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Viscosity Solutions of a Degenerate Parabolic-Elliptic System Arising in the Mean-Field Theory of Superconductivity

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Abstract

In a Type-II superconductor the magnetic field penetrates the superconducting body through the formation of vortices. In an extreme Type-II superconductor these vortices reduce to line singularities. Because the number of vortices is large it seems feasible to model their evolution by an averaged problem, known as the mean-field model of superconductivity. We assume that the evolution law of an individual vortex, which underlies the averaging process, involves the current of the generated magnetic field as well as the curvature vector. In the present paper we study a two-dimensional reduction, assuming all vortices to be perpendicular to a given direction. Since both the magnetic field $\boldsymbol{H}$ and the averaged vorticity $\boldsymbol{\omega}$ are curl-free, we may represent them via a scalar magnetic potential q and a scalar stream function ψ, respectively. We study existence, uniqueness and asymptotic behaviour of solutions (ψ, q) of the resulting degenerate elliptic-parabolic system (with curvature taken into account or not) by means of viscosity and weak solutions. In addition we relate (ψ, q) to solutions $(\boldsymbol{\omega}, \boldsymbol{H})$ of the mean-field equations without curvature. Finally we construct special solutions of the corresponding stationary equations with two or more superconducting phases.

1. Introduction

We study the degenerate parabolic elliptic system

$$psi_t - |\nabla\psi|\left(\sigma\nabla\cdot\left(\frac{\nabla\psi}{|\nabla\psi|}\right) + q - \psi\right) = 0 \quad \text{in } \Omega\times]0,T[,$$
$$-\Delta q + \chi_\Omega q = \chi_\Omega\psi \qquad \text{in } \mathbb{R}^2\times]0,T[,$$

together with a homogeneous Neumann boundary condition for ψ on $\partial\Omega\times]0,T[$ and a Neumann condition q_∞ for q at infinity. Here Ω is a bounded domain in $\mathbb{R}^2$ and σ is a non-negative constant.

This system couples the level-set method for evolving interfaces by curvature to a nonlocal term. The nonlocal term contains information not only of the level line in question, but of all other level lines.

This model arises in the mean field theory of superconductivity. To this end, assume that a Type-II superconductor occupies a region $D \subset \mathbb{R}^3$. The three-dimensional mean-field model consists in determining a vorticity field $\boldsymbol{\omega} : D\times]0,T[\to \mathbb{R}^3$, a magnetic field $\boldsymbol{H} : \mathbb{R}^3\times]0,T[\to \mathbb{R}^3$ and a velocity field $\boldsymbol{v} : D\times]0,T[\to \mathbb{R}^3$ which satisfy

$$\boldsymbol{\omega}_t + \operatorname{curl}(\boldsymbol{\omega}\times\boldsymbol{v}) = 0 \quad \text{in } D,$$
$$\boldsymbol{v} = \operatorname{curl}\boldsymbol{H}\times\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \quad \text{in } D,$$
$$\operatorname{curl}^2\boldsymbol{H} + \boldsymbol{H} = \boldsymbol{\omega} \quad \text{in } D, \qquad \operatorname{curl}\boldsymbol{H} = 0 \quad \text{in } \overline{D}^c, \qquad [\boldsymbol{H}\times\boldsymbol{\nu}] = 0 \quad \text{on } \partial D,$$
$$\nabla\cdot\boldsymbol{H} = 0 \quad \text{in } \mathbb{R}^3.$$

Here $\boldsymbol{\nu}$ is the exterior normal to D. These equations are supplemented by the boundary conditions

$$\boldsymbol{\omega}\times\boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega,$$
$$\boldsymbol{H}\to\boldsymbol{H}_\infty \quad \text{as } \boldsymbol{x}\to\infty.$$

Here $\boldsymbol{H}_\infty$ is a given divergence-free applied field. This mean field model was derived formally by CHAPMAN [2] as an average of the following discrete problem: find a finite collection of evolving line vortices $\boldsymbol{\Gamma}_i$ and magnetic fields $\boldsymbol{H}_i$ in D such that

$$\operatorname{curl}^2\boldsymbol{H}_i + \boldsymbol{H}_i = \sum_{j\neq i}\boldsymbol{\delta}_{\boldsymbol{\Gamma}_j},$$
$$\boldsymbol{V}_i = \operatorname{curl}\boldsymbol{H}_i\times\boldsymbol{\tau}_i,$$

where $\boldsymbol{V}_i$ and $\boldsymbol{\tau}_i$ denote the velocity and the tangent of $\boldsymbol{\Gamma}_i$, respectively. For the case of straight-line vortices treated as point vortices in two dimensions, the model has also been derived by E [7].

This discrete model in return has been obtained by RICHARDSON [14] via asymptotic expansions of the time dependent Ginzburg-Landau equations assuming that the Ginzburg-Landau parameter κ is large. For the case of straight-line vortices we refer as well to E [6] and PERES & RUBINSTEIN [13]. In fact RICHARDSON derives a more general rule for the velocity of an individual vortex that contains a self-induced current proportional to the curvature vector and to a nonlocal term. In the present paper we incorporate the curvature term, but we neglect the current given by the nonlocal term. The latter is justified since the nonlocal term is of higher order (in κ) and thus smaller than the curvature part of the self-induced energy. Thus we use the velocity law

$$\boldsymbol{V}_i = \sigma C_i \boldsymbol{\nu}_i + \operatorname{curl} \boldsymbol{H}_i \times \boldsymbol{\tau}_i$$

where C_i and $\boldsymbol{\nu}_i$ denote the curvature and normal of $\boldsymbol{\Gamma}_i$, respectively and σ is a non-negative constant. Using the modified velocity law for the discrete model changes the velocity law of the mean field model into

$$\boldsymbol{v} = \sigma \operatorname{curl}\left(\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\right) \times \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} + \operatorname{curl} \boldsymbol{H} \times \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}.$$

The averaging process may be understood in the following way: assume that the number of vortices is n and that the order of vortex spacing is ζ. Then the averaged vorticity $\boldsymbol{\omega}$ is obtained as the limit $n \to \infty$ (and consequently $\zeta \to 0$) of the scaled discrete vortex densities $\boldsymbol{\omega}_n(\boldsymbol{x}) := \zeta^2 \sum_{i=1}^n \frac{1}{|B_\eta(\boldsymbol{x})|} \langle \boldsymbol{\delta}_{\boldsymbol{\Gamma}_i}, B_\eta(\boldsymbol{x}) \rangle = \frac{3}{4\pi} \frac{\zeta^2}{\eta^2} \sum_{i=1}^n \frac{1}{\eta} \int_{\boldsymbol{\Gamma}_i \cap B_\eta(\boldsymbol{x})} \boldsymbol{\tau}_i \, d\mathscr{H}^1$. Here $\eta = \eta(\zeta)$ is chosen much larger than ζ, but still converging to 0. Thus $\boldsymbol{\omega}$ is the L^1 density of the weak* limit of the measures $\zeta^2 \sum_{i=1}^n \boldsymbol{\delta}_{\boldsymbol{\Gamma}_i}$. In the same manner $\boldsymbol{H}$ is obtained as a properly scaled averaged limit of any of the $\boldsymbol{H}_i$. We observe that $\boldsymbol{\omega}_n$ is approximately parallel to $\boldsymbol{\tau}_i$. Thus we may replace $\boldsymbol{\tau}_i$ by $\frac{\boldsymbol{\omega}_n}{|\boldsymbol{\omega}_n|}$. In addition, we note that the curvature vector $C_i \boldsymbol{\nu}_i$ is given by $\operatorname{curl} \boldsymbol{\tau}_i \times \boldsymbol{\tau}_i$. Thus, as long as there are no drastic changes in the direction of the vortices, it seems reasonable to pass to the limit $n \to \infty$ in the discrete model, thereby justifying the mean-field model. We finally observe that the first equation of the averaged model is a conservation law for the vorticity, which has its discrete analogue in the fact that no vortices merge or nucleate.

