *then* $\mathrm{val}(a + b) = \min(\mathrm{val}(a), \mathrm{val}(b))$.

Proof. Without loss of generality we may assume that $\mathrm{val}(b) > \mathrm{val}(a)$. Since $1^2 = 1$, we have $\mathrm{val}(1) = 0$, and so $(-1)^2 = 1$ implies $\mathrm{val}(-1) = 0$ as well. Thus $\mathrm{val}(-b) = \mathrm{val}(b)$, so $\mathrm{val}(a) \geq \min(\mathrm{val}(a + b), \mathrm{val}(-b)) = \min(\mathrm{val}(a + b), \mathrm{val}(b))$, and so $\mathrm{val}(a) \geq \mathrm{val}(a + b)$. But $\mathrm{val}(a + b) \geq \min(\mathrm{val}(a), \mathrm{val}(b)) = \mathrm{val}(a)$, and thus $\mathrm{val}(a + b) = \mathrm{val}(a)$. □

Given a valuation val we define the *valuation ring*

$$R = \{a \in K : \mathrm{val}(a) \geq 0\} \cup \{0\}.$$

This is closed under addition and multiplication, since $\mathrm{val}(a), \mathrm{val}(b) \geq 0$ implies $\mathrm{val}(ab), \mathrm{val}(a + b) \geq 0$. It has a unique maximal ideal

$$\mathfrak{m} = \{a \in K : \mathrm{val}(a) > 0\} \cup \{0\}.$$

To see that $\mathfrak{m}$ is the unique maximal ideal, it suffices to note that if $a \in R \setminus \mathfrak{m}$ then a is a unit in R. Indeed, if $a \in R \setminus \mathfrak{m}$, then $\mathrm{val}(a) = 0$, so $\mathrm{val}(a^{-1}) = -\mathrm{val}(a) = 0$, so $a^{-1} \in R$. The *residue field* is

$$\Bbbk = R/\mathfrak{m}.$$

Example: If $K = \Bbbk((x))$ is the quotient ring of $\Bbbk[[x]]$, then $R = \Bbbk[[x]]$, and $R/\mathfrak{m} = \Bbbk$.

Example: In the case that $K = \mathbb{Q}$ and val is the p-adic valuation, we have $R = \{p^j a/b : j \geq 0\} \cup \{0\}$. Exercise: Check that the residue field is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Example: Let $R_n = \Bbbk[[t^{1/n}]]$, and let $\Bbbk((t^{1/n}))$ be its quotient field. Let $K = \bigcup_{n\geq 1} \Bbbk((t^{1/n}))$, which we denote by $\Bbbk\{\{t\}\}$. Note that K is closed under addition and multiplication, and is thus a field. The field K is the ring of *Puiseux series*. An element of K has the form $\sum_{q\in\mathbb{Q}} a_q t^q$ where $\{q : a_q \neq 0\}$ is bounded below and has a common denominator.

The field $\Bbbk((t^{1/n}))$ has a valuation like that on the ring of rational functions. This induces a valuation $\mathrm{val} : K \to \mathbb{R} \cup \infty$. If $a = \sum_{q\in\mathbb{Q}} a_q t^q \in K$, then $\mathrm{val}(a) = \min\{q : a_q \neq 0\}$.

Puiseux series are useful because they are algebraically closed, as we now prove.

Theorem 5.2. *If $\Bbbk$ is an algebraically closed field of characteristic zero, then $K = \Bbbk\{\{t\}\}$ is algebraically closed.*

I learned the following proof from Thomas Markwig, and it is closely modelled on the one he gives in his paper [Mar07] on a generalization of the Puiseux series.

Proof. We need to show that given a polynomial $F = \sum_{i=0}^{n} c_i x^i \in S = K[x]$ there is $y \in K$ with $F(y) = \sum_{i=0}^{n} c_i y^i = 0$. In principle the idea is to build y up as a Puiseux series by successive powers of t.

We first note that we may assume the following properties of F:

(1) $\mathrm{val}(c_i) \geq 0$ for all i,
(2) There is some j with $\mathrm{val}(c_j) = 0$,
(3) $c_0 \neq 0$, and
(4) $\mathrm{val}(c_0) > 0$.

To see this, note that if $\alpha = \min\{\mathrm{val}(c_i) : 0 \leq i \leq n\}$ then multiplying F by $t^{-\alpha}$ does not change the existence of a root of F, which deals with the first two properties. If $c_0 = 0$ then $y = 0$ is a root so there is nothing to prove.

To make the last assumption, suppose that F satisfies the first three assumptions but $\mathrm{val}(c_0) = 0$. If $\mathrm{val}(c_n) > 0$ then we can form $G(x) = x^n F(1/x) = \sum_{i=0}^{n} c_{n-i} x^i$, which has the desired form, and if $G(y') = 0$ then $F(1/y') = 0$. If $\mathrm{val}(c_0) = \mathrm{val}(c_n) = 0$ then consider the polynomial $f := \overline{F} \in \Bbbk[x]$ that is the image of F in $K[x]/\mathfrak{m}K[x]$. This which is not constant since $\mathrm{val}(c_n) = 0$. Since $\Bbbk$ is algebraically closed, we can choose a root $\lambda \in \Bbbk$ of f. Then

$$F'(x) := F(x + \lambda) = \sum_{i=0}^{n} (\sum_{j=i}^{n} c_j \binom{j}{i} \lambda^{j-i}) x^i$$

has constant term $F'(0) = F(\lambda)$ with positive valuation, and F' still satisfies the first three properties. If y' is a root of F', then $y' + \lambda$ is a root of F.

Set $F_0 = F$. We will construct a sequence of polynomials $F_i = \sum_{j=0}^{n} c_j^i x^j$. Suppose, as we have shown we may assume for $i = 0$, that F_i satisfies conditions 1 to 4 above. The *Newton polygon* of F_i is the convex hull of the points $\{(i,j) :$ there is k with $k \leq i, \mathrm{val}(c_k) \leq j\} \subset \mathbb{R}^2$. There is an edge of the Newton polygon with negative slope connecting the vertex $(0, \mathrm{val}(c_0^i))$ to a vertex $(k_i, \mathrm{val}(c_{k_i}^i))$. Let

$$w_i = \frac{\mathrm{val}(c_0^i) - \mathrm{val}(c_{k_i}^i)}{k_i}.$$

Let f_i be the image in $\Bbbk[x]$ of the polynomial $t^{-\mathrm{val}(c_0^i)} F(t^{w_i} x) \in K[x]$. Note that f_i has degree k_i, and has nonzero constant term. Since $\Bbbk$ is algebraically closed we can find a root λ_i of f_i. Let r_{i+1} be the multiplicity of λ_i as a root of f_i, so $f_i = (x - \lambda_i)^{r_{i+1}} g_i(x)$, where $g_i(\lambda_i) \neq 0$. Set

$$F_{i+1}(x) = t^{-\mathrm{val}(c_0^i)} F_i(t^{w_i}(x + \lambda_i)) = \sum_{j=0}^{n} c_j^{i+1} x^j.$$

Note that the coefficients c_j^{i+1} are given by the formula

$$c_j^{i+1} = \sum_{l=j}^{n} c_l^i t^{lw_i - \mathrm{val}(c_0^i)} \binom{l}{j} \lambda_i^{l-j}. \tag{1}$$

The image of this in $\Bbbk$ is

$$\overline{c_j^{i+1}} = \frac{1}{j!}\frac{\partial^j f_i}{\partial x^j}(\lambda_i).$$

Note that we are using the characteristic zero assumption here. For $0 \leq j < r_{i+1}$ this is zero, since λ_i is a root of f_i of multiplicity r_{i+1}. For $j = r_{i+1}$ this is nonzero. Thus $\mathrm{val}(c_j^{i+1}) > 0$ for $0 \leq j \leq r_{i+1}$, and $\mathrm{val}(c_j^{i+1}) = 0$ for $j = r_{i+1}$. Note that we are using the fact that $\mathrm{char}(\Bbbk) = 0$ here.

If $c_0^{i+1} = 0$ then $x = 0$ is a root of F_{i+1}, so $\lambda_i t^{w_i}$ is root of F_i and so by recursing we get $\sum_{j=0}^{i} \lambda_i t^{w_0+\dots+w_j}$ is a root of $F_0 = F$, and we are done. Thus we may assume that for each i we have $c_0^{i+1} \neq 0$, so F_{i+1} satisfies conditions 1 to 4 above, so we can continue.

The observation above on $\mathrm{val}(c_j^{i+1})$ implies that $k_{i+1} \leq r_{i+1} \leq k_i$. Since n is finite, the value of k_i can only drop a finite number of times, so there is $1 \leq k \leq n$ and m for which for $i \geq m$ we have $k_i = k$. This means that $r_i = k$ for all $i > m$, so $f_i = \mu_i(x - \lambda_i)^k$ for all $i > m$, and some $\mu_i \in \Bbbk$.

Let N_i be such that $c_j^i \in \Bbbk((t^{1/N_i}))$ for $0 \leq j \leq n$. We can take N_{i+1} to be the least common denominator of N_i and w_i by Equation 1. Let $y_i = \sum_{j=0}^{i} \lambda_i t^{w_0+\dots+w_j} \in \Bbbk((t^{1/N_i}))$. We now show that we can take $N_{i+1} = N_i$ for $i > m$. In that case, we have $w_i = \mathrm{val}(c_0^i)/k$, so it suffices to show that for $i > m$ we have $\mathrm{val}(c_0^i) \in k/N_i\mathbb{Z}$. This follows from the fact that f_i is a pure power, so $\mathrm{val}(c_j^i) = (k-j)/k\,\mathrm{val}(c_0^i)$ for $1 \leq j \leq k$, and in particular $\mathrm{val}(c_{k-1}^i) = 1/k\,\mathrm{val}(c_0^i) \in 1/N_i\mathbb{Z}$. Thus there is an N for which $y_i \in \Bbbk((t^{1/N}))$ for all i, and so the limit

$$y = \sum_{j \geq 0} \lambda_j t^{w_0+\dots+w_j} \in \Bbbk((t^{1/N})).$$

It remains to show that that y is a root of F. To see this, consider $z_i = \sum_{j \geq i} \lambda_j t^{w_i+\dots+w_j}$, and note that $y = y_{i-1} + t^{w_0+\dots+w_{i-1}} z_i$ for $i > 0$, so

$$F_i(z_i) = t^{\mathrm{val}(c_0^i)} F_{i+1}(z_{i+1}).$$

Since $z_0 = y$, it follows that

$$\mathrm{val}(F(y)) = \sum_{j=0}^{i} \mathrm{val}(c_0^j) + \mathrm{val}(F_{i+1}(z_{i+1})) > \sum_{j=0}^{i} \mathrm{val}(c_0^j)$$

for all $i > 0$. Since $\mathrm{val}(c_0^j) \in 1/N\mathbb{Z}$, we conclude that $\mathrm{val}(F(y)) = \infty$, so $F(y) = 0$ as required. □

If $\Bbbk$ has characteristic $p > 0$ then $\Bbbk\{\{t\}\}$ is not algebraically closed. This is because the Artin-Schreier polynomial $x^p - x - t^{-1}$ has p "roots" of the form

$$(t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + \dots) + c$$

for c in the prime field $\mathbb{F}_p$ of $\Bbbk$. These are not Puiseux series, since there is no common denominator of the exponents, but do live in the field of *generalized power series* which we now define.

Fix an algebraically closed field $\Bbbk$, and a divisible group $G \subset \mathbb{R}$. The *Mal'cev-Neumann ring* $K = \Bbbk((G))$ of generalized power series is the set of formal sums $\alpha = \sum_{g \in G} \alpha_g t^g$ in an indeterminant t with the property that $\operatorname{supp}(\alpha) := \{g \in G : \alpha_g \neq 0\}$ is a well-ordered set.

If $\beta = \sum_{g \in G} \beta_g t^g$ then we set $\alpha+\beta = \sum_{g \in G}(\alpha_g+\beta_g)t^g$, and $\alpha\beta = \sum_{h \in G}(\sum_{g+g'=h} \alpha_g \beta_{g'})t^h$. Then $\operatorname{supp}(\alpha+\beta) \subseteq \operatorname{supp}(\alpha) \cup \operatorname{supp}(\beta)$, so is well-ordered, and thus $\alpha+\beta$ is well-defined. For $\alpha\beta$, define $\operatorname{supp}(\alpha)+\operatorname{supp}(\beta)$ to be the set $\{g+g' : g \in \operatorname{supp}(\alpha), g' \in \operatorname{supp}(\beta)\}$. Then $\operatorname{supp}(\alpha)+\operatorname{supp}(\beta)$ is well-ordered, and the set $\{(g,g') : g+g' = h\}$ is finite for all $h \in G$, so multiplication is well-defined.

The field of generalized power series is the most general field with valuation we need to consider in the following sense.

Theorem 5.3. *Fix a divisible group G and a residue field $\Bbbk$. Let K be a field with a valuation* val *with value group G such that* val *is trivial on the prime field ($\mathbb{F}_P$ or $\mathbb{Q}$) of K, and K has residue field $\Bbbk$. Then K is isomorphic to a subfield of $\Bbbk((t^G))$.*

One reference for these topics is [Poo93].

6. Lecture 5

The goal for today is to discuss Gröbner bases in our contexts. From now on we will always have K being an algebraically closed field with a nontrivial valuation (such as $\mathbb{C}\{\{t\}\}$) with residue field $\Bbbk$. We will discuss Gröbner bases in three different contexts.

The homogeneous case. We first consider the case where there is an inclusion of $\Bbbk$ into K with the image having valuation zero. This is the case for the Puiseux series, but not for all possible K. When an ideal has generators in $\Bbbk \subset K$, we say that the corresponding variety is defined over $\Bbbk$, and that we are in the *constant coefficients* case.

In this case, we first let $S = \Bbbk[x_0, \dots, x_n]$. Fix $w \in \mathbb{R}^n$. Given a polynomial $f = \sum_{u \in \mathbb{N}^{n+1}} c_u x^u \in S$, set $W = \min\{(0,w) \cdot u : c_u \neq 0\}$. Then

$$\operatorname{in}_w(f) = \sum_{(0,w)\cdot u = W} c_u x^u.$$

If I is a homogeneous ideal, then we set

$$\operatorname{in}_w(I) = \langle \operatorname{in}_w(f) : f \in I \rangle.$$

Example: Let $f = x_0^2 + 3x_0x_1$. When $w = 2$, $\operatorname{in}_w(f) = x_0^2$. When $w = 0$, $\operatorname{in}_w(f) = x_0^2 + 3x_0x_1$. When $w = -3$, $\operatorname{in}_w(f) = 3x_0x_1$. If $I = \langle x_0^2 + 3x_0x_1 \rangle$, then $\operatorname{in}_2(I) = \langle x_0^2 \rangle$.
Example: $I = \langle x_0x_2 - x_1^2, x_0x_1 - x_2^2 \rangle$. Take $w = (2,3)$. Then $\operatorname{in}_w(x_0x_2 - x_1^2) = x_0x_2$, and $\operatorname{in}_w(x_0x_1 - x_2^2) = x_0x_1$. *However* $\operatorname{in}_w(I) \neq \langle x_0x_2, x_0x_1 \rangle$, since $x_1^3 - x_2^2 \in I$, and $\operatorname{in}_w(x_1^3 - x_2^3) = x_1^3$. In this case $\operatorname{in}_w(I) = \langle x_0x_2, x_0x_1, x_1^3 \rangle$.

Definition 6.1. A set $\{g_1, \dots, g_s\} \subset I$ is a Gröbner basis for w if $\operatorname{in}_w(I)$ is generated by $\{\operatorname{in}_w(g_1), \dots, \operatorname{in}_w(g_s)\}$.

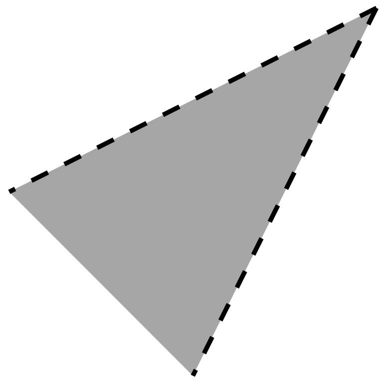

FIGURE 12.

Example: With I as above, and $w = (-1,-1)$, $\mathrm{in}_w(I) = \langle x_1^2, x_2^2\rangle$. With $w = (0,0)$, $\mathrm{in}_w(I) = \langle x_0x_2 - x_1^2, x_0x_1 - x_2^2\rangle$. With $w = (1,2)$, $\mathrm{in}_w(x_0x_2 - x_1^2) = x_0x_2 - x_1^2$, and $\mathrm{in}_w(x_0x_1 - x_2^2) = x_0x_1$, and we have $\mathrm{in}_w(I) = \langle x_0x_2 - x_1^2, x_0x_1\rangle$.

Definition 6.2. An ideal in S is monomial if it is generated by monomials. We say that w is generic with respect to I if $\mathrm{in}_w(I)$ is a monomial ideal.

Lemma 6.3. *If w is generic then the monomials not in $\mathrm{in}_w(I)$ form a $\mathbb{k}$-basis for S/I.*

We put an equivalence relation on $\mathbb{R}^n$ by setting $w \sim w'$ if $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$.
Example: With I as above, for $w = (-2,-3)$, we have $\mathrm{in}_w(I) = \langle x_1^2, x_2^2\rangle$, so $(-1,-1) \sim (-2,-3)$.

Theorem 6.4. *The set*

$$C[w] := \{w' \in \mathbb{R}^n : \mathrm{in}_w(I) = \mathrm{in}_{w'}(I)\}$$

is a relatively open polyhedral cone. This means that it can be described by equations and strict inequalities.

For a proof of this, see [Stu96, Chapter 2], or [Mac].
Example: Let I be as above, and $w = (-1,-1)$. Then

$$C[w] = \{w' \in \mathbb{R}^2 : 2w_1' < w_2', 2w_1' < w_2'\}.$$

This is shown in Figure 12.

Definition 6.5. A polyhedral cone is the intersection of finitely many halfspaces with the corresponding hyperplanes passing through the origin. It is thus of the form

$$C = \{x : Ax \leq 0\}$$

where A is a $d \times n$ matrix. A hyperplane H in $\mathbb{R}^n$ is supporting for a cone C if C lies in one of the two halfspaces determined by H. A face of C is the intersection of C with a supporting hyperplane.

A fan is a collection of polyhedral cones, the intersection of any two of which is a face of each.

Theorem 6.6. *For a fixed ideal I the collection $\{\overline{C[w]} : w \in \mathbb{R}^n\}$ is a polyhedral fan.*

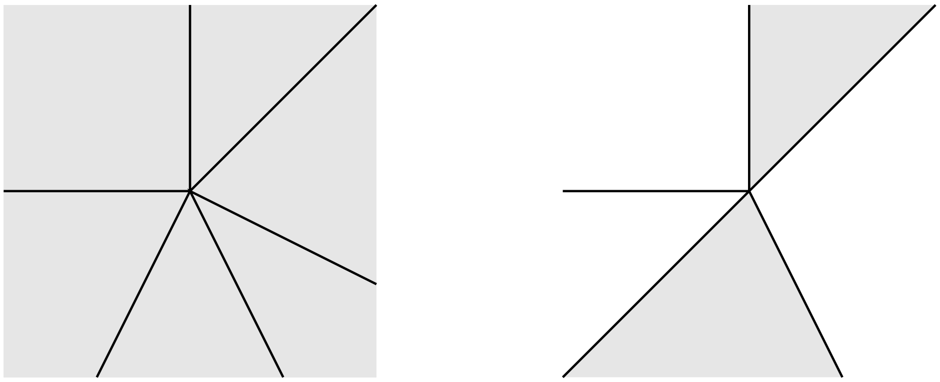

FIGURE 13. Polyhedral fans

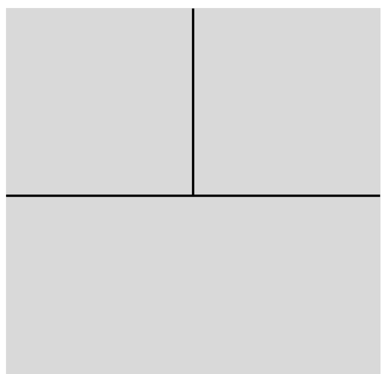

FIGURE 14. Not a polyhedral fan

Definition 6.7. This fan is called the Gröbner fan of I.

Example: Let $I = \langle x_0x_1 - x_2^2, x_0x_2 - x_1^2 \rangle$. Then the Gröbner fan for I is shown in Figure 15, where the ideals corresponding to the cones are given in the following table.

Cone	Initial ideal	Cones	Initial ideal
A	$\langle x_0x_1, x_1^2, x_0^2x_2 \rangle$	a	$\langle x_0x_1, x_0x_2 - x_1^2 \rangle$
B	$\langle x_0x_2, x_0x_1, x_2^3 \rangle$	b	$\langle x_0x_2, x_0x_1, x_1^3 - x_2^3 \rangle$
C	$\langle x_0x_2, x_0x_1, x_1^3 \rangle$	c	$\langle x_0x_2, x_0x_1 - x_2^2 \rangle$
D	$\langle x_0x_2, x_2^2, x_0^2x_1 \rangle$	d	$\langle x_0x_2, x_2^2, x_0^2x_1 - x_1^2x_2 \rangle$
E	$\langle x_0x_2, x_2^2, x_2^2x_2, x_0^3x_1 \rangle$	e	$\langle x_0x_2, x_2^2, x_1^2x_2, x_0^3x_1 - x_1^4 \rangle$
F	$\langle x_0x_2, x_2^2, x_1^2x_2, x_1^4 \rangle$	f	$\langle x_0x_2 - x_1^2, x_2^2 \rangle$
G	$\langle x_1^2, x_2^2 \rangle$	g	$\langle x_0x_1 - x_2^2, x_1^2 \rangle$
H	$\langle x_0x_1, x_1^2, x_1x_2^2, x_2^4 \rangle$	h	$\langle x_0x_1, x_1^2, x_1x_2^2, x_0^3x_2 - x_2^4 \rangle$
I	$\langle x_0x_1, x_1^2, x_1x_2^2, x_0^3x_2 \rangle$	i	$\langle x_0x_1, x_1^2, x_0^2x_2 - x_1x_1^2 \rangle$

There is software, called `gfan` [Jen], written by Anders Jensen, that will compute the Gröbner fan of an ideal.

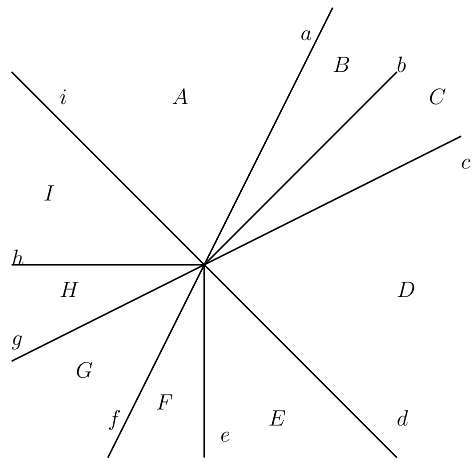

FIGURE 15.

The torus case. We now consider the case when $S = \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Given a Laurent polynomial $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$, and $w \in \mathbb{R}^n$, we set $W = \min\{w \cdot u : c_u \neq 0\}$, and then

$$\mathrm{in}_w(f) = \sum_{w \cdot u = W} c_u x^u.$$

If I is an ideal in S, then the initial ideal is

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle.$$

We make the same caveats as before on the fact that the initial ideal is not necessarily generated by the initial terms of generators.

Example: Let $f = x + 1 \in \Bbbk[x^{\pm 1}]$, and let $I = \langle x + 1 \rangle$. When $w = 1$, we have $\mathrm{in}_w(f) = 1$, and $\mathrm{in}_w(I) = \langle 1 \rangle$. When $w = -1$, $\mathrm{in}_w(f) = x$ and $\mathrm{in}_w(I) = \langle x \rangle = \langle 1 \rangle$.

Example: Let $f = x + y + 1 \in \Bbbk[x^{\pm 1}, y^{\pm 1}]$, and let $I = \langle f \rangle$. For $w = (1, 1)$, $\mathrm{in}_w(f) = 1$, and $\mathrm{in}_w(I) = \langle 1 \rangle$. For $w = (1, 0)$, $\mathrm{in}_w(f) = y + 1$, and $\mathrm{in}_w(I) = \langle y + 1 \rangle$. If $w = (1, -1)$ then $\mathrm{in}_w(f) = y$, and $\mathrm{in}_w(I) = \langle y \rangle = \langle 1 \rangle$.

The Gröbner fan does not exist in the same fashion, as we can see from this example that $\{w : \mathrm{in}_w(I) = \langle 1 \rangle\}$ is not convex. However ignoring these cases gives a cone. The support of a polyhedral fan in $\mathbb{R}^n$ is the set of those $w \in \mathbb{R}^n$ lying in some cone of the fan. It follows from the following proposition that the set of w with $\mathrm{in}_w(I) \neq \langle 1 \rangle$ has the support of a polyhedral fan.

If $f = \sum c_u x^u \in \Bbbk[x_1, \dots, x_n]$, let $W = \max\{|u| : c_u \neq 0\}$, where $|u| = \sum_{i=1}^n u_i$. The homogenization, $\tilde{f}$ of f is $\tilde{f} = \sum c_u x_0^{W - |u|} x^u \in \Bbbk[x_0, \dots, x_n]$.

Proposition 6.8. *Let I be an ideal in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $\bar{I} = I \cap \Bbbk[x_1, \dots, x_n]$, and let $J \subset \Bbbk[x_0, \dots, x_n] = \langle \tilde{f} : f \in \bar{I} \rangle$. Then*

$$\mathrm{in}_w(J)|_{x_0 = 1} = \mathrm{in}_w(I).$$

Proof. Exercise. □

Remark 6.9. The variety in $\mathbb{P}^n$ of the ideal J from Proposition 6.8 is the *projective closure* of the variety of I in T^n. This is the Zariski closure in $\mathbb{P}^n$ of the image of the variety of $I \subset T^n$ under the map $i : T^n \to \mathbb{P}^n$ given by $x \mapsto (1 : x)$.