In the present paper we study a two-dimensional reduction of this three dimensional model. We assume the superconductor to occupy a domain $D = \Omega \times \mathbb{R}$, and we look for solutions of the mean field model that have the special two-dimensional structure $\boldsymbol{\omega} = (\boldsymbol{\omega}, 0)$ and $\boldsymbol{H} = (\boldsymbol{H}, 0)$ with $\boldsymbol{\omega} : \Omega \times]0, T[\to \mathbb{R}^2$ and $\boldsymbol{H} : \mathbb{R}^2 \times]0, T[\to \mathbb{R}^2$. We are thus studying a geometry where we assume all vortices to be perpendicular to the z direction. The two-dimensional mean-field model with curvature thus consists in solving

$$\boldsymbol{\omega}_t - \nabla^\perp\Big(|\boldsymbol{\omega}| \Big(\sigma \nabla^\perp \cdot \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} + \nabla^\perp \cdot \boldsymbol{H}\Big)\Big) = 0 \quad \text{in } \Omega,$$
$$-\nabla^\perp(\nabla^\perp \cdot \boldsymbol{H}) + \boldsymbol{H} = \boldsymbol{\omega} \quad \text{in } \Omega, \quad \nabla^\perp \cdot \boldsymbol{H} = 0 \quad \text{in } \overline{\Omega}^c,$$
$$[\boldsymbol{H} \cdot \boldsymbol{\nu}^\perp] = 0 \quad \text{on } \partial\Omega, \quad \nabla \cdot \boldsymbol{H} = 0 \quad \text{in } \mathbb{R}^2,$$

with boundary conditions

$$\boldsymbol{\omega} \times \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega,$$
$$\boldsymbol{H} \to \boldsymbol{H}_\infty \quad \text{as } \boldsymbol{x} \to \infty.$$

Here $\nabla^\perp = (-\partial_y, \partial_x)$ and $\boldsymbol{\nu}$ is the exterior normal to Ω.

In fact, any solution of this two-dimensional reduction is a solution of the fully three-dimensional problem in the infinite cylinder D with an applied field $\boldsymbol{H}_\infty$ and with an initial datum $\boldsymbol{\omega}_0$ perpendicular to the z-direction and only depending on (x, y).

Since $\boldsymbol{H}$ is divergence-free, we can find a scalar magnetic potential $q : \mathbb{R}^2\times]0, T[\to \mathbb{R}$ such that $\boldsymbol{H} = \nabla^\perp q$. Owing to the stationary equation for $\boldsymbol{H}$ in Ω, which is known as London's equation, we see that $\boldsymbol{\omega}$ as well is divergence free in Ω, and we may thus find a scalar stream function $\psi : \Omega\times]0, T[\to \mathbb{R}$ such that $\boldsymbol{\omega} = \nabla^\perp \psi$. This last observation is only valid as long as Ω is simply-connected. Substituting these relations into the system of equations implies

$$\nabla^\perp\left(\psi_t - |\nabla\psi|\left(\sigma\nabla\cdot\left(\frac{\nabla\psi}{|\nabla\psi|}\right) + \Delta q\right)\right) = 0,$$

$$\nabla^\perp(-\Delta q + q - \psi) = 0 \quad \text{in } \Omega, \quad -\Delta q = 0 \quad \text{in } \overline{\Omega}^c, \quad [\nabla q\cdot\boldsymbol{\nu}] = 0 \quad \text{on } \partial\Omega,$$

$$\nabla\psi\cdot\boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega,$$

$$\nabla q \to \nabla q_\infty \quad \text{as } \boldsymbol{x}\to\infty.$$

Upon changing ψ and q by adding functions depending only on time, we eventually find the system of equations we consider in the present paper.

In return, any solution (ψ, q) gives a solution $(\boldsymbol{\omega}, \boldsymbol{H})$ of the two-dimensional mean-field equations. This holds true even for domains Ω which are not simply-connected. But it restricts the class of initial data we may consider.

We stress that assuming the two-dimensional structure of the solution we are able to show well-posedness of the problem, even without the curvature term. This may no longer be the case without the assumption on the two-dimensional structure as suggested by an eigenvalue analysis of RICHARDSON [15] for the evolution law of an individual vortex. In addition, the fully three-dimensional problem is substantially more difficult than the two-dimensional reduction, owing to the fact that we then have to work with vector potentials $\boldsymbol{\psi}$ and $\boldsymbol{q}$ of $\boldsymbol{\omega}$ and $\boldsymbol{H}$, respectively. In this case the system of equations in D, with the curvature term taken into account, reads

$$\boldsymbol{\psi}_t + \operatorname{curl}\boldsymbol{\psi}\times\left(\sigma\operatorname{curl}\left(\frac{\operatorname{curl}\boldsymbol{\psi}}{|\operatorname{curl}\boldsymbol{\psi}|}\right)\times\frac{\operatorname{curl}\boldsymbol{\psi}}{|\operatorname{curl}\boldsymbol{\psi}|} + \boldsymbol{q} - \boldsymbol{\psi}\right) = 0,$$

$$\operatorname{curl}^2\boldsymbol{q} + \boldsymbol{q} - \boldsymbol{\psi} = 0.$$

We finally note that there exists another two-dimensional reduction of the mean-field model which was studied by CHAPMAN, RUBINSTEIN & SCHATZMAN [3] and STYLES [17]. It consists in searching for solutions of the form $\boldsymbol{\omega} = \omega\boldsymbol{e}_z$, $\boldsymbol{H} = H\boldsymbol{e}_z$, with both ω and H depending on $\boldsymbol{x}\in\mathbb{R}^2$. Hence, this two-dimensional reduction assumes that all vortices are lines pointing in the z direction. CHAPMAN, RUBINSTEIN & SCHATZMAN show existence and regularity of solutions of a resulting stationary free-boundary-value problem. STYLES proves existence of a weak solution, and uniqueness in the special case of Ω being essentially one-dimensional. In addition, she proposes a numerical scheme to solve the problem and shows its convergence in one space dimension.

We now describe our results. In Section 2, we prove the existence of a unique solution (ψ, q) of the two-dimensional degenerate parabolic-elliptic system, either assuming that $\sigma = 0$ or that σ is a positive constant. The dynamic equation is solved in the viscosity sense while the elliptic equation is solved in the weak sense. To this end we introduce the solution operator $q = H\psi$ of the elliptic equation with homogeneous Neumann condition at ∞. Then we interpret the dynamic equation in the viscosity sense, replacing q by $H\psi + q_\infty$. The solution operator has the property that $\inf \psi \leqq H\psi \leqq \sup \psi$. This observation is crucial for a comparison principle to hold for the dynamic equation. Due to this comparison principle we are able to prove existence for initial data of ψ which are only continuous. In addition we show that as $t \to \infty$ the solution tends uniformly to a stationary state.

In Section 3, we relate (ψ, q) to solutions of the two-dimensional mean-field model derived by CHAPMAN. We show (for $\sigma = 0$) that $(\boldsymbol{\omega}, \boldsymbol{H}) = (\nabla^\perp \psi, \nabla^\perp q)$ locally in Ω is a weak solution of the mean-field model. In addition, we show that this $(\boldsymbol{\omega}, \boldsymbol{H})$ is the unique limit of a viscous approximation of the mean-field equations. The mean-field model is hyperbolic in nature, and we show that it only admits very few entropy pairs. As a consequence, we are not able to give a notion of general weak solutions of the mean-field model, incorporating the boundary condition $\boldsymbol{\omega} \times \boldsymbol{\nu} = 0$.

In Section 4 we study the corresponding stationary problem with $\sigma = 0$. We present special model solutions, admitting a functional dependence of the stream function ψ on the magnetic potential q. In particular, we construct domains Ω that allow for solutions with more than two superconducting phases.

2. Existence and Uniqueness

2.1. Notation. Let $0 < T < \infty$ and $\emptyset \neq \Omega \subset\subset \mathbb{R}^2$ be open with $\partial\Omega \in C^{2,\beta}$ for some $\beta > 0$, and let ν be the outer unit normal to $\partial\Omega$. We use the notation $\Omega_T := \Omega \times]0, T[$ and $\Omega_\infty := \Omega \times]0, \infty[$. We study the system of equations

$$\begin{gathered} \psi_t - |\nabla\psi| \left(\sigma \nabla \cdot \left(\frac{\nabla\psi}{|\nabla\psi|} \right) + q - \psi \right) = 0 \quad \text{in } \Omega \times]0, T[, \\ -\Delta q + \chi_\Omega q = \chi_\Omega \psi \quad \text{in } \mathbb{R}^2 \times]0, T[, \end{gathered} \tag{1}$$

with $\sigma \geqq 0$. On ψ, we impose a homogeneous Neumann boundary condition

$$\nabla\psi \cdot \nu = 0 \quad \text{on } \partial\Omega \times]0, T[, \tag{2}$$

and on q, we impose

$$\nabla(q - q_\infty) \in \mathrm{L}^2(\mathbb{R}^2) \tag{3}$$

for all $0 < t < T$. Here q_∞ is a given function which we assume to be harmonic outside a large ball. We will later normalize q_∞ to satisfy a differential equation in $\mathbb{R}^2$ (cf. 2.3(a)).

The initial values of ψ are given by ψ_0, which we assume to be Lipschitz continuous (with the exception of Remark 2.17, where we discuss existence and uniqueness for continuous initial data).

We choose $\Lambda > 0$ such that

$$\sigma, \ \| \nu \|_{C^1(U_{\Lambda^{-1}}(\partial\Omega))}, \ \| \psi_0 \|_{C^{0,1}(\overline{\Omega})}, \ |\Omega| \leqq \Lambda, \tag{4}$$

where $U_{\Lambda^{-1}}(\partial\Omega)$ is the set of all points whose distance to $\partial\Omega$ is less than Λ^{-1}.

We first investigate the elliptic equation.

2.2. Lemma. *For any $\psi \in L^2(\Omega)$, there exists a unique solution $q \in H^{1,2}_{loc}(\mathbb{R}^2)$ of the elliptic equation*

$$-\Delta q + \chi_\Omega q = \chi_\Omega \psi \quad in \ \mathbb{R}^2, \quad \nabla(q - q_\infty) \in L^2(\mathbb{R}^2). \tag{1}$$

Proof. Since only the values of q_∞ near infinity are relevant for the equation, and since q_∞ is harmonic outside a large ball, we may assume that $q_\infty \in C^\infty(\mathbb{R}^2)$. When q is replaced by $q - q_\infty$, (1) is equivalent to

$$-\Delta q + \chi_\Omega q = f \quad \text{in } \mathbb{R}^2, \quad \nabla q \in L^2(\mathbb{R}^2),$$

where $f := \chi_\Omega \psi - \Delta q_\infty + \chi_\Omega q_\infty$. Thus $f \in L^2(\mathbb{R}^2)$ and f has compact support in $\mathbb{R}^2$. We consider the Hilbert space $X := \{p \in H^{1,2}_{loc}(\mathbb{R}^2) \mid \nabla p \in L^2(\mathbb{R}^2)\}$ with scalar product $\langle q, p \rangle := \int_{\mathbb{R}^2} \nabla q \cdot \nabla p + \int_\Omega qp$. It is easily seen that X is indeed a Hilbert space with this scalar product. Further $F(p) := \int_{\mathbb{R}^2} fp$ is a continuous linear functional on X. By the Riesz Representation Theorem, there is a unique $q \in X$ with $\langle q, p \rangle = F(p)$ for all $p \in X$, which gives the unique solution to the original elliptic equation. □

2.3. Remarks. (a) Since the elliptic problem 2.2(1) remains unchanged when q_∞ is replaced by a $\tilde{q}_\infty$ that satisfies $\nabla(q_\infty - \tilde{q}_\infty) \in L^2(\mathbb{R}^2)$ we may normalize q_∞. We choose $\tilde{q}_\infty$ to be the unique solution of 2.2(1) with $\psi = 0$. Thus in the following, we will always assume that

$$-\Delta q_\infty + \chi_\Omega q_\infty = 0. \tag{1}$$

Elliptic regularity theory implies that $q_\infty \in H^{2,p}_{loc}(\mathbb{R}^2)$ for all $p \geqq 1$, and we may enlarge Λ so that

$$\| q_\infty \|_{C^{1,\alpha}(\overline{\Omega})} \leqq \Lambda, \tag{2}$$

for some $0 < \alpha < 1$.

(b) Assume that $q_\infty = 0$ in 2.2(1). Then the solution operator is a linear continuous operator $H : L^2(\Omega) \to X$. We see that

$$\int_\Omega \psi \ H\tilde{\psi} = \langle H\psi, H\tilde{\psi} \rangle. \tag{3}$$

This yields that $H : \mathrm{L}^2(\Omega) \to X \hookrightarrow \mathrm{L}^2(\Omega)$ is a positive, selfadjoint operator on $\mathrm{L}^2(\Omega)$ with $\| H \| = 1$, since $\| H\psi \|_X \geqq \| H\psi \|_{\mathrm{L}^2(\Omega)}$ and $H1 = 1$.

Finally, we establish

$$\inf \psi \leqq H\psi \leqq \sup \psi \quad \text{for } \psi \in \mathrm{L}^\infty(\Omega). \tag{4}$$

Since $H1 = 1$, it suffices to prove that $q = H\psi \geqq 0$ for $\psi \geqq 0$. Multiplying equation 2.2(1) by $q_- := \min(q, 0)$, we get

$$\int_{\mathbb{R}^2} |\nabla q_-|^2 + \int_\Omega |q_-|^2 = \int_\Omega \psi q_- \leqq 0.$$

Therefore q_- is constant on $\mathbb{R}^2$, $q_- = 0$ on Ω, and hence vanishes on the whole of $\mathbb{R}^2$. This yields $q \geqq 0$. □

(c) For general q_∞ as in (a), we can represent the solution q of 2.2(1) in terms of H by

$$q = H\psi + q_\infty. \tag{5}$$

2.4. Remark. In the same way as in Lemma 2.2, we may consider the equation

$$-\Delta q = \chi_\Omega \psi \quad \text{in } \mathbb{R}^2, \quad \nabla(q - q_\infty) \in \mathrm{L}^2(\mathbb{R}^2). \tag{1}$$

Replacing q by $q - q_\infty$ as before, we see that (1) is equivalent to

$$-\Delta q = f \quad \text{in } \mathbb{R}^2, \quad \nabla q \in \mathrm{L}^2(\mathbb{R}^2), \tag{2}$$

where $f := \chi_\Omega \psi - \Delta q_\infty \in \mathrm{L}^2(\mathbb{R}^2)$, with compact support in $\mathbb{R}^2$. Now, since coercivity is lacking on the whole of X, we instead consider $Y := \{p \in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \mid \nabla p \in \mathrm{L}^2(\mathbb{R}^2),\ \int_\Omega p = 0\} \subset X$. As in Lemma 2.2 there exists a unique $q \in Y$ such that $\int_{\mathbb{R}^2} \nabla q \cdot \nabla p = \int_{\mathbb{R}^2} fp$ for all $p \in Y$. We conclude that there is a solution of (1) if and only if

$$\int_\Omega \psi = \int_{\mathbb{R}^2} \Delta q_\infty,$$

and this solution is unique up to a constant.

As a corollary, for two functions q_∞ and $\tilde{q}_\infty$ which are smooth in $\mathbb{R}^2$, which are harmonic outside a large ball and which satisfy $\nabla(q_\infty - \tilde{q}_\infty) \in \mathrm{L}^2(\mathbb{R}^2)$, we get

$$\int_{\mathbb{R}^2} \Delta q_\infty = \int_{\mathbb{R}^2} \Delta \tilde{q}_\infty.$$

In particular, the normalization as in Remark 2.3(a) does not change the integral of the Laplacian of q_∞ over $\mathbb{R}^2$, provided the original q_∞ was smooth.

2.5. Definition. We will consider 2.1(1) in the viscosity sense. To this end, we have to specify the functional dependence of ψ, $\nabla\psi$ and $D^2\psi$, where we have to take into account that q in 2.1(1) appears through the coupling of the equations. For a given $q \in C^0(\overline{\Omega_T})$, we set

$$K(q)(x,t,r,a,p,X) := a - \sigma \operatorname{tr}\left(\left(I - \frac{p \otimes p}{|p|^2}\right)X\right) + |p|(r - q(x,t)),$$

where $(x,t,r,a,p,X) \in \overline{\Omega} \times [0,T] \times \mathbb{R} \times \mathbb{R} \times (\mathbb{R}^2 - \{0\}) \times S(2)$ and $S(2)$ denotes the set of all symmetric (2×2)-matrices. Associated with $K(q)$ are $K^\star(q),\ K_\star(q),\ K^+(q),\ K_-(q)$, which agree with $K(q)$ when $p \neq 0$, and which are defined for $p = 0$ by

$$\begin{aligned}
K^\star(q)(x,t,r,a,p,X) &:= a - \sigma \inf_{|b|=1} \operatorname{tr}((I - b \otimes b)X),\\
K_\star(q)(x,t,r,a,p,X) &:= a - \sigma \sup_{|b|=1} \operatorname{tr}((I - b \otimes b)X),\\
K^+(q)(x,t,r,a,p,X) &:= a - \sigma \inf_{|b|\leqq 1} \operatorname{tr}((I - b \otimes b)X),\\
K_-(q)(x,t,r,a,p,X) &:= a - \sigma \sup_{|b|\leqq 1} \operatorname{tr}((I - b \otimes b)X).
\end{aligned}$$

It requires a calculation to verify that $K^\star$ and $K_\star$ are respectively the upper and lower semicontinuous envelopes of $K(q)$ which are defined, for example, by

$$\begin{aligned}
&K_\star(q)(x,t,r,a,p,X)\\
&\quad := \inf\Big\{\liminf_{j\to\infty} K(q)(x_j,t_j,r_j,a_j,p_j,X_j) \mid (x_j,t_j,r_j,a_j,p_j,X_j)\\
&\qquad \to (x,t,r,a,p,X)\Big\}.
\end{aligned}$$

Viscosity solutions are stable under certain limit procedures (cf. [5, Lemma 6.1]). For a family of functions $K_\delta : D \subseteq \mathbb{R}^N \to \mathbb{R}$ we define

$$K := (\lim_{\delta\to 0})_\star K_\delta$$

by

$$K(z_0) := \inf\{\liminf_{j\to\infty} K_{\delta_j}(z_j) \mid \delta_j \to 0,\ z_j \to z_0\}.$$

$(\lim_{\delta\to 0})^\star K_\delta$ is defined analogously.