Corollary 6.10. *Let $I \subset \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and let J be the ideal defined in Proposition 6.8. Then the support of the subfan of the Gröbner fan of J consisting of those cones σ for which $(\mathrm{in}_w(J) : x_0^\infty) \neq \langle 1 \rangle$ for $w \in \sigma$ is $\{w \in \mathbb{R}^n : \mathrm{in}_w(I) \neq \langle 1 \rangle$.*

Proof. By Proposition 6.8, we have $\mathrm{in}_w(I) \neq \langle 1 \rangle$ if and only if there is no power of x_0 in $\mathrm{in}_w(J)$, and thus if and only if $(\mathrm{in}_w(J) : x_0^\infty) \neq \langle 1 \rangle$. ☐

Nonconstant coefficients. We now consider the case that $S = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Fix $w \in \mathbb{R}^n$, and let $f = \sum_{u \in \mathbb{Z}^n} c_u x_u \in S$. Let $W = \min\{\mathrm{val}(c_u) + w \cdot u : c_u \neq 0\}$. Then

$$\mathrm{in}_w(f) = \overline{t^{-W} \sum_{u \in \mathbb{Z}^n} c_u t^{w \cdot u} x^u} \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

and

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle.$$

Example: Let $f = (t + t^2)x + t^2 y + t^4$, and $w = (0, 0)$. Then $W = 1$, and $\mathrm{in}_w(f) = \overline{(1 + t)x} = x$. When $w = (4, 2)$, $W = 4$, and $\mathrm{in}_w(f) = y + 1$. When $w = (2, 1)$, $W = 3$, and $\mathrm{in}_w(f) = x + y$.

Definition 6.11. A polyhedron is the intersection of finitely many (affine) halfspaces. Unlike a polyhedral cones the boundary (affine) hyperplanes are not required to pass through the origin. An affine hyperplane H is supporting for a polyhedron P if $P \cap H \neq \emptyset$ and P lies on one side of H. A face of P is the intersection of P with a supporting hyperplane, or P itself. A *polyhedral complex* in $\mathbb{R}^n$ is a collection of polyhedra in $\mathbb{R}^n$, the intersection of any two of which is a face of each.

As in the constant coefficient $\Bbbk$ case we can also consider initial ideal in the homogenized polynomial ring $K[x_0, \dots, x_n]$. Instead of a Gröbner fan, though, there is now a *Gröbner complex*. Each $w \in \mathbb{R}^n$ determines a relatively open polyhedron in $\mathbb{R}^n$ on which the initial ideal is constant. Throwing away those polyhedra for which the corresponding initial ideal contains a power of x_0, we obtain the following theorem, whose proof we will omit.

Theorem 6.12. *The set of $w \in \mathbb{R}^n$ for which $\mathrm{in}_w(I) \neq \langle 1 \rangle$ is the support of a polyhedral complex.*

Example: Let $f = tx^2 + 2xy + 3ty^2 + 4x + 5y + 6t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$, and let $I = \langle f \rangle$ be the ideal generated by I. Then the set $\{w \in \mathbb{R}^2 : \langle \mathrm{in}_w(I) \neq \langle 1 \rangle\}$ is illustrated in Figure 16. The initial ideals corresponding to the labelled edges are as listed in the following table.

Cone	Initial ideal
A	$\langle x^2 + 4x \rangle$
B	$\langle 4x + 6 \rangle$
C	$\langle 5y + 6 \rangle$
D	$\langle 4x + 5y \rangle$
E	$\langle 2xy + 4x \rangle$
F	$\langle x^2 + 2xy \rangle$
G	$\langle 2xy + 5y \rangle$
H	$\langle 3y^2 + 5y \rangle$
I	$\langle 2xy + 3y^2 \rangle$

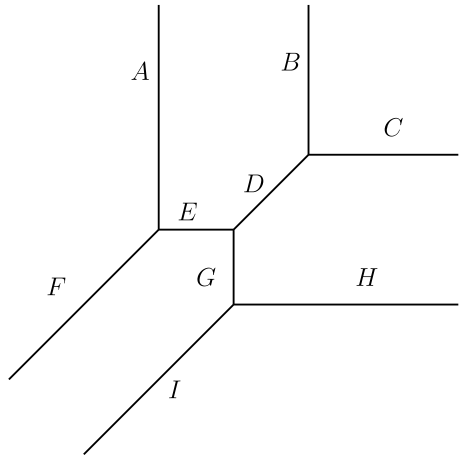

Figure 16.

7. Exercises

Valuations

(1) Show that the residue field of $\Bbbk\{\{t\}\}$ is isomorphic to $\Bbbk$.
(2) Let $K = \mathbb{Q}$ with the p-adic valuation. Show that the residue field of K is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.
(3) Show that if K is an algebraically closed field with a valuation $\mathrm{val} : K^* \to \mathbb{R}$, and $\Bbbk = R/\mathfrak{m}$ is its residue field, then $\Bbbk$ is algebraically closed. Give an example to show that if $\Bbbk$ is algebraically closed it does not automatically follow that K is algebraically closed.
(4) In the proof that $\Bbbk\{\{t\}\}$ is algebraically closed, explain why f_i has degree k_i and has a nonzero constant term.
(5) Apply the algorithm implicit in the proof that $\mathbb{C}\{\{t\}\}$ is algebraically closed to compute (the start of) a solution to the equation $x^2 + t + 1 = 0$. Check your answer with a computer algebra package (eg `puiseux` in `maple`).

Gröbner bases

(1) Let $I = \langle f \rangle \subset \Bbbk[x_0, \dots, x_n]$ be a principal ideal. Show that f is a Gröbner basis for I.
(2) Compute all the initial ideals $\mathrm{in}_w(f)$ of $f = 7x_0^2 + 8x_0x_1 - x_1^2 + x_0x_2 + 3x_2^2$ as w varies in $\mathbb{R}^2$. Draw the Gröbner fan of $\langle f \rangle$. (Hint: start by choosing some particular values of w).
(3) Show that if $\mathrm{in}_w(I)$ is a monomial ideal for $I \subset S = \Bbbk[x_0, \dots, x_n]$ then the monomials not in $\mathrm{in}_w(I)$ form a $\Bbbk$-basis for S/I.
(4) In this question you will compute the Gröbner fan of a principal ideal. The *Newton polytope* of a polynomial $f = \sum_{u \in \mathbb{N}^{n+1}} c_u x^u \in \Bbbk[x_0, \dots, x_n]$ is the convex hull in $\mathbb{R}^{n+1}$ of the exponents $\{u : c_u \neq 0\}$.
 (a) Draw the Newton polytope of $x_0^2 + x_0x_1 + x_1^2 + x_2^2$.
 If P is a polytope in $\mathbb{R}^n$, a point $\mathbf{v} \in P$ is a *vertex* if there is $\mathbf{w} \in \mathbb{R}^n$ for which $\mathbf{w} \cdot \mathbf{v} < \mathbf{w} \cdot x$ for all $x \in P \setminus \mathbf{v}$. The *normal cone* to P at $\mathbf{v}$ is the closure of the set of all such $\mathbf{w}$.

(b) Let $P = \operatorname{conv}((0,0),(2,0),(0,2),(1,1),(2,2))$. What are the vertices of P? Draw the normal cone to each.

The *normal fan* of P is the union of the normal cones to vertices of P. It is a polyhedral fan.

(c) Draw the normal fan to the P of the previous question.

(d) Show that the Gröbner fan (as we have defined it) of $\langle f \rangle$ is the $x_0 = 0$ slice of the normal fan to the Newton polytope of f.

(5) (For people who already knew something about Gröbner bases). It is more standard to define an initial ideal using a term order on the polynomial ring.

(a) Let $f = x_0^2 + x_0x_1 + x_1^2 + x_2^2$. For each lexicographic or degree reverse lexicographic term order $\prec$ find $w \in \mathbb{R}^2$ with $\mathrm{in}_w(f) = \mathrm{in}_\prec(f)$.

(b) In fact every term order can be represented by a vector w. You can read a proof, for example, in Proposition 2.4.4 of the notes available at `www.warwick.ac.uk/staff/D.Maclagan/papers/indialectures.pdf.gz` . See elsewhere in that chapter for hints on how to compute $\mathrm{in}_w(I)$ using your favourite computer algebra package.

(6) Let $f = t^2x + 3ty + t^4 \in K[x^{\pm 1}, y^{\pm 1}]$, where $K = \mathbb{C}\{\{t\}\}$. Compute $\mathrm{in}_w(f)$ for $w = (2,5)$, and $w = (1,2)$.

(7) Let $f = x + y + 1$. Draw $\{w \in \mathbb{R}^2 : \mathrm{in}_w(f) \neq \langle 1 \rangle\}$. Repeat this with $f = tx^2 + xy + ty^2 + x + y + t$. Compare your pictures with $\mathrm{trop}(V(f))$ in each case.

(8) Fix $I \subset \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $\bar{I} = I \cap \Bbbk[x_1, \ldots, x_n]$, and let $\tilde{J} = \langle \tilde{f} : f \in \bar{I} \rangle \subset \Bbbk[x_0, \ldots, x_n]$, where $\tilde{f}$ is the homogeneization of f using the variable x_0. Show that

$$\mathrm{in}_w(J)|_{x_0=1} = \mathrm{in}_w(I).$$

Optional extra: repeat with K.

(9) (Open ended for the more computationally minded:) Play with the software `gfan` (freely available from `http://www.math.tu-berlin.de/~jensen/software/gfan/gfan.html`).

(10) (Less open ended). If you don't download gfan, find someone else in the class who has.

8. Lectures 6 and 7

The goal for today is to define tropical varieties and state the fundamental theorem of tropical varieties.

As always, K is an algebraically closed field with a nontrivial valuation $\mathrm{val} : K \to \mathbb{R} \cup \infty$. It will never be wrong to take $K = \mathbb{C}\{\{t\}\}$.

We first recall the definition of Gröbner bases in $S = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ from last time. Technically the definition I gave in the previous lecture used the notation t^{-W}, so only made sense in the Puisuex series field. We first define a notion of t^a for $a \in \mathrm{im}(\mathrm{val})$ for an arbitrary algebraically closed field K with a valuation val.

Lemma 8.1. *Let K be an algebraically closed field with a valuation $\mathrm{val} : K \to \mathbb{R} \cup \infty$, and let $\mathrm{im}(\mathrm{val})$ be the additive subgroup of $\mathbb{R}$ that is the image of K^* under val. The surjection of abelian groups $K^* \twoheadrightarrow \mathrm{im}(\mathrm{val})$ splits, so there is a group homomorphism $\phi\colon \mathrm{im}(\mathrm{val}) \to K^*$ with $\mathrm{val}(\phi(w)) = w$.*

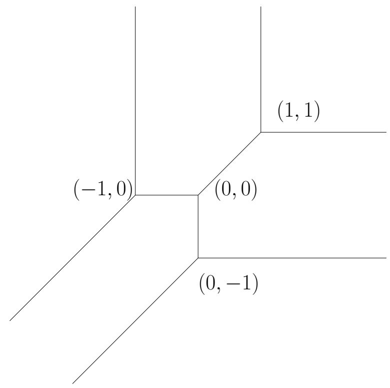

FIGURE 17.

Proof. Since K is algebraically closed, it contains the nth roots of all of its elements. Thus K^*, and so im(val) are divisible abelian groups. Since im(val) is an additive subgroup of $\mathbb{R}$ it is torsionfree, so im(val) is a torsionfree divisible group, and thus isomorphic to a (possibly uncountable) direct sum of copies of $\mathbb{Q}$ (see, for example, [Hun80, Exercise 8, p198]). Given any summand isomorphic to $\mathbb{Q}$, with $w \in \mathrm{im(val)}$ taken to 1 by the isomorphism, and any $a \in K^*$ with $\mathrm{val}(a) = w$, there is a homomorphism $\phi \colon \mathbb{Q} \to K^*$ taking w to $a \in K^*$. By construction this homomorphism satisfies $\mathrm{val}(\phi((m/n)w)) = (m/n)w$. The universal property of the direct sum then implies the existence of a homomorphism $\mathrm{im(val)} \to K^*$ with the desired property. □

We use the notation t^w to denote the element $\phi(w) \in K^*$. We always assume $1 \in \mathrm{im(val)}$, and so $\mathbb{N} \subseteq \mathrm{im(val)}$, so t^n makes sense for any $n \in \mathbb{N}$.

Fix $w \in \mathbb{R}^n$. Given $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in S$, let $W = \min\{\mathrm{val}(c_u) + w \cdot u : c_u \neq 0\}$. Then

$$\mathrm{in}_w(f) = \overline{t^{-W} \sum_{u \in \mathbb{Z}^n} c_u t^{w \cdot u} x^u} \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

and

$$\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle.$$

Example: Let $f = 3tx^2 + 5xy + 7ty^2 + 9x + y + 2t$, and let $I = \langle f \rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. Fix $w = (w_1, w_2) \in \mathbb{R}^2$. Then

$$\mathrm{in}_w(f) = \overline{t^{-W}(3t^{2w_1+1}x^2 + 5t^{w_1+w_2}xy + 7t^{2w_2+1}y^2 + 9t^{w_1}x + t^{w_2}y + 2t)} \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}],$$

where

$$W = \min(2w_1 + 1, w_1 + w_2, 2w_2 + 1, w_1, w_2, 1).$$

For example, if $W = 2w_1 + 1$ and all other terms are larger, then $\mathrm{in}_w(f) = 3x^2$, and $\mathrm{in}_w(I) = \langle 3x^2 \rangle = \langle 1 \rangle$. So for $\mathrm{in}_w(I) \neq \langle 1 \rangle$, a necessary condition is that the minimum in the definition of W is achieved twice!

For example, if $2w_1 + 1 = w_1 \leq w_1 + w_2, 2w_2 + 1, w_2, 1$, then $w_1 = -1, w_2 \geq 0$. In this case $\mathrm{in}_w(f) = 3x^2 + 9x$, so $\mathrm{in}_w(I) = \langle 3x^2 + 9x \rangle = \langle x + 3 \rangle \neq \langle 1 \rangle$. The set of w for which $\mathrm{in}_w(I) \neq \langle 1 \rangle$ is illustrated in Figure 17.

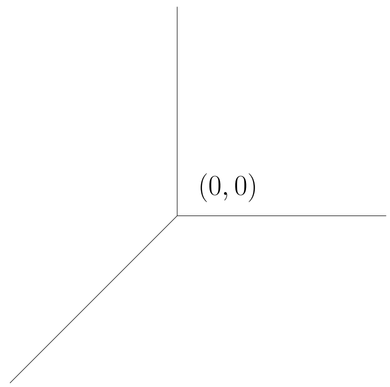

FIGURE 18.

We now recall from the first day of class the definition of the tropical hypersurface. Given $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we defined

$$\mathrm{trop}(f)(w) = \min(\mathrm{val}(c_u) + w \cdot u : c_u \neq 0),$$

and

$$\mathrm{trop}(V(f)) = \{w \in \mathbb{R}^n : \text{ the minimum in } \mathrm{trop}(f)(w) \text{ is achieved twice }\}.$$

Recall that if $X \subset T^n$ is a variety with (radical) ideal I then $X = \bigcap_{f \in I} \{x \in T^n : f(x) = 0\}$.

Definition 8.2. Let $X \subseteq T^n$ be a subvariety of T^n with (radical) ideal I. Then the *tropical variety* or *tropicalization* of X is

$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f)).$$

Warning: In the "classical" world we have

$$X = \bigcap_{f \in \mathcal{G}} V(f)$$

where $\mathcal{G}$ is *any* generating set for the ideal of X. The analogue is not true tropically.
Example: Let $X = V(x+y+1, x+2y+3) \subseteq T^2$. Note that $X = V(y+2, x-1) = \{(1, -2)\} \subseteq T^2$. However $\mathrm{trop}(V(x+y+1)) = \mathrm{trop}(V(x+2y+3)) = \{(u,v) \in \mathbb{R}^2 : u = v \leq 0\} \cup \{(u,v) \in \mathbb{R}^2 : u = 0 \leq v\} \cup \{(u,v) \in \mathbb{R}^2 : v = 0 \leq u\}$ as shown in Figure 18. Thus $\mathrm{trop}(V(x+y+1)) \cap \mathrm{trop}(V(x+2y+3))$ is the union of these three line segments. However $\mathrm{trop}(V(y+2))$ is the line $y = 0$, while $\mathrm{trop}(V(x-1))$ is the line $x = 0$, so their intersection is the point $(0, 0)$.

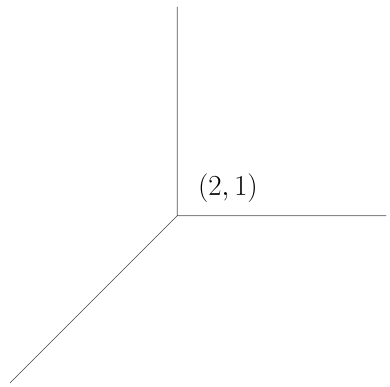

FIGURE 19.

Fundamental Theorem of Tropical Geometry.

Theorem 8.3. *Let $X \subseteq T^n_K$ be a subvariety of T^n with (radical) ideal $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then the following subsets of $\mathbb{R}^n$ coincide:*

(1) $\operatorname{trop}(X)$*;*
(2) $\{w \in \mathbb{R}^n : \operatorname{in}_w(I) \neq \langle 1 \rangle\}$*;*
(3) *The closure in $\mathbb{R}^n$ of* $\{(\operatorname{val}(x_1), \dots, \operatorname{val}(x_n)) \in \mathbb{R}^n : x = (x_1, \dots, x_n) \in X\}$.

We will sketch a proof of this theorem below, starting in the hypersurface case (when $I(X)$ is a principal ideal). We first illustrate the theorem with an example.
Example: Let $X = V(f) \subseteq T^2_K$ for $f = tx + 3t^2y + t^3 \in K[x^{\pm 1}, y^{\pm 1}]$, and let $I = \langle f \rangle$. Here, as always if it is not otherwise indicated, we take $K = \mathbb{C}\{\{t\}\}$. The three sets of Theorem 8.3 are constructed as follows.

(1) We have $\operatorname{trop}(f) = \min(x+1, y+2, 3)$, so the set $\operatorname{trop}(V(f)) = \{(u,v) \in \mathbb{R}^2 : u = 2 \leq v\} \cup \{(u,v) \in \mathbb{R}^2 : v = 1 \leq u\} \cup (u,v) \in \mathbb{R}^2 : u = v+1 \leq 2\}$. This is illustrated in Figure 19.

We are using here that the two possible definitions of $\operatorname{trop}(V(f))$ coincide, so the set where the minimum in the definition of $\operatorname{trop}(f)$ is achieved twice is equal to the intersection over *all* $g \in I$ of the set where the minimum in the definition of $\operatorname{trop}(g)$ is achieved twice. This is an exercise in the second exercise set.

(2) Given $w \in \mathbb{R}^2$, the initial term $\operatorname{in}_w(f)$ is the image in $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ of

$$t^{-W}(t^{w_1+1}x + 3t^{w_2+2}y + t^3),$$

where $W = \min(w_1+1, w_2+2, 3)$. If this minimum is achieved only once then $\operatorname{in}_w(f)$ is a monomial, so $\operatorname{in}_w(I) = \langle 1 \rangle$. So if $\operatorname{in}_w(I) \neq \langle 1 \rangle$, then the minimum is achieved at least twice. Conversely, if the minimum is achieved at least twice, then $\mathrm{in}_w(f)$ is not a monomial. It is

$$\begin{aligned} x+3y \text{ if } w_1+1=w_2+2<3; \\ x+1 \text{ if } w_1+1=3<w_2+2; \\ 3y+1 \text{ if } w_2+2=3<w_1+1; \\ x+3y+1 \text{ if } (w_1,w_2)=(2,1). \end{aligned}$$

Thus, since $\mathrm{in}_w(I)=\langle \mathrm{in}_w(f)\rangle$, we have

$$\{w\in\mathbb{R}^2 : \mathrm{in}_w(f)\neq\langle 1\rangle\}=\mathrm{trop}(X).$$

(3) The variety X is

$$\begin{aligned} X &= \{(x,y)\in T_K^2 : tx+3t^2y+t^3=0\} \\ &= \{(-t^2-3ty,y) : y\in K^*, 3ty+t^2\neq 0\}. \end{aligned}$$

Thus

$$\begin{aligned} &\overline{\{(\mathrm{val}(x),\mathrm{val}(y)) : (x,y)\in X\}} \\ &= \overline{\{(\mathrm{val}(-t^2-3ty),\mathrm{val}(y)) : y\in K^*, y\neq -t/3\}}. \end{aligned}$$

Now $\mathrm{val}(-t^2-3ty)=\min(2,\mathrm{val}(y)+1)$ if $y\neq -t/3+z$ with $\mathrm{val}(z)>1$. Thus $\overline{(\mathrm{val}(x),\mathrm{val}(y)) : (x,y)\in X\}}=\{(2,w):w\geq 1\}\cup\{(w+1,w):w\leq 1\}\cup\{(w,1):w\geq 2\}$. So all three sets coincide in this example.

We now begin the proof of Theorem 8.3. We first prove this in the hypersurface case.

Proposition 8.4. *Let K be an algebraically closed field with a nontrivial valuation* val*, and let $f\in K[x_1^{\pm1},\dots,x_n^{\pm1}]$. Then the following three sets coincide.*

1. $\mathrm{trop}(V(f))$*;*
2. *The set $\{w\in\mathbb{R}^n : \mathrm{in}_w(f)$ is not a monomial $\}$.*
3. *The closure of the set $\{(\mathrm{val}(v_1),\dots,\mathrm{val}(v_n)) : \mathbf{v}\in T_K^n, f(\mathbf{v})=0\}$;*

Proof. Let $(w_1,\dots,w_n)\in\mathrm{trop}(V(f))$. Then by definition the minimum $W=\min(\mathrm{val}(c_u)+\mathbf{u}\cdot w : c_u\neq 0)$ is achieved at least twice. This then means that $\mathrm{in}_w(f)=\sum_{\mathbf{u}:c_u\neq 0,\mathrm{val}(c_u)+\mathbf{u}\cdot w=W}\overline{c_u}x^u$ is not a monomial, and thus set (1) is contained in set (2). Conversely, if $\mathrm{in}_w(f)$ is not a monomial, then $\min(\mathrm{val}(c_u)+\mathbf{u}\cdot w : c_u\neq 0)$ is achieved at least twice, so $w\in\mathrm{trop}(V(f))$. This shows the other containment, so the first two sets are equal.

We now prove the inclusion of set (3) in set (1). Since set (1) is closed, it is enough to consider points in (3) of the form $\mathrm{val}(v):=(\mathrm{val}(v_1),\dots,\mathrm{val}(v_n))$ where $\mathbf{v}=(v_1,\dots,v_n)\in T_K^n$ satisfies $f(\mathbf{v})=0$. Let $\mathbf{v}\in T_K^n$ satisfy $f(\mathbf{v})=0$, so $\sum_{\mathbf{u}\in\mathbb{Z}^n}c_u\mathbf{v}^{\mathbf{u}}=0$. We first reduce to the case where $\mathrm{val}(c_u\mathbf{v}^{\mathbf{u}}))\geq 0$ for all $\mathbf{u}$, so $c_u\mathbf{v}^{\mathbf{u}}\in R$. Let $W=\min\{\mathrm{val}(c_u\mathbf{v}^{\mathbf{u}}) : c_u\neq 0\}$, and let $g=t^{-W}f$. Then $g(\mathbf{v})=0$, and $\mathrm{trop}(V(g))=\mathrm{trop}(V(f))$, so it suffices to prove the inclusion with f replaced by g. We can thus assume ($\star$) that $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) \geq 0$, and that there is at least one $\mathbf{u}$ with $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) = 0$. Then $f(\mathbf{v}) = \sum c_u \mathbf{v}^{\mathbf{u}} = 0$ is the sum of elements of R, and so we can consider their image in the residue field $\Bbbk = R/\mathfrak{m}$. This is $\sum \overline{c_u \mathbf{v}^{\mathbf{u}}} = 0 \in \Bbbk$. By assumption ($\star$) at least one of the terms $\mathbf{v}^{\mathbf{u}}$ is nonzero. Since the sum of all such terms is $0 \in \Bbbk$, we conclude that there must in fact be at least two terms with $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) = 0$. We have $\operatorname{val}(c_u \mathbf{v}^{\mathbf{u}}) = \operatorname{val}(c_u) + \sum u_i \operatorname{val}(v_i) \geq 0$ for all u by assumption $\star$, so this means that the minimum $0 = \min_{c_u \neq 0}(\operatorname{val}(c_u) + \mathbf{u} \cdot \operatorname{val}(\mathbf{v}))$ is achieved twice, where $\operatorname{val}(\mathbf{v}) = (\operatorname{val}(v_1), \ldots, \operatorname{val}(v_n))$. Thus $\operatorname{val}(\mathbf{v}) \in \operatorname{trop}(V(f))$ as required.

Finally, we prove the inclusion of set (1) into set (3). Since the image of the valuation val is dense in $\mathbb{R}$ (see Exercises), and the set (3) is closed by definition, it suffices to consider a point in (1) of the form $w = \operatorname{val}(\mathbf{y})$ for some $\mathbf{y} \in T_K^n$. We want to construct $\beta = (\beta_1, \ldots, \beta_n) \in T_K^n$ with $\operatorname{val}(\beta) = (\operatorname{val}(\beta_1), \ldots, \operatorname{val}(\beta_n)) = w$ and $f(\beta) = 0$.

Let $\operatorname{in}_w(f) = \sum a_u x^u \in \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Since $\operatorname{in}_w(f)$ is not a monomial, there is some variable x_i that appears to different powers in those x^u with $a_u \neq 0$. After reordering, we can assume that this is x_1. Replacing f by $x_1^m f$ for some $m \in \mathbb{Z}$ does not change $\operatorname{trop}(V(f))$ or whether $\operatorname{in}_w(f)$ is a monomial, so we may assume that x_1 appears in some, but not all, of the monomials occuring in $\operatorname{in}_w(f)$, and that if x_1 occurs, the exponent is positive.

Since $\Bbbk$ is algebraically closed and $\operatorname{in}_w(f)$ is not a monomial, we can find $\alpha \in (\Bbbk^*)^n$ with $\operatorname{in}_w(f)(\alpha) = 0$. Let $\beta_i = \alpha_i t^{w_i}$ for $2 \leq i \leq n$. Let $g(y) = f(y, \beta_2, \ldots, \beta_n) \in K[y^{\pm 1}]$. The assumption that the exponent of x_1 in every monomial of f is non-negative and that some such exponents are zero means that in fact $g \in K[y]$ is a polynomial of positive degree with nonzero constant term. Since K is algebraically closed we can thus factor g into linear factors:

$$g = \lambda \prod_{i=1}^{m} (y - b_i).$$

Write $\mathbf{u}' = (u_2, \ldots, u_n) \in \mathbb{Z}^{n-1}$ for the projection of $\mathbf{u}$ onto the last $n-1$ components. Then $g(y) = \sum_{\mathbf{u} \in \mathbb{Z}^n} (c_u \beta^{\mathbf{u}'}) y^{u_1}$. Note that $\operatorname{val}(\beta^{\mathbf{u}'}) = \sum_{i=2}^n u_i \operatorname{val}(\beta_i) = \sum_{i=2}^n u_i w_i$, so $\operatorname{val}(c_u \beta^{\mathbf{u}'}) + w_1 u_1 = \operatorname{val}(c_u) + w \cdot \mathbf{u}$.

Thus $\operatorname{in}_{w_1}(g)(y) = \sum_{\mathbf{u}: \operatorname{val}(c_u) + w \cdot \mathbf{u} = W} \overline{t^{w_1 u_1 - W} c_u \beta^{\mathbf{u}'}} y^{u_1} = \sum_{\mathbf{u}: \operatorname{val}(c_u) + w \cdot \mathbf{u} = W} a_u \alpha^{\mathbf{u}'} y^{u_1}$, and so $\operatorname{in}_{w_1}(g)(\alpha_1) = \sum a_u \alpha^{\mathbf{u}} = 0$.

Now $\operatorname{in}_{w_1}(g) = \overline{t^{-\operatorname{val}(\lambda)} \lambda} \operatorname{in}_w(y - b_1) \cdots \operatorname{in}_w(y - b_m)$ (see Exercises). Thus there is some j for which $\operatorname{in}_w(y - b_j)(\alpha_1) = 0$. For this j we must have $\operatorname{val}(b_j) = w_1$, as otherwise $\operatorname{in}_w(y - b_j)$ is a monomial, and so then $\operatorname{in}_w(y - b_j)(\alpha_1) \neq 0$ (since $\alpha_1 \neq 0$ because g has a nonzero constant term). Let $\beta_1 = b_j$. Then $g(\beta_1) = 0$ by construction. Thus if $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, we have $f(\beta) = 0$ and $\operatorname{val}(\beta) = w$, so β is the desired point. $\square$

The proof of Theorem 8.3 in the general (non-hypersurface) case proceeds by reduction to the hypersurface case, as we now sketch.

Sketch of proof of Theorem 8.3. The equality of the first two sets is immediate from Proposition 8.4, as $w \in \operatorname{trop}(X) = \cap_{f \in I} \operatorname{trop}(f)$ if and only if $\operatorname{in}_w(f)$ is not a monomial for all $f \in I$, which occurs if and only if $\operatorname{in}_w(I) \neq \langle 1 \rangle$.

The inclusion of the third set into the first is also an immediate corollary of Proposition 8.4. If $w = \mathrm{val}(y)$ for $y \in X$, then $f(y) = 0$ for all $f \in I$, so $w \in \mathrm{trop}(V(f))$ for all $f \in I$, so $w \in \mathrm{trop}(X)$.

We are left with showing that the second set is contained in the third set. The key idea is to project X to a hypersurface. We can (proof skipped) assume that X is irreducible of dimension d. We then claim (proof skipped) that for a generic choice of projection $\phi : T^n \to T^{d+1}$ the image $\phi(X)$ is a hypersurface in T^{d+1}. Applying the map $\mathrm{Hom}(K^*, -)$ to the projection $\phi : T^n \to T^{d+1}$ we get a map $\psi : \mathbb{Z}^n \to \mathbb{Z}^{d+1}$. We write ϕ for both the projection $T_K^n \to T_K^{d+1}$ and for the corresponding projection $T_{\Bbbk}^n \to T_{\Bbbk}^{d+1}$, and also for the maps of coordinate rings $\Bbbk[y_1^{\pm 1}, \ldots, y_{d+1}^{\pm 1}] \to \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. For a generic projection we have

$$\phi(\mathrm{in}_w(I))) = \mathrm{in}_{\psi(w)}(\phi(I)).$$

Thus if w lies in the second set of the Theorem, so $\mathrm{in}_w(I) \neq \langle 1 \rangle$, then $\phi(\mathrm{in}_w(I)) = \mathrm{in}_{\psi(w)}(\phi(I)) \neq \langle 1 \rangle \subseteq \Bbbk[y_1^{\pm 1}, \ldots, y_{d+1}^{\pm 1}]$. Applying Proposition 8.4 we see that $\psi(w) \in \mathrm{trop}(\phi(X))$, so there is $y \in \phi(X)$ with $\mathrm{val}(y) = \psi(w)$. Since $y \in \phi(X)$ there is $\tilde{y} \in X$ with $\phi(\tilde{y}) = y$, and $\psi(\mathrm{val}(\tilde{y})) = \psi(w)$. We claim (proof skipped) that we can choose $\tilde{y}$ with $\mathrm{val}(\tilde{y}) = w$, which shows that w lies in the third set of the theorem. □

Remark 8.5. Theorem 8.3 says that for a dense set of $w \in \mathrm{trop}(X)$ then there is $y \in X$ with $\mathrm{val}(y) = w$. Sam Payne has shown that the set

$$\{y \in X : \mathrm{val}(y) = w\}$$

is Zariski dense in X for all $w \in \mathrm{trop}(X)$. See [Pay07] for details.

We finish this lecture with examples of tropical varieties for which the classical variety is not a hypersurface.

Example: Let $X = V(x_1 + x_2 + x_3 + 1, x_2 + 2x_3 + 3\rangle \subset T^3$. Then

$$\begin{aligned} X &= V(x_1 - x_3 - 2, x_2 + 2x_3 + 3) \\ &= \{(2+s, -3-2s, s) : s \in K^* : t \neq -2, -3/2\}. \end{aligned}$$

Now

$$\mathrm{val}(2+, -3-2s, s) = \begin{cases} (0, 0, \mathrm{val}(s)) & \text{if } \mathrm{val}(s) > 0 \\ (\mathrm{val}(s), \mathrm{val}(s), \mathrm{val}(s)) & \text{if } \mathrm{val}(s) < 0 \\ (w, 0, 0) & \text{if } s = -2 + s', \mathrm{val}(s') = w > 0 \\ (0, w, 0) & \text{if } s = -3/2 + s', \mathrm{val}(s') = w > 0 \\ (0, 0, 0) & \text{if } \mathrm{val}(s) = 0, \overline{s} \neq -2, -3/2 \end{cases}$$

Thus $\mathrm{trop}(X)$ is the union of the rays through $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(-1,-1,-1)$ in $\mathbb{R}^3$.

Example: Let $I = \langle x_1 + x_2 + x_3 + x_4 + 1, x_2 + 2x_3 + 3x_4 + 4\rangle \subseteq \Bbbk[x_1^{\pm 1}, \ldots, x_4^{\pm 1}]$, and let $X = V(I) \subseteq T_K^4$. Then $\mathrm{trop}(X)$ is the two-dimensional fan in $\mathbb{R}^4$ with

$$\{(1,0,0,0), (0,1,0,0,), (0,0,1,0), (0,0,0,1), (-1,-1,-1,-1)\}$$

and two-dimensional cones spanned by any two of these. The intersection of $\mathrm{trop}(X)$ with the sphere $S^3 \subseteq \mathbb{R}^4$ is then a graph with five vertices and ten edges (the complete graph K_5).

It is hard to draw pictures of tropical varieties that do not lie in $\mathbb{R}^2$. For two-dimensional tropical varieties we will often resort to this trick of intersecting with the sphere and drawing the corresponding graph.

9. Exercises

(1) Let K be an algebraically closed field with a nontrivial valuation val : $K \to \mathbb{R} \cup \infty$. Show that im(val) is dense in $\mathbb{R}$.
(2) Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Show that $\text{in}_w(fg) = \text{in}_w(f)\,\text{in}_w(g)$.
(3) Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Show that if $g \in \text{in}_w(I)$ then $g = \text{in}_w(f)$ for some $f \in I$.
(4) Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $I = \langle f \rangle$. Show that $\text{trop}(V(f)) = \cap_{g \in I} \text{trop}(V(g))$. This can be rephrased as "hypersurfaces tropicalize to hypersurfaces".
(5) Let $f = tx_1^2 + x_1x_2 + tx_2^2 + x_0x_1 + x_0x_2 + t^4x_0^2 \in \mathbb{C}\{\{t\}\}[x_0, x_1, x_2]$. Compute the Gröbner complex of $I = \langle f \rangle$ (ie compute the polyhedra on which $\text{in}_w(f)$ is constant). (Part of this exercise is taking the common generalization of Gröbner bases in $\mathbb{C}[x_0, x_1, x_2]$ and those in $\mathbb{C}\{\{t\}\}[x_1^{\pm 1}, x_2^{\pm 1}]$). Use your answer to draw the tropical variety of $V(f) \subseteq \mathbb{C}\{\{t\}\}[x_1^{\pm 1}, x_2^{\pm 1}]$.
(6) Verify (as much as possible) the fundamental theorem of tropical geometry for $X = V(f)$ for the following polynomials $f \in \mathbb{C}\{\{t\}\}[x_1^{\pm 1}, x_2^{\pm 1}]$:
   (a) $f = 3x_1 + t^2x_2 + 2t$;
   (b) $f = tx_1^2 + x_1x_2 + tx_2^2 + x_1 + x_2 + t$;
   (c) $f = x_1^3 + x_2^3 + 1$.
(7) Let $S = K[x_1^{\pm 1}, \dots, x_4^{\pm 1}]$. Describe $\text{trop}(X)$ for the following subvarieties of T^4. Hint: Both are two-dimensional, so you could draw the graph of $\text{trop}(X) \cap S^3$.
   (a) $X = V(x_1 + x_2 + x_3 + x_4 + 1, x_2 + 2x_3 + 3x_4 + 4)$;
   (b) $X = V(x_1 + x_2 + x_3 + x_4 + 1, x_2 + x_3 + 2x_4 + 2)$.
(8) Let $\phi : T^2 \to T^4$ be given by

$$\phi(t_1, t_2) = (t_1, t_1t_2, t_1t_2^2, t_1t_2^3).$$

   Let $X = \text{im}(\phi)$. Compute $\text{trop}(X)$.

10. Lecture 8

The goal for today is to start describing the structure of the tropical variety.
Example: Let $f = x + y + 1 \in K[x^{\pm 1}, y^{\pm 1}]$. Then $V(f)$ is a line in T^2, and $\text{trop}(V(f))$ is the standard "tropical line" we have seen multiple times, as shown in Figure 20

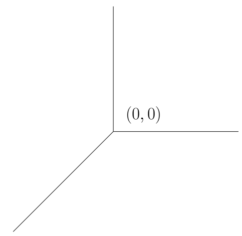

FIGURE 20.

Example: Let $f = x + y + z + 1 \in K[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. Then $V(f)$ is a surface in T^3. We have $w \in \operatorname{trop}(V(f))$ if and only if

$$\begin{aligned} & w_1 = w_2 \leq w_3, 0 \\ \text{or } & w_1 = w_3 \leq w_2, 0 \\ \text{or } & w_1 = 0 \leq w_2, w_3 \\ \text{or } & w_2 = w_3 \leq w_1, 0 \\ \text{or } & w_2 = 0 \leq w_1, w_3 \\ \text{or } & w_3 = 0 \leq w_1, w_2. \end{aligned}$$

This is a fan with rays

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \right\}.$$

The fan consists of all two-dimensional cones in $\mathbb{R}^3$ generated by any two of these rays. It intersects the sphere S^3 in the complete graph K_4.

Example: Let $\phi : T^d \to T^n$ be a subtorus embedded by

$$\phi : s = (s_1, \ldots, s_d) \mapsto (s^{a_1}, \ldots, s^{a_n}),$$

where $a_j \in \mathbb{Z}^d$ for $1 \leq i \leq n$, and $s^{a_j} = \prod_{i=1}^d s_i^{a_{ij}}$. We assume that the $d \times n$ matrix $A = (a_{ij})$ has rank d, so that ϕ is an embedding. Let $X = \operatorname{im}(\phi) \cong T^d \subset T^n$. Then

$$\begin{aligned} \operatorname{trop}(X) &= \text{closure of } \{\operatorname{val}(s^{a_1}), \ldots, \operatorname{val}(s^{a_n}) : s = (s_1, \ldots, s_d) \in T_K^d\} \\ &= \text{closure of } \{\mathbf{a}_1 \cdot \operatorname{val}(s), \ldots, \mathbf{a}_n \cdot \operatorname{val}(s) : s \in T_K^d\} \\ &= \text{closure of } \{A^T \operatorname{val}(s) : s \in T_K^d\} \\ &= \operatorname{im} A^T. \end{aligned}$$

So $\operatorname{trop}(X)$ is a linear space of dimension d.

Definition 10.1. The *Minkowski sum* of two subsets $A, B \subset \mathbb{R}^n$ is the set

$$A + B = \{a + b : a \in A, b \in B\}.$$

Definition 10.2. The affine span of a polyhedron $P \subseteq \mathbb{R}^n$ is

$$\operatorname{aff}(P) = v + \operatorname{span}(u - v : u \in P)$$

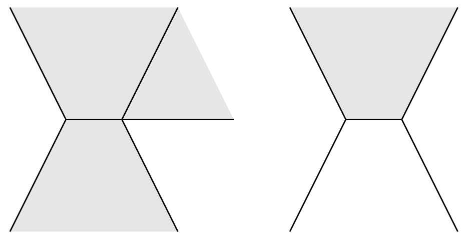

FIGURE 21. The complex on the left is pure, while the one on the right is not.

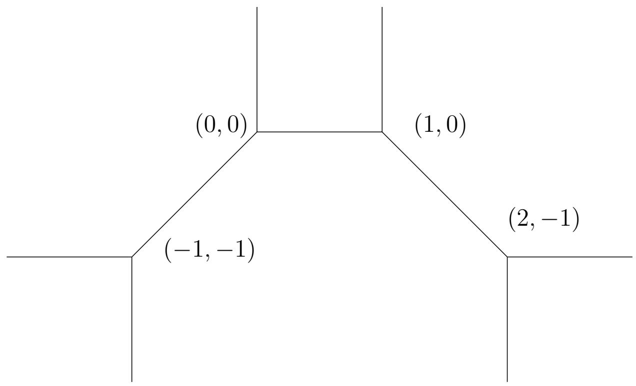

FIGURE 22.

where $v \in P$. Here the sum is an example of Minkowski addition. Note that this is independent of the choice of $v \in P$. The relative interior of P is the interior of P inside its affine span.

Definition 10.3. The dimension of a polyhedron P is the dimension of its affine span. A polyhedral complex Σ is *pure of dimension d* if all maximal polyhedra in Σ are d-dimensional.

Note: In each of the examples, $\mathrm{trop}(X)$ is a pure polyhedral complex and $\dim(X) = \dim(\mathrm{trop}(X))$. This is true in general.

Recall that the support of a polyhedral complex in $\mathbb{R}^n$ is the subset of $\mathbb{R}^n$ obtained by taking the union of all the polyhedra in the complex.

Theorem 10.4. *Let $X \subset T_K^n$ be an irreducible variety of dimension d defined by the prime ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. There is a polyhedral complex Σ that is pure of dimension d whose support is* $\mathrm{trop}(X)$.

The existence of the polyhedral complex we saw already in the discussion of the Gröbner complex last week. The new material here is that this complex is pure of dimension d.

To prove this we need some more Gröbner basics, which we will see first in an example.

Example: Let $f = tx^2y + x^2 + xy + t^2y + x + t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. Then $\mathrm{trop}(f) = \min(2x + y + 1, 2x, x + y, y + 2, x, 1)$, and $\mathrm{trop}(V(f))$ is shown in Figure 22.

For $w = (1,0)$, we have $\mathrm{in}_w(f) = xy + x + 1 \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Then $\mathrm{trop}(V(\mathrm{in}_w(f))) = \{v \in \mathbb{R}^2 : \langle \mathrm{in}_v(\mathrm{in}_w(f))\rangle \neq \langle 1 \rangle\}$ is shown in Figure 23.

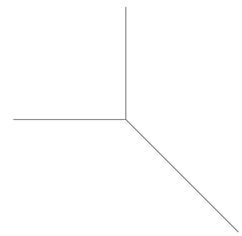

Figure 23.

For $w = (0,0)$ we have $\mathrm{in}_w(f) = x^2 + xy + x$, and $\mathrm{trop}(\mathrm{in}_w(f)) = \{v \in \mathbb{R}^2 : \langle \mathrm{in}_v(\mathrm{in}_w(f))\rangle \neq \langle 1 \rangle\}$ is shown in Figure 24.

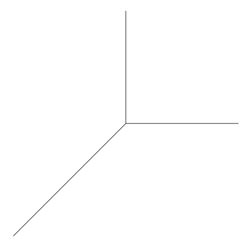

Figure 24.

If $w = (1/2, 0)$, then $\mathrm{in}_w(f) = x + xy$, and $\mathrm{trop}(V(\mathrm{in}_w(f)))$ is the x-axis $\{(x,y) \in \mathbb{R}^2 : y = 0\}$.

If $w = (1,1)$, then $\mathrm{in}_w(f) = x + 1$, and $\mathrm{trop}(V(\mathrm{in}_w(f)))$ is the y-axis $\{(x,y) \in \mathbb{R}^2 : x = 0\}$.

Note that in all cases the set $\mathrm{trop}(V(\mathrm{in}_w(f)))$ looks like the piece of $\mathrm{trop}(V(f))$ "near" the polyhedron containing w. More formally, it is the *star*, which we now define, of the polyhedron containing w in the polyhedral complex $\mathrm{trop}(V(f))$.

Definition 10.5. Let Σ be a polyhedral complex, and let $\sigma \in \Sigma$ be a polyhedron. The *star* $\mathrm{star}_\Sigma(\sigma)$ of $\sigma \in \Sigma$ is a fan in $\mathbb{R}^n$ whose cones are indexed by those $\tau \in \Sigma$ for which σ is a face of τ. Fix $w \in \sigma$. Then the cone indexed by τ is the Minkowski sum

$$\bar{\tau} = \{v \in \mathbb{R}^n : \exists \epsilon > 0 \text{ with } w + \epsilon v \in \tau\} + \mathrm{aff}(\sigma) - w.$$

Example: For the polyhedral complex Σ shown on the left of Figure 25, the affine span of the vertex σ_1 is just the vertex itself. The star is the standard tropical line shown on the right. For σ_2 the affine span is the entire y-axis, and this is also the star.

Lemma 10.6. *Let Σ be a polyhedral complex in $\mathbb{R}^n$, and $\sigma \in \Sigma$. Fix w in the relative interior of σ. Then*

$$\mathrm{star}_\Sigma(\sigma) = \{v \in \mathbb{R}^n : w + \epsilon v \in \Sigma \text{ for sufficiently small } \epsilon > 0\}.$$

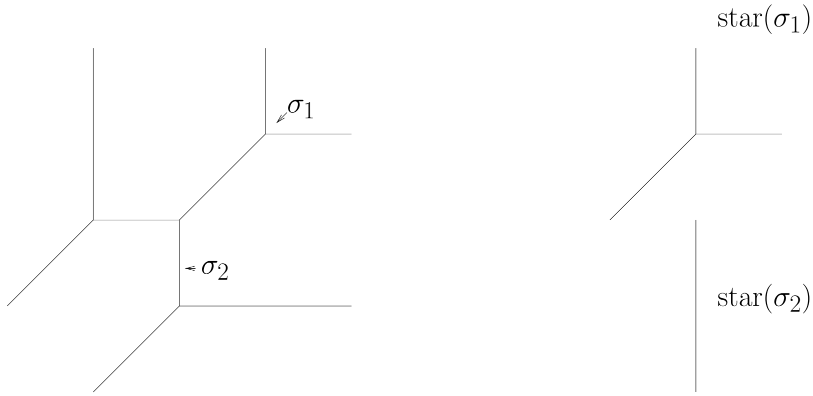

FIGURE 25.

Definition 10.7. A subspace $V \subseteq \mathbb{R}^n$ is the *lineality space* of a polyhedron $P \subseteq \mathbb{R}^n$ if

$$x \in P \text{ implies } x + v \in P \text{ for all } v \in V.$$

If V is the lineality space of a polyhedron P then we often consider P/V in $\mathbb{R}^n/V$ for ease of visualization.

Note: The affine span aff(σ) of a polyhedron $\sigma \in \Sigma$ lies in the lineality space of every cone in the fan star(σ).

Lemma 10.8. *Let $I \subseteq K[x_1^{\pm 1} \dots, x_n^{\pm 1}]$, and fix $w, v \in \mathbb{R}^n$. Then there is $\epsilon > 0$ such that*

$$\mathrm{in}_v(\mathrm{in}_w(I)) = \mathrm{in}_{w+\epsilon v}(I).$$

Proof. We first note that it suffices to check that for all $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ there is $\epsilon > 0$ such that

$$\mathrm{in}_v(\mathrm{in}_w(f)) = \mathrm{in}_{w+\epsilon' v}(f)$$

for all $\epsilon' < \epsilon$.

To see this, note that $\mathrm{in}_v(\mathrm{in}_w(I))$ is finitely generated by $g_1, \dots, g_s \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and each generator g_i is of the form $\mathrm{in}_v(\mathrm{in}_w(f_i))$ for some $f_i \in I$, so we can choose ϵ to be the minimum of the ϵ_i corresponding to these generating f_i. Then $g_i = \mathrm{in}_v(\mathrm{in}_w(f_i)) = \mathrm{in}_{w+\epsilon v}(f_i)$, so $\mathrm{in}_v(\mathrm{in}_w(I)) \subseteq \mathrm{in}_{w+\epsilon v}(I)$. Equality follows from the fact that we cannot have a proper inclusion of initial ideals.

We now prove the claim lemma for an individual polynomial. Let $f = \sum_{u \in \mathbb{Z}^n} c_u x^u$. Then

$$\mathrm{in}_w(f) = \sum_{u \in \mathbb{Z}^n} \overline{c_u t^{w \cdot u - W}} x^u,$$

where $W = \min(\mathrm{val}(c_u) + w \cdot u : c_u \neq 0) = \mathrm{trop}(f)(w)$. Let $W' = \min(v \cdot u : \mathrm{val}(c_u) + w \cdot u = W)$. Then

$$\mathrm{in}_v(\mathrm{in}_w(f)) = \sum_{v \cdot u = W'} \overline{c_u t^{w \cdot u - W}} x^u.$$

Let $\delta = \min(\mathrm{val}(c_u) + w \cdot u - W : \mathrm{val}(c_u) + w \cdot u > W)$, and let $M = \max(v \cdot u : c_u \neq 0)$. Set $\epsilon = \delta / 2M$, and $W'' = \min(\mathrm{val}(c_u) + (w + \epsilon v) \cdot u$. Then by construction we have

$$W'' = W + \epsilon W'$$

and

$$\{u : \mathrm{val}(c_u) + (w + \epsilon v) \cdot u = W''\} = \{u : \mathrm{val}(c_u) + w \cdot u = W, v \cdot u = W'\}.$$

Thus $\mathrm{in}_{w+\epsilon v}(f) = \mathrm{in}_v(\mathrm{in}_w(f))$. □

Recall that we say that $v \in \mathbb{R}^n$ is generic for I if $\mathrm{in}_v(I)$ is a monomial ideal. The following corollary allows us to compute Gröbner bases with respect to nongeneric weight vectors using standard computer algebra packages.

Corollary 10.9. *Let $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and $w \in \mathbb{R}^n$. Choose a vector $v \in \mathbb{R}^n$ that is generic for $\mathrm{in}_w(I)$. Then a Gröbner basis $\mathcal{G}$ for I with respect to $w + \epsilon v$ for sufficiently small ϵ is a Gröbner basis for I with respect to w, and $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g) : g \in \mathcal{G}\rangle$.*

Proof. Fix $\epsilon > 0$ such that $\mathrm{in}_{w+\epsilon v}(I) = \mathrm{in}_v(\mathrm{in}_w(I))$, the existence of which is guaranteed by Lemma 10.8. Let $\mathcal{G} = \{g_1, \dots, g_r\}$ be a Gröbner basis for I with respect to $\mathrm{in}_{w+\epsilon v}$. Thus $\mathrm{in}_{w+\epsilon v}(I) = \langle \mathrm{in}_{w+\epsilon v}(g_1), \dots, \mathrm{in}_{w+\epsilon v}(g_r)\rangle$. The choice of ϵ was made to guarantee that $\mathrm{in}_{w+\epsilon v}(g_i) = \mathrm{in}_v(\mathrm{in}_v(g_i))$ for all i, so $\mathrm{in}_{w+\epsilon v}(I) = \langle \mathrm{in}_v(\mathrm{in}_w(g_1)), \dots, \mathrm{in}_v(\mathrm{in}_w(g_r))\rangle = \mathrm{in}_v(\mathrm{in}_w(I))$, so $\{\mathrm{in}_w(g_1), \dots, \mathrm{in}_(g_r)\}$ is a Gröbner basis for $\mathrm{in}_w(I)$ with respect to v, and thus $\mathrm{in}_w(I) = \langle \mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_r)\rangle$. □

Corollary 10.10. *Let $X \subset T_K^n$, with $X = V(I)$ for $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and let Σ be a polyhedral complex whose support is $\mathrm{trop}(X) \subset \mathbb{R}^n$. Fix $w \in \mathrm{trop}(X)$, and let σ be the polyhedron of Σ containing w in its relative interior. Then*

$$\mathrm{trop}(V(\mathrm{in}_w(I))) = \mathrm{star}_\Sigma(\sigma).$$

Proof. We have

$$\begin{aligned}
\mathrm{trop}(V(\mathrm{in}_w(I))) &= \{v \in \mathbb{R}^n : \mathrm{in}_v(\mathrm{in}_w(I)) \neq \langle 1\rangle\} \\
&= \{v \in \mathbb{R}^n : \mathrm{in}_{w+\epsilon v}(I) \neq \langle 1\rangle \text{ for sufficiently small } \epsilon > 0\} \\
&= \{v \in \mathbb{R}^n : w + \epsilon v \in \mathrm{trop}(X) \text{ for sufficiently small } \epsilon > 0\} \\
&= \mathrm{star}_\Sigma(\sigma),
\end{aligned}$$

where the last equality is by Lemma 10.6. □

To prove Theorem 10.4 we need the following Lemma.