In our case, we approximate $K(q)$ by

$$\begin{aligned}
&K_\delta(q)(x,t,r,a,p,X)\\
&\quad := a - 2\delta f_\delta(p)\operatorname{tr}(X) - \sigma \operatorname{tr}\left(\left(I - \frac{p \otimes p}{f_\delta(p)^2}\right)X\right) + f_\delta(p)(r - q(x,t)),
\end{aligned}$$

where $f_\delta(p) := \sqrt{\delta^2 + |p|^2}$. An elementary computation yields that

$$(\lim_{\delta\to 0})_\star K_\delta(q_\delta) = K_-(q), \quad (\lim_{\delta\to 0})^\star K_\delta(q_\delta) = K^+(q) \tag{1}$$

if $q_\delta \to q$ uniformly on $\overline{\Omega_T}$.

2.6. Definition. Let $\psi \in C^0(\overline{\Omega_T})$ and set $q := H\psi + q_\infty$. Then ψ is called a *viscosity subsolution* of problem 2.1(1,2,3), if for all $(x, t) \in \Omega_T$, and for all $(a, p, X) \in P^{2,+}_{\Omega_T}\psi(x, t)$,

$$K_\star(q)(x, t, \psi(x, t), a, p, X) \leqq 0,$$

and for all $(x, t) \in (\partial\Omega)\times]0, T[$, and for all $(a, p, X) \in P^{2,+}_{\overline{\Omega_T}}\psi(x, t)$,

$$\min(K_\star(q)(x, t, \psi(x, t), a, p, X), p\, \nu(x)) \leqq 0.$$

The sets of superdifferentials $P^{2,+}_{\Omega_T}$ and $P^{2,+}_{\overline{\Omega_T}}$ are defined in the next subsection.

ψ is called a weak viscosity subsolution if the above is satisfied when $K_\star$ is replaced by K_-.

Viscosity and weak viscosity supersolutions are defined analogously by replacing $K_\star$ and K_- by $K^\star$ and K^+ and by considering the set of subdifferentials $P^{2,-}$ instead of $P^{2,+}$.

ψ is called a solution when it is both a sub- and a supersolution.

2.7. Definition. For $\emptyset \neq Q \subseteq \mathbb{R}^n\times]0, T[$, $v : Q \to \mathbb{R}$, and $(x_0, t_0) \in Q$, we define the *sets of superdifferentials*

$$\begin{aligned} P^{2,+}_Q v(x_0, t_0) := \big\{&(a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S(n) \mid v(x, t) \leqq v(x_0, t_0) + a(t - t_0) \\ &+ p(x - x_0) + \tfrac{1}{2}(x - x_0)^T X(x - x_0) + o(|t - t_0| + |x - x_0|^2) \\ &\text{as } t \to t_0,\ x \to x_0,\ (x, t) \in Q\big\}, \\ \overline{P}^{2,+}_Q v(x_0, t_0) := \big\{&(a, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S(n) \mid \exists (x_j, t_j) \in Q : \\ &\exists (a_j, p_j, X_j) \in P^{2,+} v(x_j, t_j) : (a_j, p_j, X_j) \to (a, p, X), \\ &(x_j, t_j) \to (x_0, t_0),\ v(x_j, t_j) \to v(x_0, t_0)\big\}. \end{aligned}$$

The sets of subdifferentials $P^{2,-}$ and $\overline{P}^{2,-}$ are defined analogously.

2.8. Remark. It is seen easily that for $\varphi \in C^{2,1}(U(Q))$ with $v-\varphi \leqq (v-\varphi)(x_0, t_0)$ in Q and $U(Q)$ a neighbourhood of Q, the triple of derivatives $(\partial_t\varphi, \nabla\varphi, D^2\varphi) \cdot (x_0, t_0) \in P^{2,+}_Q v(x_0, t_0)$.

Conversely, for all superdifferentials $(a, p, X) \in P^{2,+}_Q v(x_0, t_0)$, there is a $\varphi \in C^{2,1}(U(Q))$ with $v - \varphi \leqq (v - \varphi)(x_0, t_0)$ in Q and $(a, p, X) = (\partial_t\varphi, \nabla\varphi, D^2\varphi) \cdot (x_0, t_0)$. A proof of the second statement can be found in [16, Section 14A].

2.9. Theorem. *For any $\psi_0 \in \mathrm{C}^{0,1}(\overline{\Omega})$ and for any q_∞ which is harmonic outside a large ball, there exists a unique viscosity solution $\psi \in \mathrm{C}^0(\overline{\Omega}\times], 0, \mathrm{T}[)$ of problem 2.1(1,2,3) with initial data $\psi(\cdot, 0) = \psi_0$ in $\overline{\Omega}$. This solution can be extended globally in time and satisfies the estimates*

$$\| \psi \|_{\mathrm{L}^\infty(\Omega_\infty)},\ \| \nabla\psi \|_{\mathrm{L}^\infty(\Omega_\infty)},\ \int_0^\infty\int_\Omega |\psi_t|^2,\ \| \psi \|_{H^{\alpha,\alpha/2}(\overline{\Omega}_\infty)} \leqq C(\Lambda), \quad (1)$$

for some $0 < \alpha < 1$.

2.10. Remark. We will approximate this equation by a smooth quasilinear parabolic equation and pass to the limit. We will replace the modulus $|\cdot|$ by the smooth function $f_\delta(p) := \sqrt{|p|^2 + \delta^2}$, set $A_\delta(p) := \delta|p|^2 + \sigma f_\delta(p)$ and obtain the equation

$$\psi_t - 2\delta f_\delta(\nabla\psi)\Delta\psi - \sigma \operatorname{tr}\left(\left(I - \frac{\nabla\psi \otimes \nabla\psi}{f_\delta(\nabla\psi)^2}\right)D^2\psi\right) + f_\delta(\nabla\psi)(\psi - q) = 0,$$

i.e., $K_\delta(q)(x, t, \psi, \psi_t, \nabla\psi, D^2\psi) = 0$ and $q = H\psi + q_\infty$. This equation is not in divergence form, but as for the original equation, the second-order term has the form $f_\delta(\nabla\psi)\nabla A'_\delta(\nabla\psi)$. This will enable us to estimate the time derivative by multiplying with $\partial_t\psi/f_\delta(\nabla\psi)$. Then the divergence can be integrated in time, and we get

$$\int_{\Omega_T} \frac{|\partial_t\psi|^2}{f_\delta(\nabla\psi)} + \int_\Omega A_\delta(\nabla\psi(T)) + \cdots .$$

A difficulty arises when passing to the limit, since $(\lim_{\delta\to 0})^\star_\star K_\delta \neq K^\star_\star$. Therefore, our approximation yields only weak viscosity solutions. To get the full result, we will establish the uniqueness result for the weak viscosity solutions and approximate again. The second approximation will be fully quasilinear (we do not account for the partial divergence form of the equation) and will yield a proper viscosity solution. On the other hand, the second approximation does not give the estimate on the time derivative in 2.9(1) which will be important when considering the asymptotic behaviour for $t \to \infty$.

Note that the regularization using K_δ is of the form used by EVANS & SPRUCK [9] in their study of mean curvature flow and that their definition of solutions coincides with our definition of a weak viscosity solution. On the other hand, our definition of viscosity solution coincides with that of CHEN, GIGA & GOTO [4] for mean-curvature flow.