Lemma 10.11. *Let $Y \subseteq T_{\Bbbk}^n$ be equidimensional of dimension d (all irreducible components have the same dimension). Suppose that $\mathrm{trop}(Y)$ is a linear subspace of $\mathbb{R}^n$. Then there is a d-dimensional subtorus $T \subset T_{\Bbbk}^n$ such that $V(I)$ consists of finitely many T-orbits.*

Proof. For a proof, see Lemma 9.9 of [Stu02] □

Proof of Theorem 10.4. Let Σ be a polyhedral complex with support $\mathrm{trop}(X)$, and let σ be maximal polyhedron in Σ (so σ is not a proper face of any polyhedron in Σ). We need to show that $\dim(\sigma) = d$. Fix w in the relative interior of σ. By Corollary 10.10 we have $\mathrm{trop}(\mathrm{in}_w(I)) = \mathrm{star}_\Sigma(\sigma)$. Since σ is maximal, we have that $\mathrm{star}_\Sigma(\sigma) = \mathrm{aff}(\sigma)$ is a $\dim(\sigma)$-dimensional linear subspace. Since I is prime, it follows from a resultof Kalkbrener and Sturmfels [KS95] that $V(\mathrm{in}_w(I))$ is equidimensional,

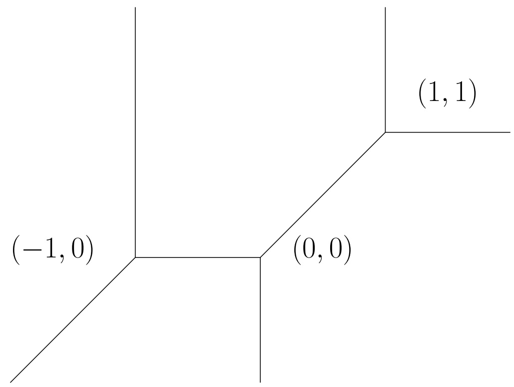

FIGURE 26.

so all irreducible components have the same dimension. By Lemma 10.11 it follows that there is a subtorus $T \subset T_K^n$ of dimension $\dim(\sigma)$ for which $V(\mathrm{in}_w(I))$ is the union of finitely many T-orbits. Since $\dim(V(\mathrm{in}_w(I))) = \dim(I) = d$ (Exercise!), it follows that $\dim(\sigma) = d$. $\square$

11. Lecture 9

In this lecture we discuss another property of tropical varieties: that they are weighted balanced polyhedral complexes.

Example: Consider the vertex $(0,0)$ of $\mathrm{trop}(V(x+y+1))$. There are three rays leaving $(0,0)$: $\mathrm{pos}((1,0))$, $\mathrm{pos}((0,1))$, $\mathrm{pos}((-1,-1))$. Note that we have

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example: Let $X = V(tx^2 + x + y + xy + t) \subset T_K^2$ for $K = \mathbb{C}\{\{t\}\}$. Then $\mathrm{trop}(X)$ is shown in Figure 26.

Then the star of the vertex $(1,1)$ has rays spanned by $(1,0)$, $(0,1)$, and $(-1-1)$, which add to $(0,0)$. This is also the star of the vertex $(-1,0)$. At the vertex $(0,0)$ the star has rays $(1,1)$, $(-1,0)$, and $(0,-1)$, which add to $(0,0)$.

Example: Let $X = V(x^2 + xy + ty + 1) \subset T_K^2$ for $K = \mathbb{C}\{\{t\}\}$. Then $\mathrm{trop}(X)$ is shown in Figure 27.

The star of the vertex $(1,-1)$ has rays spanned by $(1,0)$, $(0,-1)$, and $(-1,1)$, which add to zero. For the vertex $(0,0)$ the star has rays $(1,-1)$, $(-1,-1)$, and $(0,1)$. In this case

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

However,

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2}$$

We will define a notion of multiplicity on the top-dimensional polyhedra in $\mathrm{trop}(X)$ so that the corresponding sum is the zero vector. We review some commutative

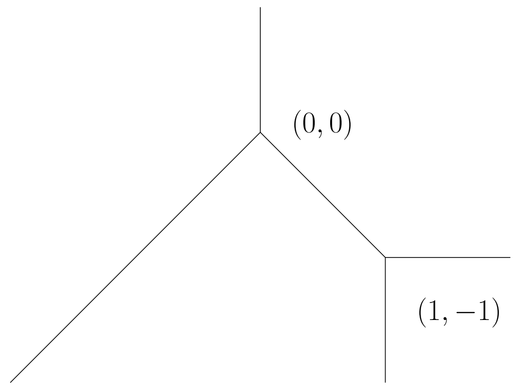

FIGURE 27.

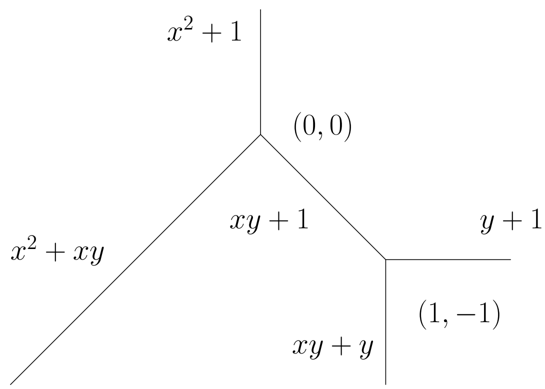

FIGURE 28.

algebra to give the precise definition, and then give the intuitive definition as given in class.

Definition 11.1. Let $S = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The localization of S at a prime ideal P is the ring with elements $\{f/g : f, g \in S, g \notin P\}$. Given an ideal $I \subset S$, the *multiplicity* of P in I is $\mathrm{mult}(P, S/I) = \dim_K(S_P/S_P I)$. If I is radical, this is one if $V(P)$ is an irreducible component of $V(I)$, and zero otherwise.

Definition 11.2. Let $X = V(I)$ be an irreducible variety of dimension d for $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then $\mathrm{trop}(X)$ is the support of a pure d-dimensional polyhedral complex Σ. Let σ be a d-dimension polyhedron in Σ, and fix w in the relative interior of σ. The *multiplicity* of the polyhedron σ is the sum

$$m_\sigma = \sum_{P \subset S} \mathrm{mult}(P, S/(\mathrm{in}_w(I))$$

where the sum is over all prime ideals P of S. If $\mathrm{in}_w(I)$ is radical, then m_σ is the number of irreducible components of $V(\mathrm{in}_w(I))$.

Example: When $f = x^2 + xy + ty + 1$, the initial terms of f corresponding to each one is shown in Figure 28.

Then $V(y+1)$ is irreducible, so this cone has multiplicity one. We have $V(xy+y) = V(x+1)$ is also irreducible, so this cone gets multiplicity one. The varieties $V(xy+1)$

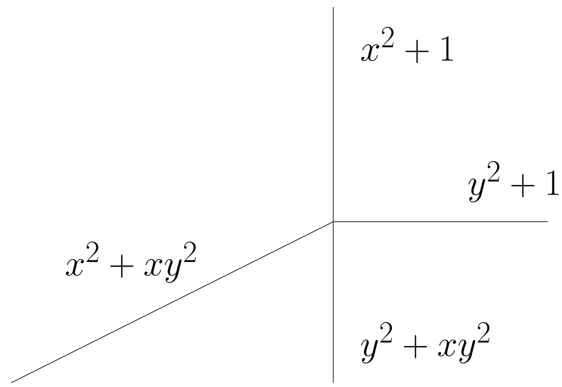

FIGURE 29.

and $V(x^2+xy) = V(x+y)$ are also irreducible, but $V(x^2+1) = V(x+i) \cup V(x-i)$, so the multiplicity of this last cone is two. This justifies Equation 2.

Definition 11.3. A weighted polyhedral complex is a polyhedral complex Σ with a positive integer on each top-dimensional polyhedron in Σ.

Let Σ be a pure d-dimensional weighted polyhedral complex, and let σ be a $(d-1)$-dimensional polyhedron in Σ. Let V be the subspace $\text{aff}(\sigma) - w$ for w in the relative interior of σ. Then V lies in the lineality space of $\text{star}_\Sigma(\sigma)$, so $\text{star}_\Sigma(\sigma)/V$ is a fan in $\mathbb{R}^n/V$. Let $\mathbf{u}_\tau \in \mathbb{R}^n/V$ be the image of the cone $\bar{\tau} \in \text{star}_\Sigma(\sigma)$ in $\mathbb{R}^n/V$ for a d-dimensional polyhedron $\tau \in \Sigma$ with σ a face of τ. The polyhedral complex Σ is *balanced at* σ if

$$\sum_\tau w_\tau \mathbf{u}_\tau = 0,$$

where w_τ is the multiplicity of the polyhedron τ.

The complex Σ is *balanced* if it is balanced at every $(d-1)$-dimensional polyhedron.

Example: Let $X = V(x+y+z+1) \subset T_K^3$ for $K = \mathbb{C}\{\{t\}\}$. Then $\text{trop}(X)$ is a two-dimensional fan with rays $\{(1,0,0),(0,1,0),(0,0,1),(-1,-1,-1)\}$ and cones spanned by any two of these. For σ the ray through $(1,0,0)$ the fan $\text{star}_{\text{trop}(X)}(\sigma)$ is the one-dimensional fan in $\mathbb{R}^3/\text{span}((1,0,0))$ with rays the images of $(0,1,0)$, $(0,0,1)$, and $(-1,-1,-1)$. If we identify $\mathbb{R}^3/\text{span}((1,0,0))$ with $\mathbb{R}^2$ by projecting onto the last two coordinates of $\mathbb{R}^3$, these are the vectors $(1,0)$, $(0,1)$, and $(-1,-1)$, which sum to zero, so $\text{trop}(X)$ is balanced at σ.

Theorem 11.4. *Let $X \subset T_K^n$ be an irreducible variety. Then* $\text{trop}(X)$ *together with the multiplicity is a balanced weighted polyhedral complex.*

Example: Let $f = x^2+y^2+xy^2+1$. Then $\text{trop}(V(f))$ is shown in Figure 29, together with the corresponding initial terms. Then

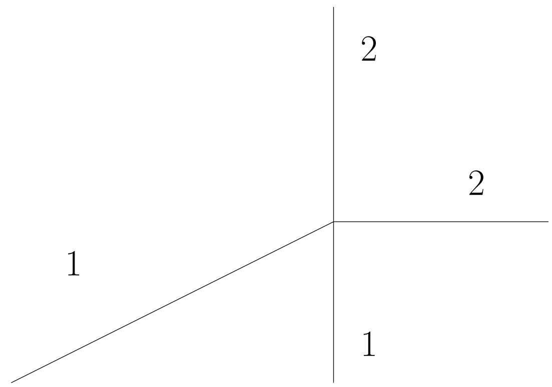

FIGURE 30.

$$\begin{aligned} V(x^2+1) &= V(x+i) \cup V(x-i) \\ V(y^2+1) &= V(y+i) \cup V(y-i) \\ V(y^2+xy^2) &= V(x+1) \\ V(x^2+xy^2) &= V(x+y^2) \end{aligned}$$

Thus the multiplicity on the first two cones is two, and on the second two is one. This is shown in Figure 30. Since

$$2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$\mathrm{trop}(X)$ is balanced.

12. LECTURE 10

In this lecture we will consider a large class of examples whose tropical varieties can be completely described. These are the $X \subset T^n$ whose equations are all linear.

Example 12.1. Let $X = V(3x_1 + 5x_2 - 7x_3) \subset T^3_K$ where $K = \mathbb{C}\{\{t\}\}$. Then $\mathrm{trop}(X) = \{(w_1, w_1, w_2) \in \mathbb{R}^3 : w_1 = w_2 \leq w_3 \text{ or } w_1 = w_3 \leq w_2 \text{ or } w_2 = w_3 \leq w_1\}$ This is the union of the three cones $\mathrm{span}((1,1,1)) + \mathrm{pos}((1,0,0))$, $\mathrm{span}((1,1,1)) + \mathrm{pos}((0,1,0))$, and $\mathrm{span}((1,1,1)) + \mathrm{pos}((0,0,1))$. Here $\mathrm{span}((1,1,1)) + \mathrm{pos}((1,0,0))$ is the set $\{a(1,1,1) + b(1,0,0) : a, b \in \mathbb{R}, b \geq 0\}$. Note $\mathrm{trop}(X)$ is a fan, rather than just a polyhedral complex. This will be the case whenever the coefficients of the polynomials defining the ideal of X live in a subfield of K (such as $\mathbb{C} \subset \mathbb{C}\{\{t\}\}$) where all elements have valuation zero. Note also that $(1,1,1)$ lies in the lineality space of all cones in $\mathrm{trop}(X)$. This will be the case whenever the equations defining the ideal of X are all homogeneous.

We assume here that there is an inclusion $i \colon \Bbbk \to K$ of the residue field $\Bbbk = R/\mathfrak{m}$ into our field K. This is true when $K = \mathbb{C}\{\{t\}\}$. Let $S = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let $I = \langle f_1, \ldots, f_r \rangle \subset S$ be an ideal in S minimally generated by linear forms $f_1, \ldots, f_r$. We now describe the tropical variety of $X = V(I) \subset T$.

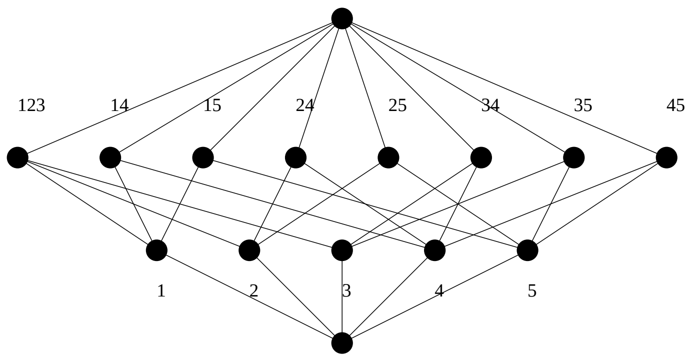

FIGURE 31. The lattice of flats for the linear space of Example 12.2

Example 12.2. Let $I = \langle x_1+x_2+x_3+x_4+x_5, x_1+2x_2+3x_3\rangle \subset \Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, x_5^{\pm 1}]$. Then $V(I)$ is a two-dimensional subvariety of $T \cong (\Bbbk^*)^5$.

Let $f_i = a_{i1}x_1 + \dots a_{in}x_n$ for $1 \leq i \leq r$. Let A be the $r \times n$ matrix with entries $a_{ij} \in \Bbbk$, and let B be a $(n-r) \times n$ matrix whose rows are a basis for $\ker(A)$. Thus $V(I)$ is equal to the intersection of the row space of B with the torus T. Let $\mathcal{B} = \{\mathbf{b}_0, \dots, \mathbf{b}_n\} \subset \Bbbk^{n-r}$ be the columns of the matrix B. While $\mathcal{B}$ depends on the choice of the matrix B, it is determined up to the action of $\mathrm{GL}(n-r, \Bbbk)$.

The lattice of flats $\mathcal{L}(B)$ of the linear space row(B) has elements the subspaces (flats) of $\Bbbk^{n-r}$ spanned by subsets of $\mathcal{B}$. We make $\mathcal{L}(B)$ into a poset (partially ordered set) by setting $S_1 \preceq S_2$ if $S_1 \subseteq S_2$ for two subspaces S_1, S_2 of $\Bbbk^{n-r}$ spanned by subsets of $\mathcal{B}$. The poset $\mathcal{L}(B)$ is actually a lattice of rank $n-r$. This means that every maximal chain in $\mathcal{L}(B)$ has length $n-r$. See, for example, [Sta97, Chapter 3] for more on lattices.

Example 12.3. We continue Example 12.2. In this case the matrices A and B are

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

We thus have $\mathbf{b}_1 = (-2,-2,-1)$, $\mathbf{b}_2 = (1,1,-1)$, $\mathbf{b}_3 = (0,0,1)$, $\mathbf{b}_4 = (1,0,1)$, and $\mathbf{b}_5 = (0,1,0)$. There are fifteen subspaces of $\Bbbk^3$ spanned by subsets of $\mathcal{B} = \{(-2,-2,-1), (1,1,-1), (0,0,1), (1,0,1), (0,1,0)\}$. These are

$$\begin{aligned} &\{0\} \cup \{\mathrm{span}(\mathbf{b}_i) : 1 \leq i \leq 5\} \\ \cup&\{\mathrm{span}(\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3), \mathrm{span}(\mathbf{b}_1,\mathbf{b}_4), \mathrm{span}(\mathbf{b}_1,\mathbf{b}_5), \mathrm{span}(\mathbf{b}_2,\mathbf{b}_4), \\ &\mathrm{span}(\mathbf{b}_2,\mathbf{b}_5), \mathrm{span}(\mathbf{b}_3,\mathbf{b}_4), \mathrm{span}(\mathbf{b}_3,\mathbf{b}_5), \mathrm{span}(\mathbf{b}_4,\mathbf{b}_5)\} \cup \Bbbk^3. \end{aligned}$$

This gives the lattice shown in Figure 31.

A *simplicial complex* on a set S is a collection of subsets of S ("simplices") that is closed under taking subsets. For example, if $S = \{1,2,3,4,5\}$, then one simplicial complex is

$$\Delta = \{\{1,2,3\}, \{2,3,4\}, \{3,4,5\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{3,5\}, \{4,5\}, \\ \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{\emptyset\}\}.$$

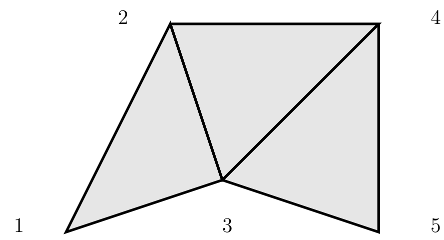

FIGURE 32.

We draw simplical complexes as collections of triangles (or higher-dimensional simplices) in $\mathbb{R}^n$. For example, the above simplicial complex is illustrated in Figure 32.

There is a simplicial complex, called the *order complex*, associated to any poset. A *chain* in a poset is a set $x_0 \prec x_1 \prec \cdots \prec x_s$. We say that this chain has length s. The order complex has vertices the elements of the poset, and simplices all proper chains, which are totally ordered subsets of the poset not using the bottom or top elements (0 or $\mathbb{R}^n$ in our case). The order complex of $\mathcal{L}(B)$ is pure of dimension $n-r-2$. In the case of our poset there is a nice geometric realization of this simplicial complex, which we now describe.

Definition 12.4. Let $\mathbf{e}_i$ be the ith standard basis vector on $\mathbb{R}^n$. Given a subset $\sigma \subset \{1, \dots, n\}$ we set $\mathbf{e}_\sigma = \sum_{i \in \sigma} \mathbf{e}_i$. If V is a subspace of $\mathbb{R}^{n-r}$ spanned by some of the $\mathbf{b}_i$, set $\sigma(V)$ to be $\{i : \mathbf{b}_i \in V\}$. Let $\Delta(\mathcal{B})$ be the fan whose cones are

$$\mathrm{pos}(\mathbf{e}_{\sigma(V_i)} : 1 \leq i \leq s) + \mathrm{span}(\mathbf{1}),$$

where $V_1 \prec V_2 \prec \cdots \prec V_s$ is a chain in $\mathcal{L}(\mathcal{B})$, and $\mathbf{1}$ is the all-ones vector in $\mathbb{R}^n$.

Example 12.5. We continue Example 12.2. The fan $\Delta(\mathcal{B}) \subset \mathbb{R}^5$ has lineality space the span of $\mathbf{1} = (1,1,1,1,1)$, meaning that every cone contains the span of this ray, so we describe the quotient fan in $\mathbb{R}^5/\mathbf{1}$. This has 13 rays, corresponding to the five rays spanned by the $\mathbf{b}_i$ and the eight planes spanned by them. There is a two-dimensional cone for every inclusion of a ray into a plane, of which there are 17 in total.

The point of this construction is that the tropical variety $\mathrm{trop}(V(I))$ is equal to $\Delta(\mathcal{B})$.

Theorem 12.6. *Let I be a linear ideal in S. The tropical variety of $X = V(I) \subset T$ is equal to $\Delta(\mathcal{B})$.*

Proof. We first show that $\mathrm{trop}(X) \subseteq \Delta(\mathcal{B})$. Suppose $w \notin \Delta(\mathcal{B})$. Let $W^j = \{\mathbf{b}_i : w_i \geq j\}$. Note that there are only finitely many subspaces $\mathrm{span}(W^j)$ as j varies, and these form a chain in the lattice of flats $\mathcal{L}(\mathcal{B})$. Let $l = \min\{j : \text{ there exists } \mathbf{b}_i \in \mathrm{span}(W^j) \setminus W^j\}$. If no such l existed, then w would live in the cone of $\Delta(\mathcal{B})$ defined by the chain $\{\mathrm{span}(W^j)\} \subset \mathcal{L}(\mathcal{B})$. Let $F = \mathrm{span}(W^l l)$. Pick $\mathbf{b}_k \in F \setminus W^l$. Then $w_k < l$ by the definition of W^l. Since $\{\mathbf{b}_i : i \in W^l\}$ spans F, we can write $\mathbf{b}_k = \sum_{i:\mathbf{b}_i \in W^l} \lambda_i \mathbf{b}_i$ for $\lambda_i \in \Bbbk$. This means that $\mathbf{e}_k - \sum \lambda_i \mathbf{e}_i \in \ker(B) = \mathrm{row}(A)$. Thus $f = x_k - \sum_{i \in W^l} \lambda_i x_i \in I$. Now $\mathrm{in}_w(f) = x_k$, so $\mathrm{in}_w(I) = \langle 1 \rangle$, and so $w \notin \mathrm{trop}(X)$.

We next show that $\Delta(\mathcal{B}) \subseteq \mathrm{trop}(X)$ by exhibiting for each $w \in \Delta(\mathcal{B})$ an element $y \in X$ with $\mathrm{val}(y) = w$. Given $w \in \Delta(\mathcal{B})$, let $V_1 \subset V_2 \subset \cdots \subset V_{n-r} = \Bbbk^{n-r}$ be the chain of flats labelling a maximal cone of $\Delta(\mathcal{B})$ containing w, so $\dim(V_i) = i$. Pick $\mathbf{b}_{i_1} \in V_1$, and $\mathbf{b}_{i_j} \in V_j \setminus V_{j-1}$ for $2 \leq j \leq n-r$. Note that

$$w_{i_j} \geq w_{i_{j+1}}. \tag{3}$$

After renumbering if necessary we may assume that $i_j = j$, and that the matrix B has been chosen so that the first $(n-r) \times (n-r)$ square submatrix is the identity, which is possible as the $\mathbf{b}_{i_j}$ are linearly independent by construction. This implies (exercise!) that the last $r \times r$ submatrix of A must be invertible, so we may assume that it is the identity matrix (since performing row operations on A corresponds to choosing a different generating set for I). We then have

$$A = \left(\; A' \; \middle| \; I_r \; \right), \quad B = \left(\; I_{n-r} \; \middle| \; -A'^T \; \right).$$

Set $y = (t^{w_1}, \ldots, t^{w_{n-r}})B$. Explicitly, $y_i = t^{w_i}$ for $1 \leq i \leq n-r$. For $n-r+1 \leq i \leq n$, $y_i = \sum_{j=1}^{n-r} -a_{j(i-n+r)} t^{w_j}$. Then $Ay = 0$ by construction, so $y \in X$. The valuation $\mathrm{val}(y_i) = w_i$ for $1 \leq i \leq n-r$ by construction. For $n-r+1 \leq i \leq n$ we have $\mathrm{val}(y_i) = \min\{w_j : a_{(i-n+r)j} \neq 0, 1 \leq j \leq n-r\} = w_s$ for $s = \max\{j : a_{(i-n+r)j} \neq 0, 1 \leq j \leq n-r\}$, by Equation 3. Now if $\mathrm{val}(y_i) = w_j$, then $\mathbf{b}_i \in V_j \setminus V_{j-1}$, so by the choice of w we have $w_i = w_j$, and thus $\mathrm{val}(y_i) = w_i$. So y is the desired element of X with $\mathrm{val}(y) = w$, and so $\Delta(\mathcal{B}) \subseteq \mathrm{trop}(X)$. □

Remark 12.7. To visualize these tropical varieties we can do two tricks. Firstly, since we are assuming that the ideals are homogeneous, we can quotient out by the lineality space span 1 when drawing pictures. Dehomogenizing by setting one variable equal to one has the same effect. Secondly, since the tropical varieties are fans, we can intersect the fan with the sphere $S^{n-1} \subset \mathbb{R}^n$ to get an (abstract) polyhedral complex of dimension $\dim(X) - 2$.

13. Lecture 11

The goal of this lecture is to describe how to compute tropical varieties in general.

We begin with an example of the algorithm outlined for linear varieties last lecture.