2.11. Proof of Theorem 2.9. According to 2.3(5) we write $q = H\psi + q_\infty$ and regularize the equation as pointed out in Remark 2.10. To this end we define $f_\delta(p) := \sqrt{\delta^2 + |p|^2}$ and $A_\delta(p) := \delta|p|^2 + \sigma f_\delta(p)$. We thus use the initial-boundary-value problem

$$\begin{gathered} \frac{\psi_t}{f_\delta(\nabla\psi)} - \nabla \cdot A'_\delta(\nabla\psi) + (\psi - H\psi - q_\infty) = 0 \quad \text{in } \Omega\times]0, T[, \\ \partial_\nu\psi = 0 \quad \text{on } \partial\Omega\times]0, T[, \\ \psi = \psi_{0,\delta} \quad \text{in } \Omega \times \{0\}, \end{gathered} \tag{1}$$

where $\psi_{0,\delta} \in C^\infty(\overline{\Omega})$ with $\| \psi_{0,\delta} \|_{C^{0,1}(\overline{\Omega})} \leqq C(\Lambda)$ and $\partial_\nu\psi_{0,\delta} = 0$ on $\partial\Omega$, and $\psi_{0,\delta} \to \psi_0$ uniformly on $\overline{\Omega}$. Since A_δ depends only on the modulus, we see that $A'_\delta(\nabla\psi) \cdot \nu = 0$ on $\partial\Omega$. Differentiating the second-order term, we write equation as

$$\psi_t - a^\delta_{ij}(\nabla\psi)\partial_{ij}\psi + f_\delta(\nabla\psi)(\psi - H\psi - q_\infty) = 0, \tag{2}$$

where $a^\delta(p) := f_\delta(p) D^2 A_\delta(p)$. We collect the following properties of a^δ:

$$\begin{aligned} a^\delta_{ij}(p) &= 2\delta f_\delta(p)\delta_{ij} + \sigma(\delta_{ij} - \frac{p_i p_j}{f_\delta(p)^2}), \\ a^\delta(p) &\geqq 2\delta f_\delta(p) I, \\ \partial_l a^\delta_{ij}(p) &= 2\delta \frac{p_l}{f_\delta(p)}\delta_{ij} + \sigma(-\frac{\delta_{il} p_j}{f_\delta(p)^2} - \frac{\delta_{jl} p_i}{f_\delta(p)^2} + 2\frac{p_i p_j p_l}{f_\delta(p)^4}). \end{aligned} \tag{3}$$

We observe that equation (2) is a smooth quasilinear parabolic equation with a compact perturbation. From standard parabolic theory (cf., e.g., [12]) and the Leray-Schauder Theorem, we obtain a solution $\psi_\delta \in H^{2+\alpha,1+\alpha/2}(\overline{\Omega_T}) \cap C^\infty_{\text{loc}}(\Omega_T)$.

To simplify the notation, we now drop the index δ.

First, we establish a uniform L^∞ bound on ψ. We set $M := \sup_\Omega \psi_{0,\delta}$ and $\tilde\psi := \psi - \delta t Q$, where Q is any number strictly bigger than $\sup_\Omega q_\infty$. We get

$$\tilde\psi_t + \delta Q - a_{ij}(\nabla\tilde\psi)\partial_{ij}\tilde\psi + f(\nabla\tilde\psi)(\tilde\psi - H\tilde\psi - q_\infty) = 0.$$

We choose $\tilde\psi(x_0, t_0) = \sup_{\Omega_T} \tilde\psi$, and we assume this supremum to be bigger than M. We obtain $0 < t_0 \leqq T$; Hopf's Theorem implies that $x_0 \in \Omega$. This yields $\partial_t \tilde\psi \geqq 0$, $\nabla\tilde\psi = 0$, and $D^2\tilde\psi \leqq 0$ at (x_0, t_0). Since $f_\delta(0) = \delta$, we conclude by using the equation

$$0 \geqq \delta Q + \delta(\tilde\psi - H\tilde\psi - q_\infty)(x_0, t_0).$$

From 2.2(4), we infer that $H\tilde\psi(x_0, t_0) \leqq \sup H\tilde\psi(., t_0) \leqq \sup \tilde\psi(., t_0) = \tilde\psi(x_0, t_0)$. Therefore $0 \geqq Q - q_\infty(x_0, t_0)$. This contradicts the choice of Q, and thus $\sup_{\Omega_T} \tilde\psi \leqq M$. Using a similar argument for the infimum eventually gives

$$\sup_{\Omega_T} |\psi| \leqq \sup_\Omega |\psi_{0,\delta}| + \delta T \sup_\Omega |q_\infty|. \tag{4}$$

Next, we establish a uniform L^∞ bound on $\nabla\psi$. We define $z := f_\delta(p)^2 = |\nabla\psi|^2 + \delta^2$. We know that $z \in H^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega_T}) \cap C^\infty_{\text{loc}}(\Omega_T)$. The derivatives of z are given by

$$\begin{aligned} \partial_t z = 2\partial_t\psi_k\psi_k, &\quad \partial_i z = 2\partial_i\psi_k\psi_k, \\ \partial_{ij} z &= 2\partial_{ij}\psi_k\psi_k + 2\partial_i\psi_k\partial_j\psi_k, \end{aligned}$$

where $\psi_k = \partial_k\psi$. Differentiating equation (2) and multiplying its derivative by $2\psi_k$, we obtain

$$\begin{aligned} &\partial_t z - a_{ij}(\nabla\psi)\partial_{ij} z + 2a_{ij}(\nabla\psi)\partial_i\psi_k\partial_j\psi_k - \partial_l a_{ij}(\nabla\psi)\partial_{ij}\psi\,\partial_l z \\ &\quad + \frac{\partial_k z}{\sqrt{z}}\partial_k\psi(\psi - q) + 2\sqrt{z}(|\nabla\psi|^2 - \nabla q\nabla\psi) = 0 \end{aligned}$$

in Ω_t. Using (3), we get

$$
\begin{aligned}
&2a_{ij}(\nabla\psi)\partial_i\psi_k\partial_j\psi_k - \partial_l a_{ij}(\nabla\psi)\partial_{ij}\psi\partial_l z \\
&\quad \geqq 2\delta\sqrt{z}|D^2\psi|^2 - 2\delta\frac{\partial_l\psi\ \partial_l z\Delta\psi}{\sqrt{z}} \\
&\qquad + \sigma\left(\frac{\partial_i z\partial_{ij}\psi\partial_j\psi}{z} + \frac{\partial_j z\partial_{ij}\psi\partial_i\psi}{z} - 2\frac{\partial_i\psi\partial_j\psi\partial_l\psi}{z^2}\partial_{ij}\psi\partial_l z\right) \\
&\quad = 2\delta\sqrt{z}|D^2\psi|^2 - 2\delta\frac{(\nabla\psi\ \nabla z)\Delta\psi}{\sqrt{z}} + \sigma(\frac{|\nabla z|^2}{z} - \frac{|\nabla\psi\nabla z|^2}{z^2}) \\
&\quad \geqq -C(\Lambda)\delta\frac{|\nabla z|^2}{\sqrt{z}}.
\end{aligned}
$$

Since $q = H\psi + q_\infty$ according to 2.3(5), we get from 2.3(2) and (4) that $|q|,\ |\nabla q| \leqq C(\Lambda)$. If $z > \max(C(\Lambda), 1)$, we obtain $|\nabla\psi|^2 = z - \delta^2 \geqq \frac{2}{3}z \geqq C(\Lambda)$, and hence

$$
\frac{\partial_k z}{\sqrt{z}}\partial_k\psi(\psi - q) + 2\sqrt{z}(|\nabla\psi|^2 - \nabla q\nabla\psi) \geqq -C(\Lambda)|\nabla z| + z\sqrt{z}.
$$

On $\{z > C(\Lambda)\} \cap \Omega_T$ these estimates yield

$$
\partial_t z - a_{ij}(\nabla\psi)\partial_{ij}z \leqq -z\sqrt{z} + C(\Lambda)|\nabla z| + C(\Lambda)\delta\frac{|\nabla z|^2}{\sqrt{z}}. \tag{5}
$$

Next we estimate $\partial_\nu z$ on $\partial\Omega_T$. We know that $\nabla\psi\ \cdot \nu = \partial_\nu\psi = 0$. Therefore $\nabla\psi$ is a tangential vector at $\partial\Omega$. This yields

$$
0 = \nabla\psi\cdot\nabla(\nabla\psi\cdot\nu) = \nabla\psi\cdot D^2\psi\nu + \nabla\psi\cdot D\nu\nabla\psi,
$$

$$
\partial_\nu z = 2\nabla\psi\cdot D^2\psi\nu = -2\nabla\psi\cdot D\nu\nabla\psi \leqq C(\Lambda)z, \tag{6}
$$

since we have $|D\nu| \leqq C(\Lambda)$ using 2.1(4).