Example: Let $X = V(x_0 - x_1 + x_3, x_0 - x_2 + x_4, x_1 - x_2 + x_5) \subset T^6$. Then

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}.$$

If we add the additional first row of $(1, 1, 1, 0, 0, 0)$ to B, which lies in the row space of B, then the columns of B are the positive roots of the root lattice of A_3. We have

$$\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

The lattice of flats $\mathcal{L}(\mathcal{B})$ is then illustrated in Figure 33. Here 013 means $\mathrm{span}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_3)$. The order complex then has 18 cones, which are indexed by the edges in Figure 33 connecting two proper nontrivial flats. The cones in $\mathrm{trop}(X) \subset \mathbb{R}^6$ are then three-dimensional, as expected. When we quotient by the lineality space $\mathrm{span}(\mathbf{1})$, we get a two-dimensional fan in $\mathbb{R}^5 \cong \mathbb{R}^6/\mathrm{span}(\mathbf{1})$. Intersecting this with the sphere $S^4 \subset \mathbb{R}^5$ gives a the graph that is shown in Figure 34. This is a refinement (subdividing the

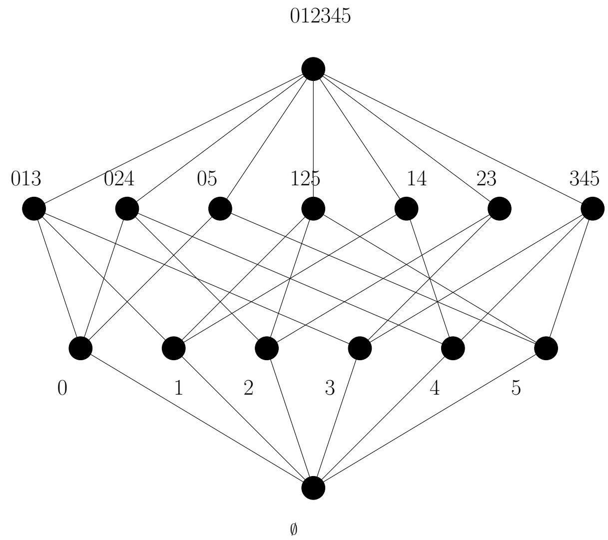

FIGURE 33.

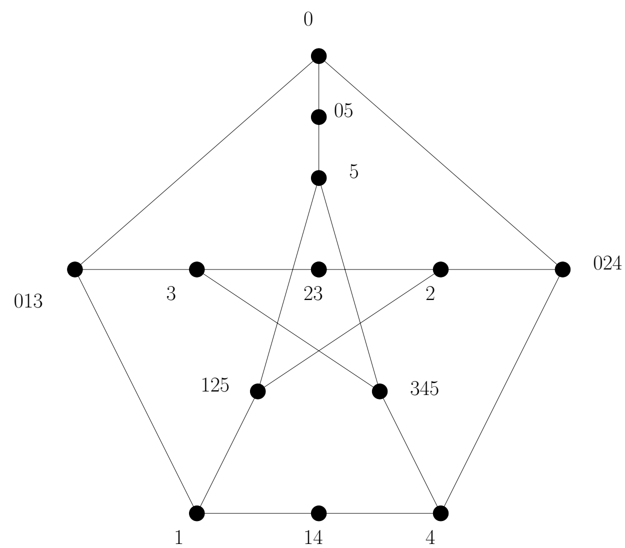

FIGURE 34.

edges 05, 14, and 23) of a well-studied graph called the *Peterson graph*.

We now describe how to use a computer to compute tropical varieties. That first raises the following question:

Question: What sort of examples can we expect to do on a computer?

We cannot enter a general Puiseux series into a computer, as it cannot be described by a finite amount of information. Similarly, we cannot describe most real numbers in finite space. This suggests that the best field we can hope to work with is the algebraic closure $\overline{\mathbb{Q}(t)}$ of the ring of rational functions in t with coefficients in $\mathbb{Q}$. Note that every example we have seen so far as lived in this field!

Any ideal with coefficients in $\overline{\mathbb{Q}(t)}$ actually has generators with coefficients in some finite extension L of $\mathbb{Q}(t)$, since the ideal is finitely generated, and a finite generating

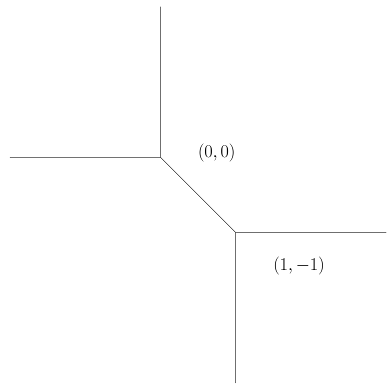

Figure 35.

set has only a finite number of coefficients. We can write $L \cong \mathbb{Q}(t)[a_1, \ldots, a_s]/J$, where J is an ideal. As a point of comparison, recall that $\mathbb{C} = \mathbb{R}[x]/\langle x^2+1\rangle$. We'll now discuss how to reduce this case to computing in $\mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

If $I \subset \mathbb{Q}(t)[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then let $J = I \cap \mathbb{Q}[t^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Fix $w \in \mathbb{R}^n$. Then $w \in \operatorname{trop}(V(I)) \subset \mathbb{R}^n$, where $V(I) \subset T^n$, if and only if $(1, w) \in \operatorname{trop}(V(J)) \subset \mathbb{R}^{n+1}$, where $V(J) \subset T^{n+1}$. This is the case because if $p(t)$ is a polynomial in t, then $\operatorname{val}(p(t))$ is the smallest exponent of t occuring in p.

If $I \subset L[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where $L = \mathbb{Q}(t)[a_1, \ldots, a_s]/J$ is a finite extension of $\mathbb{Q}(t)$, then let $v_i = \operatorname{val}(a_i)$. Consider the map

$$\phi\colon \mathbb{Q}[a_1^{\pm 1}, \ldots, a_s^{\pm 1}, t^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to L[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Fix $w \in \mathbb{R}^n$. Then $w \in \operatorname{trop}(V(I))$ if and only if $(v_1, \ldots, v_s, 1, w) \in \operatorname{trop}(V(J))$, where $J = \phi^{-1}(I)$, and $V(J) \subset T^{n+s+1}$.

Example: Let $f = 1 + x + xy + ty \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ or in $\overline{Q(t)}[x^{\pm 1}, y^{\pm 1}]$. Then $\operatorname{trop}(V(f))$ is shown in Figure 35. Let $g = 1 + x + xy + ty \in \overline{Q}[t^{\pm 1}, x^{\pm 1}, y^{\pm 1}]$. Then $\operatorname{trop}(g) = \min(0, x, x+y, y+1)$, and

$$\begin{aligned} \operatorname{trop}(V(g)) = \{(w_1, w_2, w_3) :\, & w_2 = 0 \leq w_2 + w_3, w_1 + w_3 \text{ or} \\ & w_2 + w_3 = 0 \leq w_2, w_1 + w_3 \text{ or} \\ & w_1 + w_3 = 0 \leq w_2, w_2 + w_3 \text{ or} \\ & w_2 = w_2 + w_3 \leq 0, w_1 + w_3 \text{ or} \\ & w_2 = w_1 + w_3 \leq 0, w_2 + w_3 \text{ or} \\ & w_2 + w_3 = w_1 + w_3 \leq 0, w_2\}. \end{aligned}$$

For example $\{(w_1, w_2, w_3) : w_2 = 0 \leq w_2 + w_3, w_1 + w_3\} = \{(w_1, w_2, w_3) : w_2 = 0, w_3 \geq 0, w_1 + w_3 \geq 0\} = \operatorname{pos}((-1, 0, 1), (1, 0, 0))$.

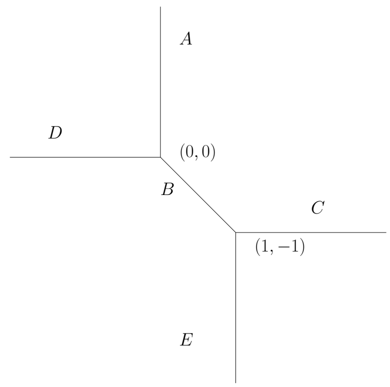

FIGURE 36.

Then

$$\begin{aligned}\operatorname{trop}(V(g)) \cap \{w_1 = 1\} = & \{(1, 0, w_3) : w_3 \geq 0\} \\ & \cup \{(1, w_2, -w_2) : 0 \leq w_2 \leq 1\} \\ & \cup \{(1, w_2, -1) : w_2 \geq 1\} \\ & \cup \{(1, w_2, 0) : w_2 \leq 0\} \\ & \cup \emptyset \\ & \cup \{(1, 1, w_3) : w_3 + 1 \leq 0\}.\end{aligned}$$

Labelling these cones A through E, these are shown in Figure 36.

These reductions mean that if $I \subset \overline{Q(t)}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ then we can compute $\operatorname{trop}(V(I))$ by doing computations in a Laurent polynomial ring with coefficients in $\mathbb{Q}$, so we need only discuss how to compute tropical varieties in this context.

A first algorithm to compute $\operatorname{trop}(V(I))$ in this context is to first homogenize the ideal I to get an ideal $I^h \in \mathbb{Q}[x_0, \ldots, x_n]$. We can then compute the Gröbner fan of I^h, and throw away those cones containing some w in their relative interior with $\operatorname{in}_w(I^h)$ containing a monomial. A problem with this, however, is that it is quite inefficient, as there can be many more cones in the Gröbner fan than there are in the tropical variety. In [BJS+07] an example is given of a family of ideals in three variables indexed by $p \in \mathbb{N}$ where the tropical varieties all have four rays, but the Gröbner fan of the ideal I_p has at least $1/4(p+1)$ rays.

The key fact for the algorithm used by `gfan` is the following theorem.

Theorem 13.1. [BJS+07] *Let $X \subset T_K^n$, where $\operatorname{char}(K) = 0$, be an irreducible variety of dimension d. Then X is a pure polyhedral complex of dimension d that is connected in codimension one.*

Here "connected in codimension one" means that there is a path

$$\sigma = \sigma_1 \succ \tau_1 \prec \sigma_2 \succ \tau_2 \prec \sigma_3 \succ \tau_3 \ldots \tau_s \prec \sigma_s = \sigma'$$

connecting any two d-dimensional polyhedra σ, σ' of $\operatorname{trop}(X)$, where $\tau \prec \sigma$ means "τ is a facet ((d-1)-dimensional face) of σ. This is illustrated in Figure 37.

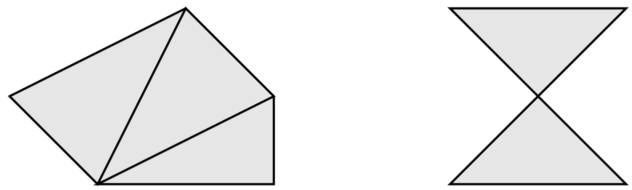

FIGURE 37.

The algorithm used by gfan starts by computing a d-dimensional cone in the Gröbner fan of I^h for which the initial ideal does not contain a monomial. For each facet of this cone, it computes the neighbouring cones in the tropical variety, and then “walks” around the tropical variety until all cones have been visited. Theorem 13.1 guarantees that every cone will be found in this fashion. The computation of computing neighbouring cones can be effectively reduced to the case $d = 1$. For more details, see [BJS+07].

The program gfan [Jen], written by Anders Jensen, can be accessed via your schwartz accounts, or by using Andrew Hoefel's applet available at

http://www.mathstat.dal.ca/~handrew/gfan/index.php

To use it, create a file containing content like:

```
Q[x_1,x_2,x_3,x_4,x_5]
{x_1+x_2+x_3+x_4+x_5, x_1+2*x_2+3*x_3}
```

Note that unlike M2 there is only one Q in the name of the rational numbers. To get the starting cone, type gfan_tropicalstartingcone < myfile >myfile.cone, where myfile is the name of the file you created above, and you can change the name of myfile.cone. This will give output like:

```
Q[x_1,x_2,x_3,x_4,x_5]
{
x_2+2*x_3,
x_1-x_3}
{
x_2-x_5-x_4+2*x_3,
x_1+2*x_5+2*x_4-x_3}
```

This is a list of the initial ideal, and then the ideal. If you are using the web-applet, you will enter the input in the dialogue box, and the output will be on the screen below, which you will need to paste back into the dialogue box for the next step. Next type gfan_tropicaltraverse <myfile.cone >myfile.output. This will give output of the form:

```
_application PolyhedralFan
_version 2.2
_type PolyhedralFan

AMBIENT_DIM
5
```

```
DIM
3

LINEALITY_DIM
1

RAYS
-1 0 0 0 0 # 0
0 -1 0 0 0 # 1
-1 -1 -1 0 0 # 2
0 0 -1 0 0 # 3
1 1 1 1 0 # 4
0 0 0 -1 0 # 5

N_RAYS
6

LINEALITY_SPACE
1 1 1 1 1

ORTH_LINEALITY_SPACE
0 0 0 1 -1
0 0 1 0 -1
0 1 0 0 -1
1 0 0 0 -1

F_VECTOR
1 6 10

CONES
{} # Dimension 1
{0} # Dimension 2
{1}
{2}
{3}
{4}
{5}
{0 2} # Dimension 3
{2 3}
{1 2}
{0 4}
{1 4}
{0 5}
{1 5}
{4 5}
```

```
{3 4}
{3 5}

MAXIMAL_CONES
{0 2} # Dimension 3
{2 3}
{1 2}
{0 4}
{1 4}
{0 5}
{1 5}
{4 5}
{3 4}
{3 5}

PURE
1

MULTIPLICITIES
1 # Dimension 3
1
1
1
1
1
1
1
1
1
```

This says that the tropical variety has lineality space $\text{span}(\mathbf{1})$, and is three-dimensional The quotient by the lineality space gives a fan with six rays, and ten two-dimensional cones, which are listed.

Note that this is not what we got when we did this example on Friday. Firstly this is because `gfan` uses the max convention instead of the min convention. Secondly, it is because gfan has amalgamated some of the cones we constructed. We had rays $\mathbf{e}_i$ for $1 \leq i \leq 5$, $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, and then $\mathbf{e}_i + \mathbf{e}_j$ for $1 \leq i < j \leq 5$ with $\{i, j\} \not\subset \{1, 2, 3\}$. Two of the cones were $\text{span}(\mathbf{1}) + \text{pos}(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_4)$, and $\text{span}(\mathbf{1}) + \text{pos}(\mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_4)$. The union of these two cones is $\text{span}(\mathbf{1}) + \text{pos}(\mathbf{e}_1, \mathbf{e}_4)$.

Warning: The program `gfan` assumes that the ideal generated by your input polynomials is prime, so the variety is irreducible. It will not (necessarily) give the correct answer if your input variety is not irreducible.

Tropical Bases.

Definition 13.2. Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. A set $\{f_1, \dots, f_s\} \subset I$ is a *tropical basis* for I if

$$\mathrm{trop}(V(I)) = \bigcap_{i=1}^{s} \mathrm{trop}(V(f_i)).$$

Example: Let $I = \langle x+y+z, x+2y \rangle \subset \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. Then $\mathrm{trop}(V(I)) = \mathrm{span}(\mathbf{1})$. Any two polynomials in the set

$$\{x+2y, y-z, x+2z\}$$

form a tropical basis for I, as $\mathrm{trop}(V(x+2y)) = \{w \in \mathbb{R}^3 : w_1 = w_2\}$, $\mathrm{trop}(V(y-z)) = \{w \in \mathbb{R}^3 : w_2 = w_3\}$, and $\mathrm{trop}(V(x+2z)) = \{w \in \mathbb{R}^3 : w_1 = w_3\}$. The set $\{x+y+z, x+2y\}$ is not a tropical basis, as the vector $(0,0,1) \in \mathrm{trop}(V(x+y+z)) \cap \mathrm{trop}(V(x+2y))$, but $(0,0,1) \notin \mathrm{trop}(V(I))$. Note that $\mathrm{trop}(V(x+y+z)) \cap \mathrm{trop}(V(x+2y))$ is not a balanced polyhedral complex.

Example: Let $I = \langle f \rangle$ be a principal ideal. Then $\{f\}$ is a tropical basis for I. This was Q4 of Exercises 3.

Proposition 13.3. *Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then I has a finite tropical basis.*

Proof. Let I^h be the homogenization of I in $K[x_0, \dots, x_n]$, and let Σ be the Gröbner complex of I^h. There are only finitely many polyhedra $\sigma \in \Sigma$, and for each $\sigma \in \Sigma$ the initial ideal $\mathrm{in}_w(I^h)$ is constant for all w in the relative interior of σ, so we may denote it by $\mathrm{in}_\sigma(I^h)$. Given $\sigma \in \Sigma$ with $\mathrm{in}_w(I^h)$ containing a monomial for w in the relative interior of σ (so $\mathrm{in}_w(I) = \langle 1 \rangle$), we can find $\tilde{f}_\sigma \in I^h$ with $\mathrm{in}_w(\tilde{f}_\sigma)$ a monomial. Let $f_\sigma \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ denote the dehomogenization of $\tilde{f}_\sigma$, and let

$$\mathcal{G} = \{f_\sigma : \mathrm{in}_\sigma(I^h) \text{ contains a monomial }\}.$$

Note that $\mathcal{G}$ is a finite set, and

$$\bigcap_{f_\sigma \in \mathcal{G}} \mathrm{trop}(V(f_\sigma)) \subseteq \mathrm{trop}(V(I)),$$

since $\mathrm{relint}(\sigma) \cap \mathrm{trop}(V(f_\sigma)) = \emptyset$. But since $f_\sigma \in I$, we have $\mathrm{trop}(V(I)) \subseteq \mathrm{trop}(V(f_\sigma))$, so

$$\bigcap_{f_\sigma \in \mathcal{G}} \mathrm{trop}(V(f_\sigma)) = \mathrm{trop}(V(I)).$$

□

Open Problem: Give a good algorithm to compute a tropical basis.

There is an algorithm implicit in the above proof, but it involves knowing the Gröbner complex of I^h, which is hard to compute.

Definition 13.4. Suppose $\Bbbk$ includes into K. Let $I \subset \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a linear ideal (generated by linear polynomials). A linear polynomial $f = \sum_{i=1}^n a_i x_i$ is a *circuit* for I if $\{i : a_i \neq 0\}$ is minimal with respect to inclusion. This means that there is no $g = \sum_{j=1}^n b_j x_j \in I$ with $\{j : b_j \neq 0\} \subsetneq \{i : a_i \neq 0\}$. Note that there are only a finite number of circuits up to scaling.

Exercise: Let I be a linear ideal. Then the set of circuits form a tropical basis for I.

Remark 13.5. There is also a stricter notion of tropical basis, where we require that the tropical basis in addition be a Gröbner basis for I with respect to any $w \in \operatorname{trop}(V(I))$. This is also finite.

14. Lecture 12

In this lecture we discuss the tropicalization of the Grassmannian. We first review this beautiful classical variety.

Let V be an n-dimensional vector space. We will use $\mathbb{k}$ to denote the field of V by default, but the constructions will make sense over an arbitrary vector space. The Grassmannian $G(d,n)$ parameterizes all d-dimensional subspaces of $V \cong \mathbb{k}^n$. It is a smooth projective variety of dimension $d(n-d)$ (so "locally" looks like affine space of dimension $d(n-d)$), whose points correspond to d-dimensional subspaces of V.

Example: The Grassmannian $G(1,n)$ parameterizes all one-dimensional subspaces of V, which are lines through the origin in $\mathbb{k}^n$. Thus $G(1,n) \cong \mathbb{P}^{n-1}$.

We'll describe two ways to see that $G(d,n)$ is a variety.

Approach 1: We'll describe $G(d,n)$ by giving an *affine cover*. An affine cover is open cover (in the Zariski topology) of the set $G(d,n)$ by affine varieties. It is analogous to giving charts for a manifold.

Fix a basis for V, and thus an isomorphism of V with $\mathbb{k}^n$. Given a d-dimensional subspace $W \subseteq V \cong \mathbb{k}^n$, choose a basis for W and write this basis, in the basis for V, as the rows of a $d \times n$ matrix W_{mat} with entries in $\mathbb{k}$. For each set $\sigma \subset \{1,\dots,n\}$ of size d we consider the set

$$G_\sigma = \{W \subset V : \text{ the submatrix of } W_{mat} \text{ indexed by the columns of } \sigma \text{ is invertible}\}.$$

Given $W \in G_\sigma$, there is a unique basis for W for which $W_{mat}|_\sigma$ is the identity matrix.

Example: Consider $G(2,4)$, and $\sigma = \{1,2\}$. Then if $W \in G_\sigma$, then there is a choice of basis for W in which

$$W_{mat} = \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}, \tag{4}$$

where $a,b,c,d \in \mathbb{k}$. So $G_\sigma \cong \mathbb{A}^4$. We can glue together all the different G_σ to get a variety.

Example: $G_{12} \cap G_{13}$. Then if $W \in G_{12} \cap G_{13}$ then there is a choice of basis for W as above, where $c \neq 0$. Thus $G_{12} \cap G_{13} \cong \mathbb{A}^3 \times \mathbb{k}^*$. There is also a version of W_{mat} of the form

$$W_{mat} = \begin{pmatrix} 1 & -a/c & 0 & b - ad/c \\ 0 & 1/c & 1 & d/c \end{pmatrix}.$$

We identify the two copies of $\mathbb{A}^4$ on their overlap using the above transitions.

Approach 2: We can embed $G(d,n)$ into $\mathbb{P}^{\binom{n}{d}-1}$ by sending $W \in G(d,n)$ to the vector of $d \times d$ minors of W_{mat}. This is the Plücker embedding of $G(d,n)$. Note that if we choose a different basis for W, then the vector of minors changes by at most a multiplicative constant, so the map to $\mathbb{P}^{\binom{n}{d}-1}$ is well-defined. This map is injective, and the image is a subvariety cut out by the *Plücker relations*, which we explain in the case $d = 2$.

Example: Consider $G(2,4)$. Then $\binom{4}{2} = 6$. Label the coordinates of $\mathbb{P}^5$ as $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$. The Plücker relation is $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}$. For example, for the matrix W_{mat} of Equation 4, the Plücker image is $(1 : c : d : -a : -b : ad - bc)$. This satisfies the Plücker relation.

Theorem 14.1. *Let*

$$I_{2,n} = \langle p_{ijkl} = x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk} : 1 \leq i < j < k < l \leq n \rangle \subset \Bbbk[x_{ij} : 1 \leq i < j \leq n].$$

Then $G(2,n) = V(I_{2,n}) \subset \mathbb{P}^{\binom{n}{2}-1}$.

There is a similar description for $G(d,n)$ for $d > n$.

Let $AG(2,n) = V(I_{2,n}) \subset \mathbb{A}^{\binom{n}{2}}$. Then $AG(2,n)$ is the *affine cone* over the Grassmannian $G(2,n)$. Let T be torus of $\mathbb{A}^{\binom{n}{2}}$. Then $T = \{(x_{12}, x_{13}, \dots, x_{(n-1)n}) \in \mathbb{A}^{\binom{n}{2}} : x_{ij} \neq 0\}$. Let $X = AG(2,n) \cap T$.

Claim: The set of Plücker relations forms a tropical basis for $I_{2,n} \subseteq K[x_{ij}^{\pm 1}]$, so

$$\operatorname{trop}(X) = \bigcap_{1 \leq i < j < k < l \leq n} \operatorname{trop}(V(p_{ijkl})).$$

For a proof, see [SS04].

Thus $\operatorname{trop}(X) = \{w \in \mathbb{R}^{\binom{n}{2}} :$ for all $1 \leq i < j < k < l \leq m$ either $w_{ij} + w_{kl} = w_{ik} + w_{jl} \leq w_{il} + w_{jk}$, or $w_{ij} + w_{kl} = w_{il} + w_{jk} \leq w_{ik} + w_{jl}$, or $w_{ik} + w_{jl} = w_{il} + w_{jk} \leq w_{ij} + w_{kl}\}$. This is the *space of Phylogenetic trees*.

Definition 14.2. A *tree* is a graph with no cycles. The *degree* of a vertex in a tree is the number of edges incident to that vertex. A tree is *trivalent* if every vertex has degree three or one. The vertices of degree one are called *leaves*. A tree is *leaf-labelled* if all leaves have a label from some set S. A *phylogenetic tree* is a trivalent leaf-labelled tree.

The name comes from the fact that diagrams illustrating closeness of species are trivalent graphs. See Figure 38.

Given a phylogenetic tree τ drawn in the plane, we can record the *tree-distance* between two vertices.

Example: For the tree of Figure 39, the tree-distance between vertices 1 and 2 is $a + b$. Ordering the pairs (i,j) as $12, 13, 14, 23, 24, 34$, we have the distance between vertices i and j recorded in the vector

$$(a + b, a + c + d, a + c + e, b + c + d, b + c + e, d + e).$$

This equals

$$a \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + e \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

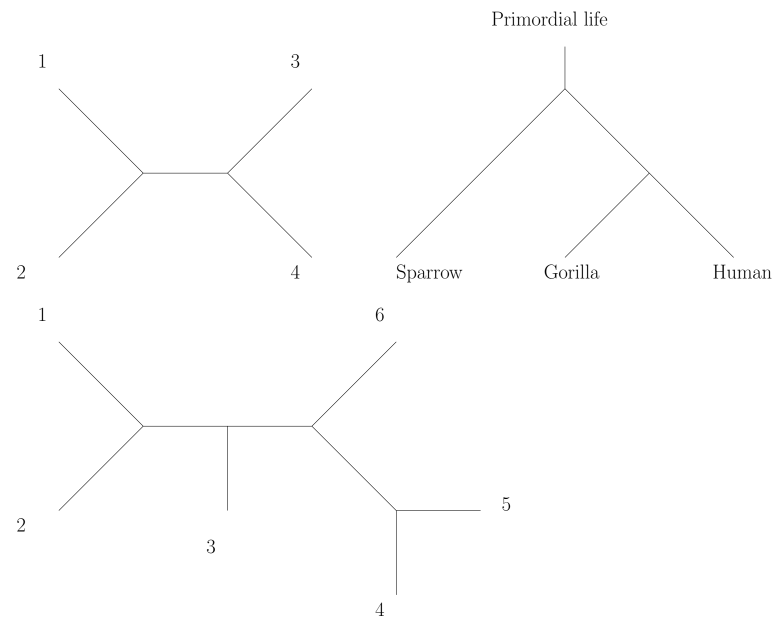

FIGURE 38. Some phylogenetic trees

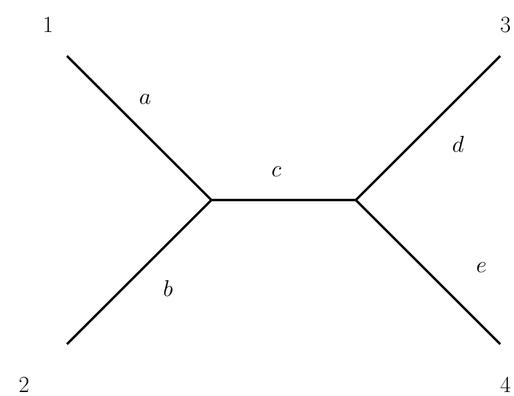

FIGURE 39.