Since $\partial\Omega \in C^2$, we again use 2.1(4) to show that there exists a function $\varphi \in C^2(\overline{\Omega})$ that satisfies

$$
\partial_\nu\varphi \geqq 1 \quad \text{on } \partial\Omega, \quad \|\varphi\|_{C^2(\overline{\Omega})} \leqq C(\Lambda), \quad \varphi \geqq 0 \quad \text{in } \Omega. \tag{7}
$$

We define $w := ze^{-C(\Lambda)\varphi}$. The derivatives of w are given by

$$
\begin{aligned}
&\partial_t w = \partial_t z e^{-C(\Lambda)\varphi}, \quad \partial_i w = \partial_i z e^{-C(\Lambda)\varphi} - wC(\Lambda)\partial_i\varphi, \\
&\partial_{ij}w = \partial_{ij}ze^{-C(\Lambda)\varphi} - \partial_i wC(\Lambda)\partial_j\varphi - \partial_j wC(\Lambda)\partial_i\varphi \\
&\qquad -wC(\Lambda)\partial_{ij}\varphi - wC(\Lambda)\partial_i\varphi\partial_j\varphi.
\end{aligned}
$$

From (6) we get

$$
\partial_\nu w = e^{-C(\Lambda)\varphi}(\partial_\nu z - C(\Lambda)z\partial_\nu\varphi) < 0. \tag{8}
$$

Since $0 \leqq w \leqq z$, we obtain from (5) that

$$\begin{aligned} e^{-C(\Lambda)\varphi}\Big(- z\sqrt{z} + C(\Lambda)|\nabla z| + \frac{\delta}{\sqrt{z}}|\nabla z|^2\Big) &\geqq e^{-C(\Lambda)\varphi}(\partial_t z - a_{ij}(\nabla\psi)\partial_{ij} z) \\ &= \partial_t w - a_{ij}(\nabla\psi)\partial_{ij} w + b_i \partial_i w - a_{ij}(\nabla\psi)\partial_{ij}\varphi C(\Lambda) w \\ &\quad - a_{ij}(\nabla\psi)\partial_i\varphi\partial_j\varphi C(\Lambda) w, \end{aligned} \tag{9}$$

on $\{w > C(\Lambda)\} \cap \Omega_T$, where b_i are functions depending on $a,\ w$, and $\nabla\varphi$. Now we choose $w(x_0, t_0) = \sup_{\Omega_T} w =: M \geqq C(\Lambda)$. Since $w(.,0) \leqq z(.,0) = |\nabla\psi_{0,\delta}|^2 + \delta^2 \leqq C(\Lambda)$, we conclude that $0 < t_0 \leqq T$, and w is of class C^2 in a neighbourhood of (x_0, t_0). From (8), we get $x_0 \in \Omega$. This yields $\partial_t w \geqq 0, \nabla w = 0,\ |\nabla z| \leqq C(\Lambda) z$, and $D^2 w \leqq 0$ at (x_0, t_0). Taking into account that $|a_{ij}(\nabla\psi)| \leqq C(\Lambda)(1 + \delta\sqrt{z})$, we get from (9) that

$$0 \leqq \partial_t w - a_{ij}(\nabla\psi)\partial_{ij} w + b_i\partial_i w \leqq e^{-C(\Lambda)\varphi}(-z\sqrt{z} + C(\Lambda)z + C(\Lambda)\delta z\sqrt{z})$$

at (x_0, t_0), which is a contradiction when $\delta < c_0(\Lambda)$. Hence $w \leqq C(\Lambda)$ and

$$\| \nabla\psi_\delta \|_{L^\infty(\Omega_T)} \leqq C(\Lambda), \tag{10}$$

for $\delta < c_0(\Lambda, T)$.

Finally, for passing to the limit, we need an estimate on the time derivative of ψ_δ. To this end we multiply (1) by $\partial_t\psi$ and, taking into account that $A'_\delta(\nabla\psi_\delta)\cdot\nu = 0$ on $\partial\Omega$, we obtain

$$\int_{\Omega_T} \frac{|\partial_t\psi|^2}{f(\nabla\psi)} + \int_\Omega A(\nabla\psi(T)) = \int_\Omega A(\nabla\psi_0) + \int_{\Omega_T} (\psi - H\psi - q_\infty)\partial_t\psi.$$

Since H is self-adjoint (see 2.3(3)) we compute for the last term that

$$\int\limits_{\Omega_T} (\psi - H\psi - q_\infty)\partial_t\psi = \int\limits_\Omega (\tfrac{1}{2}|\psi|^2 - \tfrac{1}{2}\psi H\psi - q_\infty\psi)\Big|_{t=0}^{T}.$$

Observing (4), we see that it is bounded in modulus by $C(\Lambda)$. Since $f(\nabla\psi) \leqq C(\Lambda)$ by (10), we conclude that

$$\int_{\Omega_T} |\partial_t\psi_\delta|^2 \leqq C(\Lambda). \tag{11}$$

As in the proof of Theorem 3.4 of [8], this together with (10) yields that $\| \psi_\delta \|_{H^{\alpha,\alpha/2}(\overline{\Omega_T})} \leqq C(\Lambda)$ for some $0 < \alpha < 1$. Furthermore, since $\partial_t q = H\partial_t\psi$, we get $\| \partial_t q_\delta \|_{L^2(\Omega_T)} \leqq C(\Lambda)$. Since $|\nabla q_\delta| \leqq C(\Lambda)$ follows from (4), we likewise obtain $\| q_\delta \|_{H^{\alpha,\alpha/2}(\Omega_T)} \leqq C(\Lambda)$. Therefore subsequences of $(\psi_\delta)_\delta$ and $(q_\delta)_\delta$ converge uniformly in $\overline{\Omega_T}$, and we can pass to the limit in (1) using Lemma 6.1 of [5]. In view of 2.5(1), this yields a weak viscosity solution.

From (4), (10), and (11), we get (1) for finite $T > 0$. Since Λ does not depend on T, we obtain the full estimates.

Before concluding the existence proof, we establish the comparison principle for weak viscosity solutions in the next theorem.

2.12. Theorem. *Let ψ_1 and ψ_2 be two weak viscosity solutions of problem* 2.1 (1,2,3) *with initial data $\psi_{0,1}$ and $\psi_{0,2}$, respectively. Assume that ψ_1 or ψ_2 is uniformly Lipschitz continuous in space. Then $\psi_{0,1} \leqq \psi_{0,2}$ implies that $\psi_1 \leqq \psi_2$ on $\overline{\Omega}_\infty$.*

Proof. For the sake of notational convenience we put

$$F(p, X) := -\sigma \operatorname{tr}\left(\left(I - \frac{p \otimes p}{|p|^2}\right) X\right) \quad p \neq 0,$$

$$F^+(p, X) := \begin{cases} F(p, X) & p \neq 0, \\ -\sigma \inf\limits_{0<|b|\leqq 1} \operatorname{tr}((I - b \otimes b)X) & p = 0, \end{cases}$$

and we define F_- in a similar way.

Now assume that the assertion of the theorem were not true, i.e., assume that

$$\sup_{\Omega_T}(\psi_1 - \psi_2) > 0.$$

We choose $\gamma > 0$, and for $i = 1, 2$ we set $\tilde{\psi}_i(t, x) := e^{-\gamma t}\psi_i(t, x)$. Then $\tilde{\psi}_i$ are weak viscosity solutions for

$$K^\gamma_\pm(q_i)(x, t, r, a, p, X) := a + \gamma r - F^+_-(p, X) + |p| \left(e^{\gamma t} r - q_i(x, t)\right),$$

and if γ is big enough, then

$$\sup_{\Omega}(\tilde{\psi}_1 - \tilde{\psi}_2)(\cdot, T) < M := \sup_{\Omega_T}(\tilde{\psi}_1 - \tilde{\psi}_2), \tag{1}$$

and $M > 0$.

We now drop the tilde again.

We proceed as in [10]. We set $\Phi(t, x, y) := (\psi_1(x, t) - \psi_2(y, t)) - \Psi(x, y)$, where

$$\begin{aligned} \Psi(x, y) &:= \frac{1}{\varepsilon}\Xi(x, y) + \delta g(x, y), \\ \Xi(x, y) &:= |\eta|^4 g(x, y), \\ \eta &:= x - y, \\ g(x, y) &:= \phi(x) + \phi(y) + 2\beta. \end{aligned} \tag{2}$$

We choose $\beta \in]0, c_0(\Lambda)[$ small enough, and for this β we choose $\phi \in C^2(\overline{\Omega})$ with $-\frac{1}{2}\beta \leqq \phi \leqq 0$ in $\overline{\Omega}$ as well as $\phi = 0$ and $\nabla\phi = \nu$ on $\partial\Omega$. Thus $\beta \leqq g \leqq 2\beta$. With these definitions, Lemma 3.1 and Proposition 3.2, all formulae of Section 3 of [10] as well as Proposition 4.2(i)–(iv) hold.

Obviously, for all $0 < \varepsilon \leqq \varepsilon_0$, $0 < \delta \leqq \delta_0$ there exists $(\hat{t}, \hat{x}, \hat{y}) \in [0, T] \times \overline{\Omega} \times \overline{\Omega}$ such that

$$\Phi(t, x, y) \leqq \Phi(\hat{t}, \hat{x}, \hat{y}) \quad \text{for all } (t, x, y) \in [0, T] \times \overline{\Omega} \times \overline{\Omega}. \tag{3}$$

Setting $x = y$ in (3) and first taking limits in δ and then the limit in ε implies that for any accumulation point (x_0, t_0) of $(\hat{x}_{\varepsilon,\delta}, \hat{t}_{\varepsilon,\delta})$,

$$(\psi_1 - \psi_2)(x, t) \leqq (\psi_1 - \psi_2)(x_0, t_0). \tag{4}$$

Here we used that by Proposition 4.2(iii),(iv) of [10], $\Xi(\hat{x}_{\varepsilon,\delta}, \hat{y}_{\varepsilon,\delta})/\varepsilon \to 0$ and $|\hat{x}_{\varepsilon,\delta} - \hat{y}_{\varepsilon,\delta}| \to 0$ as $\varepsilon \to 0$ uniformly in δ. Thus (x_0, t_0) is some point where $\psi_1 - \psi_2$ attains its supremum over Ω_T. By (1) we know that $t_0 < T$ and since $(\psi_1 - \psi_2)(\cdot, 0) \equiv 0$, we know that $0 < t_0$. As a consequence of this argument, we may assume that

$$\tfrac{1}{2}M < \psi_1(\hat{x}, \hat{t}) - \psi_2(\hat{y}, \hat{t}), \quad 0 < \hat{t} < T \tag{5}$$

for all $0 < \delta < \delta_0$ and all $0 < \varepsilon \leqq \varepsilon_0$.

Now, if $\hat{x} \in \partial\Omega$, $y \in \Omega$, then $\nu(\hat{x}) \cdot \nabla_x \Psi(\hat{x}, y) \geqq \delta$, whereas, if $\hat{y} \in \partial\Omega$, $x \in \Omega$, then $\nu(\hat{y}) \cdot \nabla\Psi_y(x, \hat{y}) \geqq \delta$, by Lemma 3.1 of [10]. Hence, we can apply Lemma 3.3 of [10] and obtain that for any $0 < \varepsilon \leqq \varepsilon_0$, $0 < \delta \geqq \delta_0$ and for any $\lambda > 0$ there exist symmetric matrices X and Y such that, putting

$$\begin{aligned} p^x &:= \frac{1}{\varepsilon}\nabla_x \Xi(\hat{x}, \hat{y}) + \delta\nabla\phi(\hat{x}), \\ p^y &:= \frac{1}{\varepsilon}\nabla_y \Xi(\hat{x}, \hat{y}) + \delta\nabla\psi(\hat{y}), \end{aligned} \tag{6}$$

we get

$$\begin{aligned} 0 \geqq\; & \gamma\big(\psi_1(\hat{x}, \hat{t}) - \psi_2(\hat{y}, \hat{t})\big) - F_-(p^x, X) + F^+(-p^y, -Y) \\ & + |p^x|\big(e^{\gamma\hat{t}}(\psi_1 - H\psi_1)(\hat{x}, \hat{t}) - q_\infty(\hat{x})\big) \\ & - |p^y|\big(e^{\gamma\hat{t}}(\psi_2 - H\psi_2)(\hat{y}, \hat{t}) - q_\infty(\hat{y})\big) \end{aligned} \tag{7}$$

and

$$-\left(\frac{1}{\lambda} + \|A\|\right) I \leqq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leqq A + \lambda A^2, \tag{8}$$

where $A := \frac{1}{\varepsilon}L(\hat{x}, \hat{y}) + \delta D^2 g(\hat{x}, \hat{y})$. To justify this choice of A we use that $D^2\Xi(x, y) \leqq L(x, y)$ with $L(x, y)$ as in (3.3) of [10].

Next, for all $0 < \varepsilon \leqq \varepsilon_0$ there exists a subsequence $\delta_j \to 0$ such that

$$\begin{gathered} \left(\hat{t}_{\varepsilon,\delta_j}, \hat{x}_{\varepsilon,\delta_j}, \hat{y}_{\varepsilon,\delta_j}\right) \longrightarrow \left(\hat{t}_\varepsilon, \hat{x}_\varepsilon, \hat{y}_\varepsilon\right), \\ p^x_{\varepsilon,\delta_j} \longrightarrow p^x_\varepsilon := \frac{1}{\varepsilon}\nabla_x \Xi(\hat{x}_\varepsilon, \hat{y}_\varepsilon), \quad p^y_{\varepsilon,\delta_j} \longrightarrow p^y_\varepsilon := \frac{1}{\varepsilon}\nabla_y \Xi(\hat{x}_\varepsilon, \hat{y}_\varepsilon), \\ \left(X_{\varepsilon,\delta_j}, Y_{\varepsilon,\delta_j}\right) \longrightarrow (X_\varepsilon, Y_\varepsilon) \\ A_{\varepsilon,\delta_j} \longrightarrow A_\varepsilon := \frac{1}{\varepsilon}L(\hat{x}_\varepsilon, \hat{y}_\varepsilon). \end{gathered}$$

Moreover, (8) holds for $(A_\varepsilon, X_\varepsilon, Y_\varepsilon)$ and (7) holds for $(\hat{t}, \hat{x}, \hat{y}, p^x, p^y, X, Y)_\varepsilon$.

Next, for a subsequence $\varepsilon_j \to 0$,

$$\left(\hat{t}_{\varepsilon_j}, \hat{x}_{\varepsilon_j}, \hat{y}_{\varepsilon_j}\right) \longrightarrow (t_0, x_0, y_0)\,.$$

By Proposition 4.2(iii) of [10], $x_0 = y_0$ and (5) is satisfied for $\delta = 0$.

We proceed by showing that $\hat{x}_\varepsilon \neq \hat{y}_\varepsilon$. Assume to the contrary that $\hat{x}_\varepsilon = \hat{y}_\varepsilon$. Then by formulae (3.2) and (3.3) of [10] $p_\varepsilon^x = p_\varepsilon^y = 0$ and $A_\varepsilon = 0$. Thus (8) implies that $\begin{pmatrix} X_\varepsilon & 0 \\ 0 & Y_\varepsilon \end{pmatrix} \leqq 0$ and consequently $X_\varepsilon \leqq 0 \leqq -Y_\varepsilon$. Using the monotonicity of F_- and F^+ we obtain from (7) that

$$\begin{aligned} 0 &\geqq \tfrac{1}{2}\gamma M + F_-(0, X_\varepsilon) - F^+(0, -Y_\varepsilon) \\ &\geqq \tfrac{1}{2}\gamma M + F_-(0,0) - F^+(0,0) > 0. \end{aligned} \tag{9}$$

This is a contradiction. Thus $|\eta_\varepsilon| = |\hat{x}_\varepsilon - \hat{y}_\varepsilon| > 0$ and we obtain from (3.2) and (4.2) of [10] with constants $C,\ c_0 > 0$ which are independent of ε that

$$\begin{aligned} C\frac{1}{\varepsilon}|\eta_\varepsilon|^3 \geqq |p_\varepsilon^x|, |p_\varepsilon^y| &\geqq c_0\frac{1}{\varepsilon}|\eta_\varepsilon|^3, \\ p_\varepsilon^x + p_\varepsilon^y &= \frac{1}{\varepsilon}|\eta_\varepsilon|^4\left(\nabla\phi(\hat{x}_\varepsilon) + \nabla\phi(\hat{y}_\varepsilon)\right), \\ |p_\varepsilon^x + p_\varepsilon^y| \leqq C\frac{1}{\varepsilon}|\eta_\varepsilon|^4 &\leqq C\frac{1}{\varepsilon}\Xi(\hat{x}_\varepsilon, \hat{y}_\varepsilon) \longrightarrow 0, \\ \left|\frac{p_\varepsilon^x}{|p_\varepsilon^x|} + \frac{p_\varepsilon^y}{|p_\varepsilon^y|}\right| \leqq \frac{2}{\min(|p_\varepsilon^x|, |p_\varepsilon^y|)}|p_\varepsilon^x + p_\varepsilon^y| &\leqq C|\eta_\varepsilon|. \end{aligned} \tag{10}$$