If we let the leaf lengths become negative, this cone is then

$$\operatorname{span}\left(\begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\1\\1\end{pmatrix}\right)+\operatorname{pos}\left(\begin{pmatrix}0\\1\\1\\1\\1\\0\end{pmatrix}\right),$$

which equals

$$\{w : w_{13} + w_{24} = w_{14} + w_{23} \geq w_{12} + w_{34}\}.$$

In general, a vector is the tree-distance vector for a trivalent tree with n leaves if and only if it satisfies the *four point condition:* for any four leaves i, j, k, l, if we form the three possible sums of distances between disjoint pairs $d(i, j) + d(k, l)$,

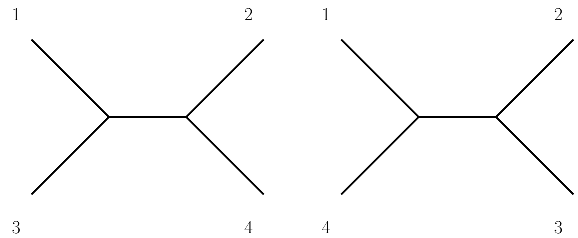

FIGURE 40.

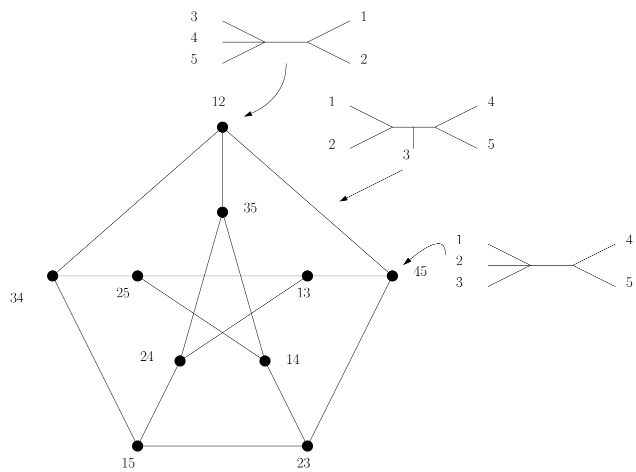

FIGURE 41.

$d(i,k) + d(j,l)$, and $d(i,l) + d(j,k)$, then two of these sums are equal and greater than or equal to the third.

For trees with four leaves the other two options are shown in Figure 40. Setting

$$V = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right),$$

these cones are $V + \operatorname{pos}((1,0,1,1,0,1))$ and $V + \operatorname{pos}((1,1,0,0,1,1))$.

Note that the union of these three cones is $-\operatorname{trop}(X)$!

Summary: The set $-\operatorname{trop}(X)$ is the set of tree-distance vectors for phylogenetic trees with n leaves. This is the *space of phylogenetic trees*, and has one cone for each combinatorial type of labelled tree. It is a polyhedral fan, with an n-dimensional lineality space. The quotient by the lineality space is a pure fan of dimension $n-3$. Note that $n-3$ is the number of internal (not adjacent to a leaf) edges of a trivalent tree with n leaves.

When $n = 4$ we get the three cones described above. When $n = 5$ the quotient by the five-dimensional lineality space is a two-dimensional fan in $\mathbb{R}^5$. The intersection of this with the sphere S^4 is the graph shown in Figure 41.

For more details on all this, see [SS04].

15. Exercises

These questions cover Wednesday of week two to Tuesday of week three.

(1) Show that the definition of the star of a polyhedron σ in a polyhedral complex Σ does not depend on the choice of the point $w \in \sigma$.

(2) Let A be an $r \times n$ matrix of rank r with entries in $\Bbbk$, and let B be a $(n-r) \times n$ matrix whose rows form a basis for $\ker(A)$. Show that if the first $(n-r) \times (n-r)$ submatrix of B is invertible, then the last $r \times r$ submatrix is also invertible.

(3) For each of the following polynomials f

  (a) Compute $\operatorname{trop}(V(f))$.

  (b) For each polyhedron σ in the polyhedral complex $\operatorname{trop}(V(f))$ compute $\operatorname{in}_w(I)$ for any w in the relative interior of σ, and compute directly $\operatorname{trop}(V(\operatorname{in}_w(I))$. Check that this is the appropriate star.

  (c) Check that $\operatorname{trop}(V(f))$ is a weighted balanced polyhedral cone. (The polynomials have been chosen so that all weights are one).

  (I do not expect to see all of these written up completely - do enough until you are satisfied you understand what is going on).

  (a) $f = x + ty + t^3 \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$;

  (b) $f = x^2 + txy + x + y + t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$;

  (c) $f = x + y + x^2y + xy^2 + x^2y^2 \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$;

  (d) $f = x + y + z + 1 \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

(4) For each of the following linear varieties compute $\operatorname{trop}(X)$. Repeat the verifications of parts (b) and (c) of the previous question in this context.

  (a) $X = V(x_1 + x_2 + x_3 + x_4, x_1 + 2x_2 + 4x_3 - x_4) \subset T^4$;

  (b) $X = V(x_1 + x_2 + x_3 + x_4 + x_5, x_1 - x_2 + 3x_3 + 4x_4 + 7x_5) \subset T^5$;

  (c) $X = V(x_1 + x_2 + x_3 + x_4 + x_5, x_1 + x_2 + x_3 + 3x_4 - x_5) \subset T^5$.

(5) Let $X \subset T^n$ be defined by linear equations with coefficients in $\Bbbk$.. Show that the multiplicity of each cone in $\operatorname{trop}(X)$ is one.

(6) The goal of this exercise is to explain in detail how to compute $\operatorname{in}_w(I)$ using a computer algebra package such as `Macaulay 2`.

  (a) Read the proof of Lemma 8 and Corollary 9 of Lecture 8.

  (b) Given $I = \langle f_1, \dots, f_r \rangle \subseteq \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $\tilde{f}_i$ be the polynomial obtained from f_i by clearing denominators. Check that $I = \langle \tilde{f}_1, \dots, \tilde{f}_r \rangle \subset \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

  (c) Let $\bar{I} = (\langle \tilde{f}_1, \dots, \tilde{f}_r \rangle : (\prod_{i=1}^n x_i)^\infty) \subset \Bbbk[x_1, \dots, x_n]$. Show that $\bar{I} = I \cap \Bbbk[x_1, \dots, x_n]$.

  (d) Let $I = \langle x^2 - y, y^2 - x \rangle \subset K[x^{\pm 1}, y^{\pm 1}]$. Use I to show that the saturation step is necessary in the previous exercise.

  (e) Let f_i' be the homogenization of $\tilde{f}_i$ using the variable x_0. Let $J = (\langle f_1', \dots, f_r' \rangle : x_0^\infty) \subseteq \Bbbk[x_0, \dots, x_n]$. Show that $V(J) \subset \mathbb{P}^n$ is the closure of the image under the map $x \mapsto (1 : x)$ of $V(\bar{I}) \subset \mathbb{A}^n$. (The saturation step is again necessary here: an optional exercise is to find an example showing this).

  (f) Show that for most choices of $v \in \mathbb{R}^n$ we have that $\operatorname{in}_v(\operatorname{in}_w(J))$ is a monomial ideal.

(g) Use the previous parts of this exercise to describe how to use a computer algebra package to compute $\mathrm{in}_w(I)$.

(h) In `Macaulay 2` we describe a ring using
`R=QQ[x_0..x_5, Weights=>{1,2,3,4,5,6}].`
In this example $\tilde{w} = (1,2,3,4,5,6)$. If $f = \sum a_u x^u$, let $\mathrm{in}_{\tilde{w}}(f) = \sum_{\tilde{w}\cdot u \text{ is maximal}} a_u x^u$. (Note the difference between min and max here!) Show that $\mathrm{in}_{\tilde{w}}(f) = \mathrm{in}_{\tilde{w}+\lambda\mathbf{1}}(f)$, where $\mathbf{1} \in \mathbb{R}^{n+1}$ is the all-ones vector, and $\lambda \in \mathbb{R}$.
The command `leadTerm(1,J)` computes generators for the ideal $\langle \mathrm{in}_{\tilde{w}}(f) : f \in J\rangle$. Weights must be positive in `Macaulay 2`. Conclude that if $w \in \mathbb{R}^n$, setting `Weights=>w2` where $w_2 = N\mathbf{1} - (0, w)$ for $N \gg 0$ lets `leadTerm(1,J)` compute $\mathrm{in}_w(I)$.

(7) The goal of this exercise is to prove that $\mathrm{trop}(X)$ is a weighted balanced complex when X is a curve in T^2. Since the balanced condition is a local condition, we only need to consider ideals in $\Bbbk[x_1, \dots, x_n]$, so $\mathrm{trop}(X)$ is a one-dimensional fan. Let

$$f = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij} x^i y^j,$$

Let

$$P = \mathrm{conv}((i,j) : a_{ij} \neq 0) \subset \mathbb{R}^2.$$

where $a_{ij} \in \Bbbk$. Then *inner normal* to an edge e of P is a vector $w \in \mathbb{R}^2$ with $w \cdot (i,j) < w \cdot (i',j')$ if $(i,j) \in e$ and $(i',j') \in P \setminus e$.

(a) Let $f = x + y + x^2y + xy^2 + x^2y^2$. Draw P, and the inner normal to each edge of P. Check that this picture is $\mathrm{trop}(V(f))$.

Note that we can always choose the inner normal w to an edge e to be in $\mathbb{Z}^2$. A vector $w \in \mathbb{Z}^2$ is *primitive* if $\gcd(w_1, w_2) = 1$.

(b) Show that every edge $e \in P$ has a unique primitive inner normal $w_e \in \mathbb{Z}^2$.

(c) Show that for any $f \in \Bbbk[x^{\pm 1}, y^{\pm 1}]$ the tropical variety $\mathrm{trop}(V(f))$ is equal to $\cup_e \mathrm{pos}(w_e)$, where the union is over all edges e of P.

(d) Let w_e be the primitive inner normal to an edge w_e of P. Show that there is a monomial $m = x^a y^b$ and a polynomial $g \in \Bbbk[m]$ for which

$$V(\mathrm{in}_{w_e}(f) = V(g)$$

and $\deg(g)$ is the *lattice length* of the edge e (one less than the number of lattice points in e). Thus conclude that the multiplicity m_e of the ray $\mathrm{pos}(w_e)$ of $\mathrm{trop}(V(f))$ is the lattice length of e.

(e) Verify the previous question for $f = x + y + x^2y + xy^2 + x^2y^2$.

(f) Conclude that $\sum_e m_e w_e = 0$, so $\mathrm{trop}(V(f))$ is a balanced weighted polyhedral fan. Hint: Let $\mathbf{e}$ be the vector corresponding to the edge e, oriented clockwise. Note that $\sum_e \mathbf{e} = \mathbf{0}$.

(8) Let $f = tx^2 + xy + ty^2 + x + y + t \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. Compute $\mathrm{trop}(V(f))$, and compare with $\mathrm{trop}(V(g)) \cap \{w_1 = 1\}$ for $g = tx^2 + xy + ty^2 + x + y + t \in \mathbb{C}[t^{\pm 1}, x^{\pm 1}, y^{\pm 1}]$.

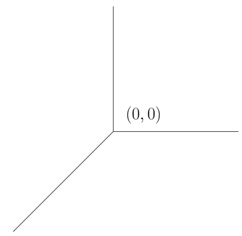

FIGURE 42.

(9) Play with `gfan`. Plug in some of the examples we have computed in lectures and from the notes and check that `gfan` agrees. Warning: Remember that `gfan` uses the max convention instead of our min convention.

(10) Use `gfan` to compute the tropical variety of $X = V(I) \subset T^{10}$, where $I = \langle x_{12}x_{34}-x_{13}x_{24}+x_{14}x_{23}, x_{12}x_{35}-x_{13}x_{25}+x_{15}x_{23}, x_{12}x_{45}-x_{14}x_{35}+x_{15}x_{24}, x_{13}x_{45}-x_{14}x_{35}+x_{15}x_{34}, x_{23}x_{45}-x_{35}x_{24}+x_{25}x_{34}\rangle \subset \Bbbk[x_{12}^{\pm 1}, x_{13}^{\pm 1}, x_{14}^{\pm 1}, x_{15}^{\pm 1}, x_{23}^{\pm 1}, x_{24}^{\pm 1}, x_{25}^{\pm 1}, x_{34}^{\pm 1}, x_{35}^{\pm 1}, x_{45}^{\pm 1}]$. Also compute the tropical variety of $X = V(J) \subset T^6$ where $J = \langle 1 - y_1 + y_3, 1-y_2+y_4, y_1-y_2+y_5\rangle \subset \Bbbk[y_1^{\pm 1}, \dots, y_5^{\pm 1}]$ (Hint: you will need to homogenize first). Compare your answers. Can you explain what you observe?

(11) Let $X = V(xy+y^2+2xz+2yz-xw-yw, x^2-y^2-3xz-3yz+3xw+3yw) \subset T^4$. Show that X is not irreducible. Hint: The `decompose` and `intersect` commands in M2 may help. What does `gfan` think is the tropical variety of X? Is this correct?

(12) (Only for the very computationally inclined). Look at the paper `math.AG/0507563` for details of the algorithm `gfan` uses to compute tropical varieties.

(13) Describe the tropical variety of the Grassmannian $G(2, 6)$. How many cones of each dimension are there?

16. LECTURES 13, 14, AND 15

For the rest of the week we consider the closure of $X \subset T^n$ in a toric variety. We start with taking the closure of X in $\mathbb{A}^n$. We will consider only the case that $\Bbbk \to K$, such as $\mathbb{C} \to \mathbb{C}\{\{t\}\}$.

Recall that $T^n \subset \mathbb{A}^n$, since $T^n = \{(x_1, \dots, x_n) : x_i \in \Bbbk, x_i \neq 0\}$, and $\mathbb{A}^n = \{(x_1, \dots, x_n) : x_i \in \Bbbk\}$. Given $X \subset T^n$, let $\overline{X}$ be the closure of X in $\mathbb{A}^n$. This is the smallest closed set in $\mathbb{A}^n$ containing X, so $\overline{X} = V(I)$ for some I and is as small as possible. Recall that if $V(I) \subsetneq V(J) \subset T^n$, then $\sqrt{J} \subsetneq \sqrt{I}$. If U is a subset of $\mathbb{A}^n$ then $\overline{U} = \bigcap_{U \subseteq V(I) : I \subseteq \Bbbk[x_1,\dots,x_n]} V(I)$.

Question: Given $X \subset T^n$ is $0 \in \overline{X}$?

The answer is given by the following theorem, which was first observed by Tevelev [Tev07].

Theorem 16.1. *Let $X \subset T^n_{\Bbbk}$, and let $\overline{X}$ be the closure of X in $\mathbb{A}^n$. Then $0 \in \overline{X}$ if and only if* $\operatorname{trop}(X) \cap \mathbb{R}^n_{>0} \neq \emptyset$, *where $\mathbb{R}^n_{>0} = \{(x_1, \dots, x_n) : x_i > 0$ for $1 \leq i \leq n\}$.*

Example: Let $X = V(x+y+1) \subset T^2$. Then $X = \{(a, -1-a) : a \in \Bbbk^*, a \neq -1\}$, so $\overline{X} = \{(a, -1-a) : a \in \Bbbk\} = X \cup \{(0, -1), (-1, 0)\} = V(x+y+1) \subset \mathbb{A}^2$. The tropical variety $\operatorname{trop}(X)$ is shown in Figure 42. Note that $0 \notin \overline{X}$, and $\operatorname{trop}(X) \cap \mathbb{R}^2_{>0} = \emptyset$.

Example: Let $X = V(x^2 - y) \subset T^2$. Then $X = \{(a, a^2) : a \in \Bbbk^*\}$, and $\overline{X} = \{(a, a^2) : a \in K\} = X \cup \{(0,0)\} = V(x^2 - y) \subset \mathbb{A}^2$. Then $\mathrm{trop}(X)$ is the line $w_2 = 2w_1$, which contains the point $(1, 2) \in \mathbb{R}^2_{>0}$, and $0 \in \overline{X}$.
Example: Let $X = V(xy - 1) = \{(a, 1/a) : a \in \Bbbk^*\} \subset T^2$. Then $\overline{X} = X$, so $0 \notin \overline{X}$. The tropical variety is the line $w_1 + w_2 = 0$, which does not intersect the positive orthant.

We first note the following description of the ideal of $\overline{X} \subset \mathbb{A}^n$.

Lemma 16.2. *If $X = V(I) \subset T^n$ for $I \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ then $\overline{X} = V(\overline{I}) \subset \mathbb{A}^n$, where $\overline{I} = I \cap \Bbbk[x_1, \dots, x_n]$.*

Proof. Let $\overline{X} = V(J)$ for $J \subset \Bbbk[x_1, \dots, x_n]$. Then since $X \subset \overline{X}$, we have $f(x) = 0$ for all $x \in X$ and $f \in J$, so $f \in \sqrt{I}$ when f is regarded as an element of $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $J' = J\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then $J' \subseteq \sqrt{I}$. Now $J \subseteq J' \cap \Bbbk[x_1, \dots, x_n] \subseteq \sqrt{I} \cap \Bbbk[x_1, \dots, x_n]$. We claim that $\sqrt{I} \cap \Bbbk[x_1, \dots, x_n] = \sqrt{\overline{I}}$. To see this, note that if $f \in \sqrt{I} \cap \Bbbk[x_1, \dots, x_n]$, then there is $N > 0$ for which $f^N \in I$, and since $f \in \Bbbk[x_1, \dots, x_n]$ we have $f^N \in \overline{I}$, so $f \in \sqrt{\overline{I}}$. Conversely, if $f \in \sqrt{\overline{I}}$, then there is $N > 0$ for which $f^N \in \overline{I}$, so $f^N \in I$, and $f \in \Bbbk[x_1, \dots, x_n]$, so $f \in \sqrt{I} \cap \Bbbk[x_1, \dots, x_n]$. Thus $J \subseteq \sqrt{\overline{I}}$, so $V(\overline{I}) \subseteq V(J) = \overline{X}$. But if $x \in X$ then $f(x) = 0$ for all $f \in \overline{I}$, so $X \subseteq V(\overline{I})$, and so $\overline{X} \subseteq V(\overline{I})$. Thus $\overline{X} = V(\overline{I})$. $\square$

A key idea of the proof of Theorem 16.1 is the *inclusion of tori*. This will let us reduce to the case where X is a curve in T^2.

Definition 16.3. A morphism $\phi : T^n \to T^m$ is determined by a $\Bbbk$-algebra homomorphism $\phi^* : \Bbbk[y_1^{\pm 1}, \dots, y_m^{\pm 1}] \to \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Note that the map ϕ^* goes in the reverse direction!

A point $a = (a_1, \dots, a_n) \in T^n$ corresponds to the maximal ideal

$$I_a = \langle x_1 - a_1, \dots, x_n - a_n \rangle \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Since the induced map $\Bbbk[y_1^{\pm 1}, \dots, y_m^{\pm 1}] \to \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/I_a \cong \Bbbk$ is surjective, the kernel $\phi^{*-1}(I_a)$ is maximal, so is of the form $\langle y_1 - b_1, \dots, y_m - b_m \rangle \in \Bbbk[y_1^{\pm 1}, \dots, y_m^{\pm 1}]$ for some $b = (b_1, \dots, b_m) \in T^m$. We thus set $\phi(a) = b$.

Note that $\phi^*(y_i)$ is an invertible polynomial in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ for $1 \leq i \leq m$, so must be a monomial. We thus have

$$\phi^*(y_i) = x^{\mathbf{u}_i} \text{ for } \mathbf{u}_i \in \mathbb{Z}^n.$$

Thus a morphism $\phi : T^n \to T^m$ corresponds to a map $\psi : \mathbb{Z}^m \to \mathbb{Z}^n$ given by $\psi(\mathbf{e}_i) = \mathbf{u}_i$, where $\mathbf{e}_i$ is the ith standard basis vector of $\mathbb{R}^m$.
Exercise: The morphism ϕ is surjective if and only if ψ is injective. The morphism ϕ is injective if and only if ψ is surjective.

We record ψ by the $n \times m$ matrix U with columns $\mathbf{u}_1, \dots, \mathbf{u}_m$, so $\psi(\mathbf{v}) = U\mathbf{v}$. The map $\phi : T^n \to T^m$ is given by $\phi(t_1, \dots, t_n) = (t^{\mathbf{u}_1}, \dots, t^{\mathbf{u}_m})$, where $t^{\mathbf{u}_i} = t_1^{u_{i1}} t_2^{u_{i2}} \dots t_n^{u_{in}}$.

Given $X \subset T^n$ with $X = V(I)$, the closure $\overline{\phi(X)}$ of the image of X under ϕ is the variety $V(\phi^{*-1}(I))$.

Proposition 16.4. *Let $\phi : T^n \to T^m$ be a morphism of tori, with associated $n \times m$ matrix U. Let $X \subset T^n$ be a variety, and let $\overline{\phi(X)}$ be the closure of its image in T^m. Then $w \in \operatorname{trop}(X)$ if and only if $U^T w \in \operatorname{trop}(\overline{\phi(X)})$.*

Proof. We denote by $X(K)$ the subvariety of T^n_K defined by the same equations as X. If $w \in \operatorname{trop}(X)$ then by the Fundamental Theorem there is $y \in X(K)$ with $\operatorname{val}(y) = w$. Then $\phi(y) = (y^{\mathbf{u}_1}, \dots, y^{\mathbf{u}_m}) \in \overline{\phi(X)}$, and $\operatorname{val}(\phi(y)) = (\mathbf{u}_1 \cdot \operatorname{val}(y), \dots, \mathbf{u}_m \cdot \operatorname{val}(y)) = U^T w$. So $w \in \operatorname{trop}(X)$ implies $U^T w \in \operatorname{trop}(\overline{\phi(X)})$.

Conversely, if $\overline{w} \in \operatorname{trop}(\overline{\phi(X)})$, then there exists $\overline{y} \in \overline{\phi(X)}$ with $\operatorname{val}(\overline{y}) = \overline{w}$. By [Pay07] the set of $\overline{y} \in \overline{\phi(X)}$ with $\operatorname{val}(\overline{y}) = \overline{w}$ is Zariski dense in $\overline{\phi(X)}$, so we may assume that $\overline{y} \in \phi(X)$. Thus there is $y \in X$ with $\overline{y} = \phi(y)$. Then $\overline{w} = \operatorname{val}(\overline{y}) = \operatorname{val}(\phi(y)) = U^T \operatorname{val}(y)$, so there is $w = \operatorname{val}(y) \in \operatorname{trop}(X)$ with $\overline{w} = U^T w$. □

We note that one direction of Proposition 16.4 can also be seen by arguments with initial ideals.

Lemma 16.5. *Let $I = \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $w \in \mathbb{R}^n$. Let $\phi : T^n \to T^m$ be given by $\phi(t)_i = t^{\mathbf{u}_i}$, and let U be the $n \times m$ matrix with columns the vectors $\mathbf{u}_i$. Let ϕ^* be the $\mathbb{k}$-algebra homomorphism $\mathbb{k}[y_1^{\pm 1}, \dots, y_m^{\pm 1}] \to \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then*

$$\operatorname{in}_{U^T w}(\phi^{*-1}(I)) \subseteq \phi^{*-1}(\operatorname{in}_w(I)).$$

Thus $w \in \operatorname{trop}(X)$ implies that $U^T w \in \operatorname{trop}(\overline{\phi(X)})$.

Proof. Let $f = \sum_{v \in \mathbb{N}^n} a_v y^v \in \phi^{*-1}(I)$, so $\phi^*(f) = \sum_{v \in \mathbb{N}^n} a_v x^{Uv} \in I$. Then $\operatorname{in}_{U^T w} = \sum_{U^T w \cdot v = W} a_v y^v$, where $W = \min\{U^T w \cdot v : a_v \neq 0\} = \min\{v^T U^T w : a_v \neq 0\}$. However $\operatorname{in}_w(\phi(f)) = \sum_{w \cdot Uv = W'} a_v x^{Uv}$ where $W' = \min\{w \cdot Uv : a_v \neq 0\} = \min\{v^T U^T w : a_v \neq 0\} = W$. Thus $\{v : w \cdot Uv = W, a_v \neq 0\} = \{v : U^T w \cdot v, a_v \neq 0\}$, so $\operatorname{in}_w(\phi(f)) = \phi(\operatorname{in}_{U^T w}(f))$. Thus $\operatorname{in}_{U^T w}(f) \in \phi^{-1}(\operatorname{in}_w(I))$.

If $w \in \operatorname{trop}(X)$, then $\operatorname{in}_w(I) \neq \langle 1 \rangle$, so $\operatorname{in}_{U^T w}(\phi^{*-1}(I)) \neq \langle 1 \rangle$, and thus $U^T w \in \operatorname{trop}(\overline{\phi(X)})$. □

We next outline how to reduce the proof of Theorem 16.1 to the case where X is a curve.

Lemma 16.6. *Let $X \subset T^n$ with ideal $I \subset \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $\phi : T^m \to T^n$ be a morphism of tori with associated $m \times n$ matrix U and $\mathbb{k}$-algebra homomorphism $\phi^* : \mathbb{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to \mathbb{k}[y_1^{\pm 1}, \dots, y_m^{\pm 1}]$. Then $Y = \phi^{-1}(X) = V(\phi^*(I)) \subseteq T^m$.*

We have the containment of sets $U^T \operatorname{trop}(Y) \subset \operatorname{trop}(X) \cap \operatorname{trop}(\operatorname{im}(\phi))$.

Proof. Note that if $f = \sum_{v \in \mathbb{Z}^n} a_v x^v$, then $\phi^*(f) = \sum_{v \in \mathbb{Z}^n} a_v x^{Uv}$, so $\phi^*(f)(y) = f(\phi(y))$ for all $y \in T^m$. Let $y \in Y$. Then $\phi^*(f)(y) = f(\phi(y)) = 0$ for all $f \in I$, so $y \in V(\phi^*(I))$. Conversely, if $g(y) = 0$ for all $g \in \phi^*(I)$, then $\phi^*(f)(y) = 0$ for all $f \in I$, so $f(\phi(y)) = 0$ for all $f \in I$, and thus $\phi(y) \in V(I) = X$, so $y \in Y$.

Let $w \in \operatorname{trop}(Y)$. Then there is $y \in Y$ with $\operatorname{val}(y) = w$. Thus $\phi(y) \in X$, and so $\operatorname{val}(\phi(y)) = U^T w \in \operatorname{trop}(X)$. Since $\operatorname{trop}(\operatorname{im}(\phi)) = \operatorname{im} U^T$, it follows that $U^T w \in \operatorname{trop}(X) \cap \operatorname{trop}(\operatorname{im}(\phi))$. □

The morphism $\phi : T^n \to T^m$ extends to a morphism $\overline{\phi} : \mathbb{A}^n \to \mathbb{A}^m$ if and only if every entry of U is nonnegative. To reduce to the case that X is a curve, one intersects with a codimension $\dim(X)-1$ subtorus of T^n that intersects X transversely in a curve $Y \subset T^m$ for $m = n - \dim(X) + 1$. We assume that the inclusion $\phi : T^m \to T^m$ has matrix U with positive entries. It then follows from Lemma 16.6 that if $0 \in \overline{X}$ then $0 \in \overline{Y}$, so given the curve case of Theorem 16.1 we know that there is $w \in \operatorname{trop}(Y)$ with $w \in \mathbb{R}^m_{>0}$. Then $U^T w \in \operatorname{trop}(X) \cap \mathbb{R}^m_{>0}$.

We will only prove Theorem 16.1 in the case where X is a curve in T^2. The elementary approach to a proof proposed in class does not work.

Proposition 16.7. *Let $X \subset T^2_{\Bbbk}$ be a curve, so $X = V(f)$ for some $f \in \Bbbk[x^{\pm 1}, y^{\pm 1}]$. Then $0 \in \overline{X}$ if and only if $\operatorname{trop}(X) \cap \mathbb{R}^2_{>0} \neq \emptyset$.*

Proof. Write $f = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij} x^i y^j$. Let $I = \langle f \rangle \subseteq \Bbbk[x^{\pm 1}, y^{\pm 1}]$, and let $\overline{I} = I \cap \Bbbk[x, y]$. Then $\overline{I} = \langle f' \rangle$ for $f' = x^{-k} y^{-l} f$, where $k = \min\{i : a_{ij} \neq 0\}$, and $l = \min\{j : a_{ij} \neq 0\}$.

Now $0 \in \overline{X}$ if and only if $f'(0,0) = 0$, which occurs if and only if $a_{kl} \neq 0$. If $a_{kl} \neq 0$, then for all $w \in \mathbb{R}^2_{>0}$, we have $\operatorname{in}_w(f') = a_{kl}$, so $\operatorname{in}_w(I) = \langle 1 \rangle$, and thus $\operatorname{trop}(X) \cap \mathbb{R}^2_{>0} = \emptyset$. Conversely, if $a_{kl} = 0$, then let $P = \operatorname{conv}((i-k, j-l) : a_{ij} \neq 0)$. By construction P lives in the positive orthant, and has a vertex v_1 of the form $(0, r)$ and a vertex v_∞ of the form $(s, 0)$. Let v_2 be the next vertex in counter-clockwise order from v_1. Let $w = (w_1, w_2)$ be the inner facet normal of the edge joining v_1 and v_2 (so $w \cdot u \geq w \cdot v_1 = w \cdot v_2$ for all $u \in P$). Note that $w_1, w_2 > 0$. Then $\operatorname{in}_w(f')$ is not a monomial, since its support contains the monomials with exponents giving rise to v_1 and v_2, and so $\operatorname{in}_w(f)$ is not a monomial, and thus $w \in \operatorname{trop}(X)$. □

Theorem 16.1 generalizes to the following theorem.

Theorem 16.8. *Let $X \subset T^n$ and let $\overline{X}$ be the closure of X in $\mathbb{A}^n$. Then for $\sigma \in \{1, \dots, n\}$*

$$\overline{X} \cap \{x \in \mathbb{A}^n : x_i = 0 \text{ for all } i \in \sigma, x_i \neq 0 \text{ for all } i \notin \sigma\} \neq \emptyset$$

if and only if there is $w \in \operatorname{trop}(X)$ with $w_i > 0$ for all $i \in \sigma$, and $w_i = 0$ for all $i \notin \sigma$.

The case $\sigma = \{1, \dots, n\}$ is Theorem 16.1. The condition on $w \in \operatorname{trop}(X)$ can be rephrased as asking that w lies in the relative interior of $\operatorname{pos}(\mathbf{e}_i : i \in \sigma)$. Examples of Theorem 16.8 can be seen by considering the subvarieties of T^2 at the start of the lecture.

One can also ask the same question about the closure of $X \subset T^n$ in $\mathbb{P}^n$. Recall that T^n embeds into $\mathbb{P}^n$ by the map $(x_1, \dots, x_n) \mapsto (1 : x_1 : \cdots : x_n)$. Given $X \subset T^n$, let $\overline{X}$ now denote the closure of X in $\mathbb{P}^n$. This is the smallest projective variety containing X.

Example: Let $X = V(x_1 + x_2 + 1) \subset T^2$. Let $\overline{X}$ be the closure of X in $\mathbb{P}^2$ under the embedding $T^2 \to \mathbb{P}^2$ given by $(t_1, t_2) \mapsto (1 : t_1 : t_2)$. Then $\overline{X} = X \cup \{(1 : 0 : -1), (1 : -1 : 0), (0 : 1 : -1)\} = V(x_1 + x_2 + x_0)$. Note that $\overline{X} \cap \{x_i = 0\} \neq \emptyset$ for $i = 0, 1, 2$, while $\overline{X} \cap \{x_i = x_j = 0\} = \emptyset$ for all choices of $0 \leq i < j \leq 2$.

Note that the torus T^n is a group, with multiplication coordinatewise, and identity the element $(1, 1, \dots, 1) \in T^n$. The torus T^n acts on $\mathbb{P}^n$ by

$$(t_1, \dots, t_n) \cdot (x_0 : x_1 : \cdots : x_n) = (x_0 : t_1 x_1 : t_2 x_2 : \cdots : t_n x_n).$$

The orbits of T^n on $\mathbb{P}^n$ are indexed by proper subsets of $\{0, 1, \ldots, n\}$ indicating which coordinates are zero.

Example: The torus T^2 acts on $\mathbb{P}^2$ by $(t_1, t_2) \cdot (x_0 : x_1 : x_2) = (x_0 : t_1x_1 : t_2x_2)$. The orbits of T^2 on $\mathbb{P}^2$ are:

$$\{T^2, \{(0 : 1 : t_2) : t_2 \neq 0\}, \{(1 : 0 : t_2) : t_2 \neq 0\}, \{(1 : t_1 : 0) : t_1 \neq 0\},$$
$$\{(1 : 0 : 0)\}, \{(0 : 1 : 0)\}, \{(0 : 0 : 1)\}.$$

These can be labelled by following subsets of $\{0, 1, 2\}$:

$$\{\emptyset, \{0\}, \{1\}, \{2\}, \{1, 2\}, \{0, 2\}, \{0, 1\}\}.$$

We denote by O_σ the orbit of $\mathbb{P}^n$ indexed by $\sigma \subset \{0, 1, \ldots, n\}$.

Question: For $X \subset T^n$, let $\overline{X}$ be the closure of X in $\mathbb{P}^n$. Given $\sigma \subsetneq \{0, 1, \ldots, n\}$, does $\overline{X}$ intersect O_σ?.

As before, then answer depends on the configuration of $\operatorname{trop}(X) \subset \mathbb{R}^n$. We will reduce this calculation to one in $\mathbb{A}^{n+1}$ that uses Theorem 16.8, by using the notion of the affine cone of X.

Let $\tilde{T}^n = \{(t_0, t_1, \ldots, t_n) : t_i \in \mathbb{k}^*\}$. Note that we have the short exact sequence

$$1 \to \mathbb{k}^* \to \tilde{T}^n \xrightarrow{\pi} T^n \to 1,$$

where 1 is the trivial group (written multiplicatively). The map $\mathbb{k}^* \to \tilde{T}^n$ is given by $t \mapsto (t, t, \ldots, t)$, and the map $\pi : (t_0, \ldots, t_n) \mapsto (t_1/t_0, \ldots, t_n/t_0)$. This short exact sequence tropicalizes to

$$0 \to \operatorname{span}(\mathbf{1}) \to \mathbb{R}^{n+1} \overset{\operatorname{trop}(\pi)}{\to} \mathbb{R}^n \to 0,$$

where $\mathbf{1}$ is the vector $(1, 1, \ldots, 1) \in \mathbb{R}^{n+1}$, the first map is the inclusion, and $\operatorname{trop}(\pi) : (w_0, \ldots, w_n) \to (w_1 - w_0, \ldots, w_n - w_0)$.

Given $X \subset T^n$, the *affine cone* over X is $\tilde{X} = \pi^{-1}(X)$. The map $\pi^* : \mathbb{k}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{k}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]$ is given by $\pi^*(x_i) = x_i/x_0$ for $1 \leq i \leq n$. If $X = V(I) \subset T^n$, then $\tilde{X} = V(\pi^*(I)) \subset \tilde{T}^n$. Note that $\mathbf{1}$ lies in the lineality space of $\operatorname{trop}(\tilde{X})$, and that $\operatorname{trop}(X) = \operatorname{trop}(\tilde{X})/\mathbf{1}$.

Example: Let $X = V(x_1 + x_2 + 1) \subset T^2$. Then $\tilde{X} = V(x_1/x_0 + x_2/x_0 + 1) = V(x_1 + x_2 + x_0) \subset \tilde{T}^2$.

Let $\overline{\tilde{X}}$ be the closure of $\tilde{X}$ in $\mathbb{A}^{n+1}$. Recall that $\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus 0)/\mathbb{k}^*$. Then $\overline{X} = (\overline{\tilde{X}}) \setminus 0)/\mathbb{k}^*$. Thus $\overline{X}$ intersects the T^n-orbit indexed by $\sigma \subsetneq \{0, \ldots, n\}$ if and only if the preimage $\operatorname{trop}(\tilde{X})$ of $\operatorname{trop}(X)$ intersects the relative interior of the appropriate $\operatorname{pos}(\mathbf{e}_i : i \in \sigma)$. We can also consider these sets in $\mathbb{R}^n$. Note that the image $\overline{\mathbf{e}}_0$ of $\mathbf{e}_0$ in $\mathbb{R}^n$ is $-\sum_{i=1}^n \overline{\mathbf{e}}_i$. Then $\overline{X}$ intersects the T^n-orbit indexed by σ if and only if $\operatorname{trop}(X) \cap \operatorname{relint}(\operatorname{pos}(\overline{\mathbf{e}}_i : i \in \sigma)) \neq \emptyset$.

Example: Figure 43 shows the fan in $\mathbb{R}^2$ whose cones are the sets $\operatorname{pos}(\overline{\mathbf{e}}_i : i \in \sigma)$ for $\sigma \subsetneq \{0, 1, 2\}$. Thus $\overline{X} \cap O_\sigma \neq \emptyset$ if and only if $\operatorname{trop}(X)$ intersects the relative interior of the appropriate cone. For an example, consider the variety $X = V(x_1 + x_2 + 1) \subset T^2$ analyzed above.

Note that T^n also acts on $\mathbb{A}^n$ by $(t_1, \ldots, t_n) \cdot (x_1, \ldots, x_n) = (t_1x_1, \ldots, t_nx_n)$, and Theorem 16.8 gave conditions for the closure $\overline{X}$ in $\mathbb{A}^n$ of a variety $X \subset T^n$ to intersect each orbit. The sets $\mathbb{A}^n$ and $\mathbb{P}^n$ are examples of *toric varieties*. These are

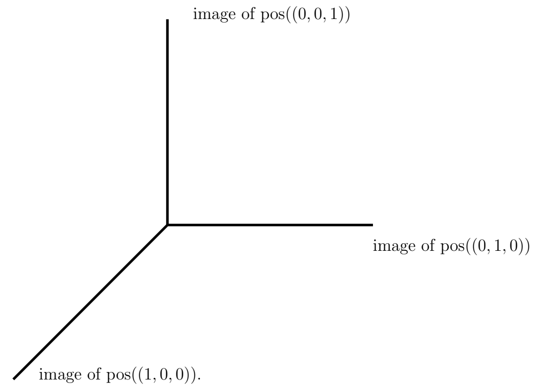

FIGURE 43.

varieties that contain a dense copy of T^n, and have an action of T^n on them extending the action of T^n on itself. They are (up to the technical notion of normalization) described by a polyhedral fan $\Sigma \subset \mathbb{R}^n$, the cones of which index T^n-orbits. If $\overline{X}$ is the closure of $X \subset T^n$ in a toric variety with fan Σ, then $\overline{X}$ intersects the T^n-orbit indexed by the cone $\sigma \in \Sigma$ if and only if $\operatorname{trop}(X)$ intersects the relative interior of σ.

The following proposition was also mentioned in class, and is used in the proof of the general case of the Fundamental Theorem.

Proposition 16.9. *Let $X \subset T^n$ be an irreducible variety of dimension d. Then for most choices of projection $\phi : T^n \to T^{d+1}$. the image $\overline{\phi(X)}$ is a hypersurface in T^{d+1}.*

Here by "most" we mean for a Zariski-open set of choices for the matrix U describing ϕ.

Proof. We first note that since X is irreducible, $\overline{\phi(X)}$ is irreducible of dimension at most d for any choice of projection ϕ. To see irreducibility, note that if $\overline{\phi(X)} = Y_1 \cup Y_2$ for $Y_1, Y_2 \subsetneq \overline{\phi(X)}$, then $X = X_1 \cup X_2$ with $X_i = \phi^{-1}(Y_i) \cap X$ for $i = 1, 2$. Then by the irreducibility of X without loss of generality we have $X = X_1$, so $X \subseteq \phi^{-1}(Y_1)$, and thus $\phi(X) \subseteq Y_1$, contradicting $Y_1 \subsetneq \overline{\phi(X)}$.

Since X is irreducible of dimension d and the generators of I have coefficients in $\Bbbk$, $\operatorname{trop}(X)$ is a pure d-dimensional fan in $\mathbb{R}^n$. Choose a d-dimensional cone $\sigma \in \operatorname{trop}(X)$, and choose an $n \times (d+1)$ rank $d+1$ matrix U with $\ker(U) \cap \operatorname{span}(\sigma) = \mathbf{0}$. Then $\{U^T w : w \in \sigma\}$ is a d-dimensional cone in $\mathbb{R}^{d+1}$, so $\operatorname{trop}(\overline{\phi(X)})$ has dimension at least d, and thus $\overline{\phi(X)}$ has dimension at least d. Since X has dimension d, $\overline{\phi(X)}$ has dimension at most d, and thus is a hypersurface in T^{d+1}. $\square$

17. EXERCISES

(1) Let I be an ideal in $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ generated by linear forms, and let $\mathcal{C}$ be the set of *circuits* of I. Recall that a circuit is a polynomial $f = \sum_{i=1}^n a_i x_i \in I$ with support $\{i : a_i \neq 0\}$ minimal with respect to inclusion. Show that $\mathcal{C}$ is a tropical basis for I.
(2) Let τ be a trivalent tree with n leaves. Show that τ has $2n - 3$ edges.

(3) Let $\phi : T^n \to T^m$ be a morphism of tori, given by $\phi(t)_i = t^{\mathbf{u}_i}$, and let U be the $n \times m$ matrix with columns $\mathbf{u}_1, \dots, \mathbf{u}_m$. Show that ϕ is surjective if and only if U has rank m.

(4) Let $X = V(x + y + z + 1) \subseteq T^3$. For each of the following maps of tori $\phi\colon T^3 \to T^m$ compute $\overline{\phi(X)} \subset T^m$, and verify that $\mathrm{trop}(\overline{\phi(X)}) = \{U^T w : w \in \mathrm{trop}(X)\}$, where U is the $3 \times m$ matrix with columns $\mathbf{u}_1, \dots, \mathbf{u}_m$ for $\phi(t)_i = t^{\mathbf{u}_i}$.

   (a) $\phi : T^3 \to T^2$ given by $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   (b) $\phi : T^3 \to T^2$ given by $U = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & 0 \end{pmatrix}$.

   (c) $\phi : T^3 \to T^4$ given by $U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

(5) For which of the following $X \subset T^3$ is $0 \in \overline{X} \subset \mathbb{A}^3$? Compute $\mathrm{trop}(X)$ and verify the statement of the Theorem.

   (a) $X = V(3x^2 + 3xy + 2x + y)$;
   (b) $X = V(3x^2 + 3xy + 2x + y + 1)$;
   (c) $X = V(x + y + z, x + 2y)$;
   (d) $X = V(x + y + z + 1, x + 2y + 3z)$;

(6) (a) Check that $T^2 \subset \mathbb{P}^2$ is a Zariski-dense subset, and that there is an action of T^2 on $\mathbb{P}^2$ that extends the action of T^2 on itself.

   (b) List the orbits of T^2 in $\mathbb{P}^2$.
   (c) Let $X = V(x + y + x^2y + xy^2) \subset T^2$, and let $\overline{X}$ be the closure of X in $\mathbb{P}^2$. How does $\overline{X}$ intersect each of the T^2-orbits of $\mathbb{P}^2$?
   (d) Draw $\mathrm{trop}(X) \subset \mathbb{R}^2$. How does this relate to your previous answer?

18. Lecture 16

Summary of Class:

We begin this lecture by summarizing what we have done in this class.

We have constructed a *tropical variety* $\mathrm{trop}(X)$ associated to a variety $X \subset T^n$ with $X = V(I)$ for $I \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. This was defined by

$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f)),$$

where for $f = \sum_{v \in \mathbb{Z}^n} c_v x^v$ the set $\mathrm{trop}(V(f))$ is the set of $w \in \mathbb{R}^n$ for which the tropicalization $\mathrm{trop}(f)(w) = \min(\mathrm{val}(c_v) + w \cdot v)$ is achieved at least twice. We have the Fundamental Theorem of Tropical Algebraic Geometry:

Theorem 18.1. *Let $X = V(I) \subset T^n$ for $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then the following sets coincide:*

(1) $\mathrm{trop}(X)$;
(2) *The closure in $\mathbb{R}^n$ of $\{w \in (\mathrm{im}\,\mathrm{val})^n : \mathrm{in}_w(I) \neq \langle 1 \rangle\}$.*

(3) *The closure in* $\mathbb{R}^n$ *of* $\{\mathrm{val}(y) : y \in X\}$.

The last part of the Fundamental Theorem means we can think of a tropical variety as a *combinatorial shadow* of the variety $X \subset T^n$. We saw that tropical varieties have the following structure:

Theorem 18.2 (Structure Theorem for Tropical Varieties)**.** *Let* $X \subset T^n$ *be an irreducible variety of dimension* d*. Then* $\mathrm{trop}(X)$ *is the support of a balanced weighted polyhedral complex that is pure of dimension* d *that is connected in codimension one.*

We saw this in examples we computed with `gfan`, and in the specific explicit examples of linear varieties and $G(2, n)$.

In the spirit of the "combinatorial shadow" philosophy, the (somewhat philosophical) question guiding work in this form of tropical geometry is:
Question: Which invariants of X can be computed from $\mathrm{trop}(X)$?

For example, we see from the main structure theorem that $\dim(X) = \dim(\mathrm{trop}(X))$, so if we know the tropical variety we know the dimension. An answer to the guiding philosophical question is a current area of active research.

Recall that a variety $\mathbb{P}$ is a *toric variety* if $\mathbb{P}$ contains a dense copy of T^n, and there is an action of T^n on $\mathbb{P}$ extending the action of $\mathbb{T}^n$ on itself. A polyhedral fan Σ defines a toric variety $\mathbb{P}_\sigma$. The cones of Σ index the T^n-orbits of $\mathbb{P}_\sigma$. Examples include T^n, $\mathbb{A}^n$, $\mathbb{P}^n$, and $\mathbb{P}^1 \times \mathbb{P}^1$.

Often we are interested not in some $X \subset T^n$ but in a projective variety $Y \subset \mathbb{P}^m$ for some m. Tropical geometry can be useful if this occurs in the following way.

(1) Choose $X \subseteq Y$ and an embedding $X \subset T^n$ for some n.
(2) Let $\mathrm{trop}(X)$ be the fan $\Sigma \subset \mathbb{R}^n$, and let $\mathbb{P}_\Sigma$ be the toric variety with fan Σ.
(3) Let $\overline{X}$ be the closure of X in $\mathbb{P}_\Sigma$.
(4) If $Y \cong \overline{X}$ then we can hope to learn about $Y = \overline{X}$ from $\mathrm{trop}(X)$.

A variety Y that arises in this fashion is called a *tropical compactification* of X. This was first considered in the work of Tevelev [Tev07] with extra conditions on the fan structure on Σ, which guarantees nicer properties for the geometry of $\overline{X}$.

One of the main invariants of $Y = \overline{X}$ we can hope to learn from $\mathrm{trop}(X)$ is the Chow ring of Y, which is the algebraic version of the cohomology ring. This area of algebraic geometry is known as intersection theory. Those unfamiliar with the cohomology ring should spend some time with an algebraic topology book, such as [Hat02] (freely electronically available from the author's webpage). The enumeration of curves problem we will discuss next is such a form of intersection problem.

Other aspects of tropical geometry

This philosophy of obtaining information about $X \subset T^n$, or a projective compactification $\overline{X}$, from the tropical variety $\mathrm{trop}(X)$ is only one portion of current research in tropical geometry. Some samples of this philosophy are found in the work of Speyer [DS05], Payne [Pay07], Tevelev [Tev07], and Hacking, Keel, and Tevelev [HKT06]. The key ideas behind the Fundamental Theorem arose from the work of Kapranov, which appears in [EKL06]. We emphasize that these references are just a (nonrepresentative) sample of the literature.

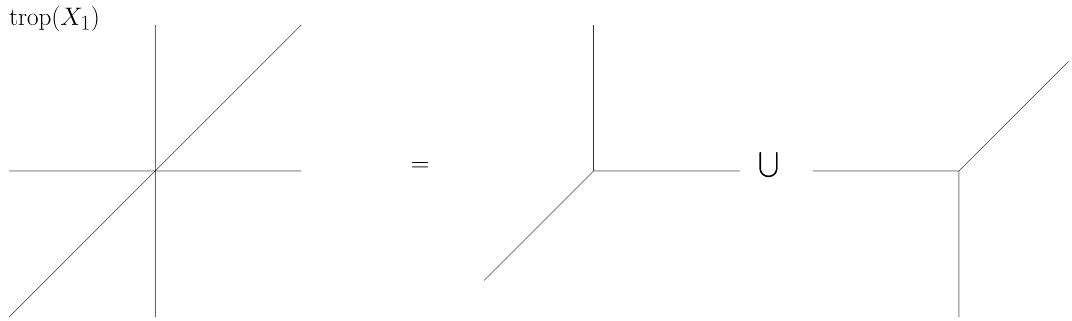

FIGURE 44.

We now give some ideas of other aspects of tropical geometry. This is only a brief overview, and references will just be a nonrepresentative sample.

Abstract tropical varieties

We can *define* a tropical variety to be a balanced weighted polyhedral complex Σ, and then define tropical versions of algebraic geometry invariants by analogy with their classical definitions. An example of this includes defining the dimension to be the dimension of the complex Σ. Current work (see [LAJR07]) is developing a version of intersection theory in this context. Another possibility would be a tropical version of irreducible components. This creates difficulties, though, as the following example shows.

Example: Let $X_1 = V((x+y+1)(x+y+xy)) = V(x+y+1) \cup V(x+y+xy) \subset T^2$. Then $\text{trop}(X_1) = \text{trop}(x+y+1) \cup \text{trop}(x+y+xy)$ is shown in Figure 44. Let $X_2 = V((x-1)(y-1)(x-y)) = V(x-1) \cup V(y-1) \cup V(x-y)$. Then $\text{trop}(X_2)$ is the union of the lines $w_1 = 0$, $w_2 = 0$ and $w_1 = w_2$, so equals $\text{trop}(X_1)$. Since we can't write any of these last three lines as the union of smaller balanced weighted polyhedral complexes, this means that $\text{trop}(X_1) = \text{trop}(X_2)$ cannot be written uniquely as the union of "irreducible" tropical varieties, so a naive definition of irreducible components of these "abstract" tropical varieties does not work.

Similarly, we will see later in the week that there are three different notions of the ranks of a matrix over the tropical semiring. One of the motivations for understanding which geometric invariants make sense for these abstract tropical varieties is that it indicates which properties we can expect combinatorial proofs and interpretations of. This can be thought of as the synthetic approach to tropical geometry. Some examples can be seen in the work of Gathmann and his students, such as [GK08], [LAJR07].

Warning: Note every balanced weighted polyhedral complex in $\mathbb{R}^n$ is $\text{trop}(X)$ for some $X \subset T_K^n$. See the example of Mikhalkin in [DS05, Figure 5.1].

Open Question: Characterize those balanced weighted polyhedral complexes in $\mathbb{R}^n$ that are $\text{trop}(X)$ for some $X \subset T_K^n$.

More general tropical geometry

Another vital branch of tropical research can be broadly classified as attempting to do *all* geometry, not merely algebraic geometry, over the tropical semifield. For example, there should be tropical manifolds, and complex and differential geometry in this setting. This the approach currently taken by Mikhalkin to establish the basics of tropical geometry. See [Mik06], [GM07], [GM] for some expositions of this approach. This also leads to intersections and applications for real algebraic geometry; see, for example, [IKS05]. See [IMS07] for more in this line.

Other aspects

Another major school of tropical geometry revolves around the approach of Gross and Siebert to mirror symmetry using tropical geometry; see [GS06] and its references.

Part of the excitement of this field is that new areas and applications are constantly arising. For example, Gubler [Gub07] recently used the connections to rigid analytic geometry and Berkovich spaces for applications in number theory. Finally, as well as the emerging tropical geometry community, there is also research in max-plus algebras in control theory and related topics, which is a more established field.

These references are only a sample of the exciting research happening in tropical geometry. For more, google "tropical geometry", or put "tropical" in the "anywhere" search box at `front.math.ucdavis.edu` or `mathscinet`.

19. Lectures 17 and 18

In these last two lectures we describe how to use tropical techniques to count rational curves in $\mathbb{P}^2$ passing through a fixed number of points.

Definition 19.1. A curve $C = V(f) \subset \mathbb{P}^2$ has degree d if $f \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree d. An irreducible curve C is rational if there is a map from an open set $U \subset \mathbb{P}^1$ to C given by

$$\phi([t_0 : t_1]) = (p_0(t_0, t_1) : p_1(t_0, t_1) : p_2(t_0, t_2))$$

where p_i is a homogeneous polynomial of degree d for $0 \leq i \leq 2$. The open set $U \subset \mathbb{P}^1$ is the set of $[t_0 : t_1]$ for which at least one of $p_0(t_0, t_1)$, $p_1(t_0, t_1)$, and $p_2(t_0, t_1)$ are nonzero.

A curve of degree one is a line in $\mathbb{P}^2$, which is rational.

Exercise: Check that every curve in $\mathbb{P}^2$ of degree two is rational.

Question: Given n general points $p_1, \dots, p_n \in \mathbb{P}^2$, how many rational curves of degree d pass through all n points?

This question clearly needs to be clarified before a clear answer can be given. For small n (such as $n = 1$) there will be an infinite number of curves. For example, there are infinitely many lines through any given point in $\mathbb{P}^2$. For large n if the p_i are not chosen specially, then there are no curves. For example, if p_1, p_2, p_3 are three points in $\mathbb{P}^2$ that do not lie on a line, then there are no curves of degree one (lines!) passing through all three points. However given any two distinct points in $\mathbb{P}^2$, there is a unique line passing through them, so for $d = 1$ and $n = 2$, with the notion of "general" being "distinct", the question has answer one.

When $d = 2$, we claim that there is also a unique curve of degree two passing through five general points in $\mathbb{P}^2$. To see this, let

$$F = ax_0^2 + bx_0x_1 + cx_0x_1 + dx_1^2 + ex_1x_2 + fx_2^2.$$

The variety $C = V(F)$ is a curve as long as one of $a, \dots, f$ is nonzero, and F and λF define the same curve, so a choice of conic corresponds to a point $[a : b : c : d : e : f] \in \mathbb{P}^5$. We observed in the exercise above that all curves of the form $V(F)$ are rational, so we need to show that given five sufficiently general points in $\mathbb{P}^2$ there is a unique point in $\mathbb{P}^5$ for which the corresponding $V(F)$ passes through all five points. Since we are requiring that the five points be general, we may assume that the first coordinate is nonzero, so they have the form $\mathcal{P}_5 = \{[1 : u_i : v_i] : 1 \le i \le 5\}$. So we need the equation

$$\begin{pmatrix} 1 & u_1 & v_1 & v_1^2 & u_1v_1 & v_1^2 \\ 1 & u_2 & v_2 & v_2^2 & u_2v_2 & v_2^2 \\ 1 & u_3 & v_3 & v_3^2 & u_3v_3 & v_3^2 \\ 1 & u_4 & v_4 & v_4^2 & u_4v_4 & v_4^2 \\ 1 & u_5 & v_5 & v_5^2 & u_5v_5 & v_5^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = 0$$

to have a one-dimensional solution space for most choices of $\mathcal{P}_5$. This means that the 5×6 coefficient matrix must have rank five for most choices of five points. The ideal of 5×5 minors of the coefficient matrix is principal, and generated by a single polynomial of degree six in $\mathbb{C}[u_1, \dots, u_5, v_1, \dots, v_6]$, so for any choice of $\mathcal{P}_5$ with this polynomial nonzero there is a unique rational curve of degree two passing through the points in $\mathcal{P}_5$.

Note that this means that for a general set of four points in $\mathbb{P}^2$ there are are an infinite number of degree two rational curves passing through the points, and for a general set of six points in $\mathbb{P}^2$ there are no degree two rational curves passing through all of the points.

Claim: For general d, there are a finite number of rational curves of degree d passing through $3d - 1$ general points in $\mathbb{P}^2$.

Idea of proof: A rational curve of degree d in $\mathbb{P}^2$ is determined by three polynomials p_0, p_1, p_2, which have the form $\sum_{j=0}^{d} a_{ij} t_0^j t_1^{d-j}$ for $0 \le i \le 2$, for a total of $3d + 3$ parameters. Since the image is in $\mathbb{P}^2$, this over-counts by one. Since we only care about the image of the curve, not the parameterization, this also over-counts by the dimension of $\mathrm{Aut}\mathbb{P}^1$, which is three. This means that the set of rational curves is determined by $3d + 3 - 1 - 3 = 3d - 1$ parameters. Thus forcing the curve to pass through $3d - 1$ general points in $\mathbb{P}^2$ will guarantee a finite number of solutions.

Definition 19.2. Let N_d be the number of irreducible rational curves passing through $3d - 1$ general points in $\mathbb{P}^2$.

Example: We saw above that $N_1 = N_2 = 1$. The numbers N_3 and N_4 were computed in the nineteenth century.

Theorem 19.3 (Kontsevich)**.** *For $d > 1$ the numbers N_d obey the following recursion:*

$$N_d = \sum_{d_A + d_B = d, d_A, d_B > 0} \left(d_A^2 d_B^2 \binom{3d-4}{3d_A - 2} - d_A^3 d_B \binom{3d-4}{3d_A - 1}\right) N_{d_A} N_{d_B}.$$

Note that this describes N_d in terms of smaller d, so knowing $N_1 = 1$ determines all larger N_d.

Example: To compute N_2, the only decomposition is $d_A = d_B = 1$. Then

$$N_2 = (1^2 1^2 \binom{2}{1} - 1^3 1 \binom{2}{2})(1)(1) = 2 - 1 = 1.$$

To compute N_3 we need to consider the pairs $(d_A, d_B) \in \{(1,2),(2,1)\}$. Thus

$$\begin{aligned} N_3 =& (1^2 2^2 \binom{5}{1} - 1^3(2)\binom{5}{2})(1)(1) \\ &+ (2^2 1^2 \binom{5}{4} - 2^3(1)\binom{5}{4})(1)(1) \\ =& 20 - 20 + 20 - 8 \\ =& 12 \end{aligned}$$

Tropical Version

We now outline how to prove Theorem 19.3 using tropical methods. Our sketch follows closely the version given in [HM06], which is based on [GM08] and [Mik05]. The idea is to define a tropical analogue N_d^{trop} of N_d, and show that this equals N_d. We then show that N_d^{trop} obeys the Kontsevich recursion Theorem 19.3.

Definition 19.4. A tropical rational curve of degree d is a one-dimensional balanced weighted polyhedral complex for which

1. the unbounded rays point in the directions $(1,0)$, $(0,1)$, and $(-1,-1)$;
2. there are d unbounded rays pointing in each of these directions (counted with multiplicity);
3. and the underlying graph of the polyhedral complex has no cycles.

Example:

Figure 45 shows contains some examples of tropical rational curves of degrees two, two, and three. Figure 46 contains some one-dimensional tropical varieties that are not tropical rational curves of degree d for some d. For the first variety, the problem is that the underlying graph contains a cycle. For the second, there are unbounded edges not pointing in one of the three prescribed directions.

Note that there is a unique tropical rational curve of degree one through two points in $\mathbb{R}^2$ unless they lie on the same vertical, horizontal, or slope-one line.

Exercise: There is a unique tropical rational curve of degree two through most sets of five points in $\mathbb{R}^2$.

One way to construct a tropical rational curve of degree d is to take a curve of the form $X = V(f) \subset T^2_{\mathbb{C}\{\{t\}\}}$, where $f \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$ has the form $\sum c_{ij} x^i y^j$ where $c_{ij} \neq 0$ if and only if $i, j \geq 0$ and $i + j \leq 2$. Then, as computed in the first exercise set, $\operatorname{trop}(X)$ is a weighted balanced polyhedral complex with d unbounded rays pointing in the prescribed directions. In [DES07] Speyer shows that every tropical rational curve of degree d in $\mathbb{R}^2$ arises in this fashion.

Definition 19.5. A rational tropical curve of degree d is trivalent if the underlying graph is trivalent. Let C be a rational tropical curve of degree d, and let V be a

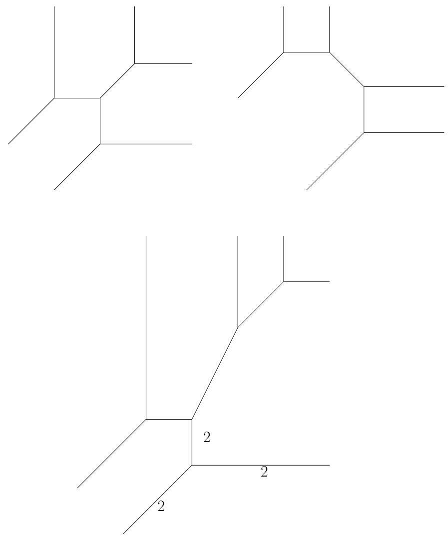

FIGURE 45.

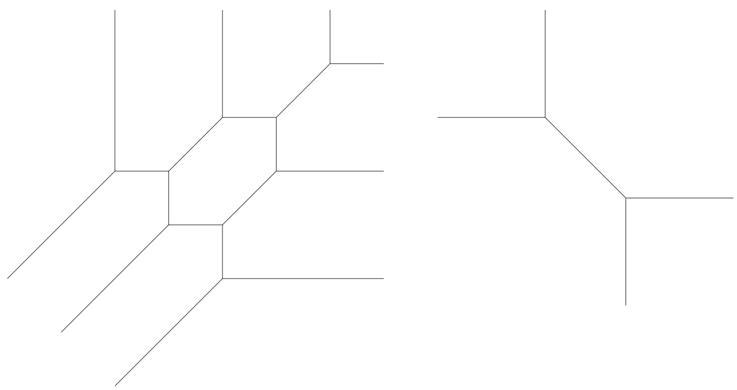

FIGURE 46.

vertex of C. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the smallest integral vectors pointing along the three rays leaving V, and let μ_1, μ_2, μ_3 be the multiplicities of the corresponding polyhedra. The *multiplicity* of V is the absolute value of the determinant of the 2×2 matrix with columns two of the $\mu_i \mathbf{v}_i$. The balancing condition guarantees that this is independent of the choice. The *multiplicity* of C is the product of the multiplicity of all vertices in C.

Example : The multiplicity of the first two tropical rational curves shown in Figure 47 is one. The multiplicity of the third curve is $8 = (2)(4)$, since the multiplicity of the bottom vertex is 4, and the multiplicity of the vertex above is 2, while the other two vertices have multiplicity one.

Definition 19.6. Fix $\mathcal{P} = \{p_1, \dots, p_{3d-1}\} \subset \mathbb{R}^2$. Then $N_d^{trop}(\mathcal{P})$ is the number of tropical rational curves counted with multiplicity passing through $p_1, \dots, p_{3d-1}$.

Proposition 19.7. *There is a Zariski-open set $U \subset (\mathbb{R}^2)^{3d-1}$ for which $N_d^{\mathrm{trop}}(\mathcal{P})$ is constant for $\mathcal{P} = \{p_1, \dots, p_{3d-1}\}$ with $(p_1, \dots, p_{3d-1}) \in U$.*

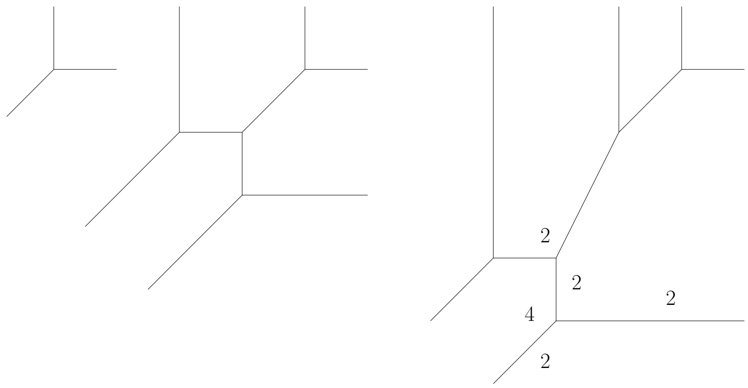

FIGURE 47.

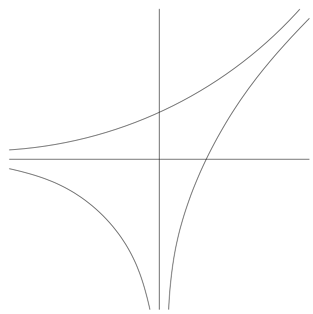

FIGURE 48.

Definition 19.8. A set $\mathcal{P} = \{p_1, \dots, p_{3d-1}\}$ with $(p_1, \dots, p_{3d-1}) \in U$ for the set U of Proposition 19.7 is said to be in tropical general position. Let $N_d^{trop} = N_d^{trop}(\mathcal{P})$ for any $\mathcal{P}$ in tropical general position

Theorem 19.9 (Mikhalkin Correspondence Theorem)**.** *We have equality*

$$N_d^{trop} = N_d.$$

The idea of the proof of Theorem 19.9 is as follows.

The *amoeba* of a variety $X \subset (\mathbb{C}^*)^2$ is the set $\{(\log(|x_1|, \log(|x_2|)) : (x_1, x_2) \in X\} \subset \mathbb{R}^2$.

Example: Let $C = V(x_1 + x_2 + 1) \subset T^2_{\mathbb{C}} = \{(a, -1-a) : a \in \mathbb{C}^*, a \neq -1\}$. Then the amoeba of C is shown in Figure 48.

When $C \subset T^2_{\mathbb{C}}$ has higher degree d, its amoeba generically has d "tentacles" pointing in each direction.

If we replace log by $\log_t$ in the definition of the amoeba, and let $t \to \infty$, then the amoeba gets thinner and thinner, and in the limit approaches a tropical curve of degree d. Given a set of $3d-1$ points we can count the number of rational curves of degree d passing through these points, or the number of tropical rational curves of degree d passing through (roughly) the logs of the points. The multiplicity of a tropical rational curve counts how many of these rational curves in T^2 limit to the tropical curve.

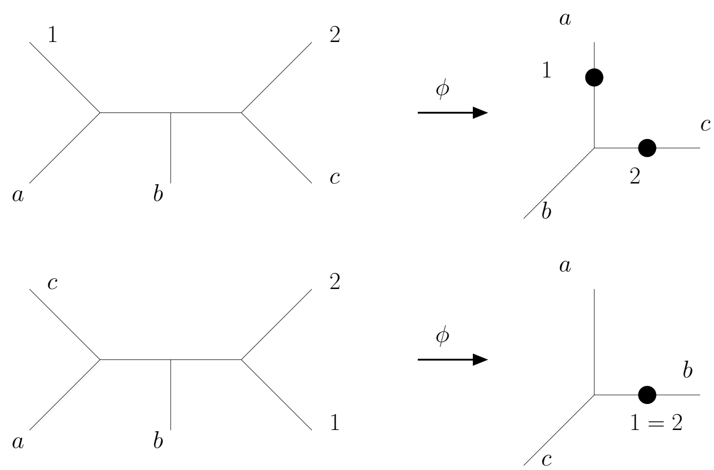

FIGURE 49.

By Theorem 19.9, in order to prove Theorem 19.3, it suffices to show that the numbers N_d^{trop} satisfy the same recursion. We show this by using the standard enumerative combinatorics trick of arguing that the equation

(5)

$$N_d + \sum_{d_A+d_B=d, d_A, d_B>0} d_A^3 d_B \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B} = \sum_{d_A+d_B=d, d_A, d_B>0} d_A^2 d_B^2 \binom{3d-4}{3d_A-2} N_{d_A} N_{d_B}$$

counts the same objects in two different ways. These objects are *parameterized tropical rational curves* with $n = 3d$ marked points.

Definition 19.10. A parameterized tropical rational curve of degree d with n marked points is a map $\phi\colon \Gamma \to \mathbb{R}^2$ where Γ is a tree with $3d + n$ leaves such that

(1) n of the leaves of Γ are labelled $1, \ldots, n$,
(2) each non-leaf edge e of Γ comes with a weight d_e,
(3) $\phi(\Gamma)$ is a rational tropical curve of degree d,
(4) the image of the labelled leaves of Γ are contracted to points, and
(5) the images of the nonlabelled leaves of Γ extended to unbounded rays.

Example: Two examples of parameterized rational curves of degree one with two marked points are shown in Figure 49. We have labelled the "unlabelled" edges a, b, and c so their image under ϕ is clear, but this is not part of the data of ϕ. Note that a non-leaf edge of Γ is contracted in the second example.

Definition 19.11. Given a parameterized tropical curve of degree d with $n \geq 4$ marked points $\phi : \Gamma \to \mathbb{R}^2$, we define the forgetful map f as follows. Let Γ' be the smallest connected subtree of Γ containing the leaves $1, 2, 3$ and 4. Then Γ' consists of two pairs of leaves which are connected by a path of non-leaf edges. The image of the forgetful map $f(\phi)$ is then the phylogenetic tree $\overline{\Gamma}$ with four leaves obtained by turning this path of edges into one edge with length the combined weights.

Example: An example of the forgetful map applied to a parameterized tropical curve of degree one is shown in Figure 50.

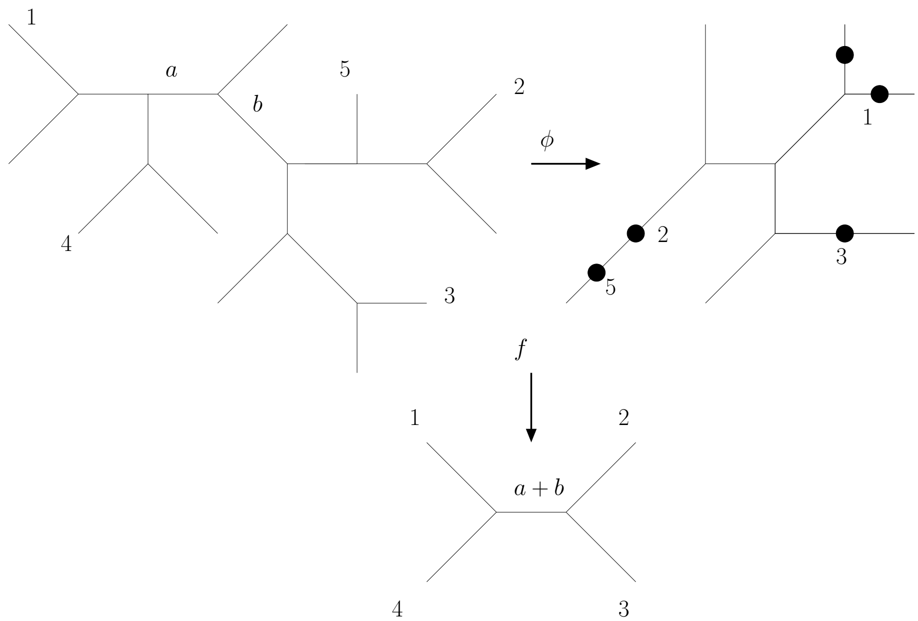

FIGURE 50.

We will count the number $M_d(p_1, \ldots, p_{3d-1}; \overline{\Gamma})$ of parameterized tropical rational curves $\phi : \Gamma \to \mathbb{R}^2$ of degree d with $n = 3d$ marked points such that

(1) The labelled edges $3, \ldots, n$ get mapped to $p_1, \ldots, p_{3d-1}$;
(2) The labelled edge 1 gets mapped to a point on the line $x = (p_1)_1$;
(3) The labelled edge 2 gets mapped to a point on the line $y = (p_1)_2$;
(4) The image $f(\phi)$ of the forgetful map is equal to $\overline{\Gamma}$.

Proposition 19.12. *The number $M_d(p_1, \ldots, p_{3d-1}; \overline{\Gamma})$ is constant for a general choice of $p_1, \ldots, p_{3d-1} \in \mathbb{R}^2$ (tropical general position) and for any choice of four-vertex phylogenetic tree $\overline{\Gamma}$. We thus denote it by M_d.*

The idea of the proof is then to choose $p_1, \ldots, p_{3d-1}$ in tropical general position, and then choose two different trees $\overline{\Gamma}$ at which to evaluate M_d to obtain equation 5.

The proof will use the tropical version of Bézout's theorem. Recall that Bézout's theorem in the plane says that if the variety $V(f_1, f_2) \subset \mathbb{P}^2$ is finite, when f_1, f_2 are homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2]$ of degree d_1 and d_2 respectively, then $V(f_1, f_2)$ consists of $d_1 d_2$ points, counted with multiplicity. The (weak) tropical version of this states that if C_1, C_2 are two tropical curves in $\mathbb{R}^2$, of degrees d_1 and d_2 respectively, that intersect in a finite number of points, then that intersection consists of $d_1 d_2$ points counted with multiplicity. An example of two tropical rational curves of degree two intersecting in four points is shown in Figure 51.

We can now outline the proof of Equation 5. First choose a phylogenetic tree $\overline{\Gamma}$ with four labelled leaves that looks like the one on the left of Figure 52, with the length a of the bounded edge large. Let $\phi : \Gamma \to \mathbb{R}^2$ be a parameterized rational tropical curve of degree d with $n = 3d$ labelled points whose image under the forgetful map is $\overline{\Gamma}$. Then one of two situations occur. The first is that the two leaves labelled 1 and 2 are adjacent to the same vertex in Γ, so the images of these two leaf edges in

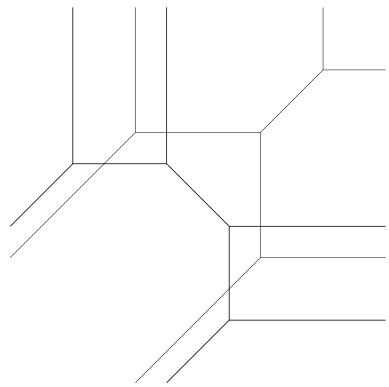

FIGURE 51.

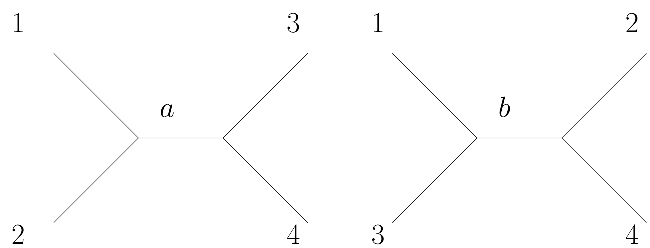

FIGURE 52.

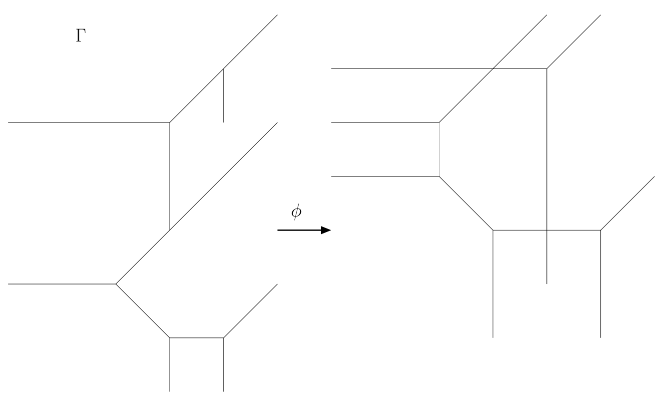

FIGURE 53.

$\mathbb{R}^2$ is the same. This means that $\phi(\Gamma)$ is a tropical rational curve of degree d in $\mathbb{R}^2$ with $\phi(1) = \phi(2) = p_1$, and $\phi(i) = p_{i-1}$ for $3 \leq i \leq n$. The number of such images is N_d. Such maps ϕ are determined by their image, so there are N_d of such ϕ.

The second possibility is that 1 and 2 are not adjacent to the same vertex in Γ. Then it can be shown (see [HM06, Remark 7.16]) that there is a contracted edge of Γ, which leads to the image $\phi(\Gamma)$ being reducible. See Figure 53 for an example with no marked points. This means it is the union of two tropical rational curves C_A and C_B, of degrees d_A and d_B respectively, where 1 and 2 live on C_A, 1 lives on the line $x = (p_1)_1$, 2 lives on the line $y = (p_1)_2$, 3 and 4 live on C_B, $3d_A - 1$ of $\{5, \dots, n\}$ live on C_A, and $\phi(i) = p_{i-1}$ for $3 \leq i \leq n$. There are $\binom{3d-4}{3d_A-1}$ choices of the $3d_A - 1$ points that live on C_A. By Bézout the tropical curve C_A intersects the line $x = (p_1)_1$ (which is itself a tropical line) d_A times, and similarly for the line $y = (p_1)_2$. There are thus d_A choices for the image of 1, and d_A choices for the image of 2. Finally, the curves C_1 and C_2 intersect in $d_A d_B$ points, so there are that many choices for the location of the contracted edge of Γ. This gives a total of

$$d_A^3 d_B \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B}$$

choices for such ϕ, so

$$M_d = N_d + d_A^3 d_B \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B}. \tag{6}$$

Alternatively, choose a phylogenetic tree $\overline{\Gamma}$ with four labelled leaves that looks like the one on the right of Figure 52, where the length b of the bounded edge is large. In this case we cannot have $\phi(1) = \phi(2)$, as that would mean that two other $\phi(i)$ would have to coincide, so two of the p_i would have to coincide, which is ruled out by the tropical general position hypothesis. So we are in the second case above, where the image $\phi(\Gamma)$ is a reducible tropical rational curve. In this case we have 1 lying on the curve C_A of degree d_A, and 2 lying on the curve C_B of degree d_B. The point 3 lives on C_A and the point 4 lives on C_B. There are $3d_A - 2$ of the points $\{5, \dots, n\}$ living on C_A. There are thus $\binom{3d-4}{3d_A-2}$ choices for these points, d_A choices for the image of 1, d_B choices for the image of 2, and $d_A d_B$ choices for the image of the contracted edge of Γ. This gives

$$M_d = d_A^2 d_B^2 \binom{3d-4}{3d_A-2} N_{d_A} N_{d_B}. \tag{7}$$

Combining Equations 6 and 7 we obtain equation 5, and thus have the outline of the tropical proof of Kontsevich's formula.

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