We use (8) and Proposition 3.2(ii) of [10] and we observe that F satisfies condition (F4) of [10] (cf. Lemma 5.1 of [10]) to conclude that

$$\begin{aligned} &F(p_\varepsilon^x, X_\varepsilon) - F(-p_\varepsilon^y, -Y_\varepsilon) \\ &\quad\geqq -K_1\nu_0\left|\frac{p_\varepsilon^x}{|p_\varepsilon^x|} + \frac{p_\varepsilon^y}{|p_\varepsilon^y|}\right|^2 - K_2\mu - K_3\zeta\left|\frac{p_\varepsilon^x}{|p_\varepsilon^x|} + \frac{p_\varepsilon^y}{|p_\varepsilon^y|}\right| - K_4\left|p_\varepsilon^x + p_\varepsilon^y\right|, \end{aligned}$$

with $\nu_0 = C\frac{1}{\varepsilon}|\eta_\varepsilon|^2$, $\mu = C\left(\frac{1}{\varepsilon}|\eta_\varepsilon|^4 + \frac{\lambda}{\varepsilon^2}\right)$ and $\zeta = C\frac{1}{\varepsilon}|\eta_\varepsilon|^3$. Thus

$$F(p_\varepsilon^x, X_\varepsilon) - F(-p_\varepsilon^y, -Y_\varepsilon) \geqq -C\left(\frac{1}{\varepsilon}|\eta_\varepsilon|^4 + \frac{\lambda}{\varepsilon^2}\right),$$

and we conclude that

$$\lim_{\varepsilon\to 0} F(p_\varepsilon^x, X_\varepsilon) - F(-p_\varepsilon^y, -Y_\varepsilon) \geqq 0, \tag{11}$$

provided that λ is chosen such that $\lambda/\varepsilon^2 \to 0$.

Next we obtain

$$\begin{aligned} &\big|\, |p_\varepsilon^x| q_\infty(\hat{x}_\varepsilon) + |p_\varepsilon^y| q_\infty(\hat{y}_\varepsilon)\, \big| \\ &\qquad \leqq |p_\varepsilon^x + p_\varepsilon^y|\, ||q_\infty||_{\mathrm{L}^\infty(\Omega)} + |p_\varepsilon^x|\, |\eta_\varepsilon|\, ||\nabla q_\infty||_{\mathrm{L}^\infty(\mathrm{conv}\,\Omega)} \longrightarrow 0 \end{aligned} \tag{12}$$

by (10).

Finally we estimate

$$\begin{aligned} &|p_\varepsilon^x|\,(\psi_1 - H\psi_1)\,(\hat{x}_\varepsilon, \hat{t}_\varepsilon) - |p_\varepsilon^y|\,(\psi_2 - H\psi_2)\,(\hat{y}_\varepsilon, \hat{t}_\varepsilon) \\ &\quad \geqq -|p_\varepsilon^x + p_\varepsilon^y|\,\big(||\psi_1||_{\mathrm{L}^\infty(\Omega)} + ||H\psi_1||_{\mathrm{L}^\infty(\Omega)}\big) \\ &\qquad + |p_\varepsilon^y|\,\big((\psi_1 - H\psi_1)(\hat{x}_\varepsilon, \hat{t}_\varepsilon) - (\psi_2 - H\psi_2)(\hat{y}_\varepsilon, \hat{t}_\varepsilon)\big)\,. \end{aligned}$$

We observe that

$$(\psi_1 - H\psi_1)(\hat{x}_\varepsilon, \hat{t}_\varepsilon) - (\psi_2 - H\psi_2)(\hat{y}_\varepsilon, \hat{t}_\varepsilon) \longrightarrow (\psi_1 - \psi_2 - H(\psi_1 - \psi_2))\,(x_0, t_0).$$

Since $(\psi_1 - \psi_2)\,(x_0, t_0) = \sup_\Omega\,(\psi_1 - \psi_2)\,(\cdot, t_0)$ by (4), 2.3(4) implies that $0 \leqq (\psi_1 - \psi_2 - H(\psi_1 - \psi_2))\,(x_0, t_0)$. Since, by assumption, one of ψ_1 and ψ_2 is Lipschitz continuous in space, we find by (3) and (6) with $\delta = 0$ that one of $|p_\varepsilon^x|$ and $|p_\varepsilon^y|$ is bounded. But, since $p_\varepsilon^x - p_\varepsilon^y \to 0$, both $|p_\varepsilon^y|$ and $|p_\varepsilon^x|$ remain bounded. Thus we eventually find that

$$\lim_{\varepsilon\to 0} |p_\varepsilon^x|\,(\psi_1 - H\psi_1)\,(\hat{x}_\varepsilon, \hat{t}_\varepsilon) - |p_\varepsilon^y|\,(\psi_2 - H\psi_2)\,(\hat{y}_\varepsilon, \hat{t}_\varepsilon) \geqq 0. \tag{13}$$

Putting (11)–(13) together leads to a contradiction. □

2.13. Remark. This comparison principle immediately implies uniqueness of weak viscosity solutions for Lipschitz continuous initial data, since the solutions constructed in 2.11 are uniformly Lipschitz continuous in space.

We use the comparison principle now to prove that the solution operator of Theorem 2.9 is a contraction semi-group on $\mathrm{L}^\infty(\Omega)$.

2.14. Corollary. *Let ψ_1 and ψ_2 be two weak viscosity solutions as defined in* 2.6 *with Lipschitz continuous initial data $\psi_{0,1}$ and $\psi_{0,2}$, respectively. Then for all $t > 0$,*

$$\|\ \psi_2(t) - \psi_1(t)\ \|_{\mathrm{L}^\infty(\Omega)} \leqq \|\ \psi_{0,2} - \psi_{0,1}\ \|_{\mathrm{L}^\infty(\Omega)}\ .$$

Proof. Since $H1 = 1$, we observe that $\psi_2 + \|\ \psi_{0,2} - \psi_{0,1}\ \|_{\mathrm{L}^\infty(\Omega)}$ is a weak viscosity solution with initial data $\psi_{0,2} + \|\ \psi_{0,2} - \psi_{0,1}\ \|_{\mathrm{L}^\infty(\Omega)} \geqq \psi_{0,1}$. The Comparison Principle 2.12 yields

$$\psi_2 - \psi_1 \geqq \|\ \psi_{0,2} - \psi_{0,1}\ \|_{\mathrm{L}^\infty(\Omega)},$$

and by symmetry, we get the assertion. □

2.15. Conclusion of the Proof of Theorem 2.9. So far, we have only established that there exists a unique weak viscosity solution. Therefore it remains to show that there also exists a proper viscosity solution. This time we approximate

$$\psi_t - \sigma\,\mathrm{tr}\left(\left(I - \frac{\nabla\psi \otimes \nabla\psi}{|\nabla\psi|^2}\right) D^2\psi\right) + |\nabla\psi|(\psi - q) = 0$$

by choosing smooth function a^δ_{ij} such that

$$a^\delta(p) = 2\delta I + \sigma \int_{\mathbb{R}^2} |q| \, \eta_\delta(p-q)dq \quad \text{for } |p| \leqq \tfrac{1}{2}, \tag{1}$$

where $\eta_\delta(p) := \delta^{-2}\eta(\frac{p}{\delta})$ with $\eta \in C_0^\infty(\mathbb{R}^2)$, $\eta \geqq 0$, and $\int_{\mathbb{R}^2} \eta = 1$. Further, we assume that

$$\begin{aligned} a^\delta_{ij}(p) &= 2\delta\delta_{ij} + \sigma\left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right), \\ \partial_l a^\delta_{ij}(p) &= \sigma\left(- \frac{\delta_{il} p_j}{|p|} - \frac{\delta_{jl} p_i}{|p|} + 2\frac{p_i p_j p_l}{|p|^4}\right) \end{aligned} \tag{2}$$

for $|p| > 1$. On the whole of $\mathbb{R}^2$, we assume that $a^\delta(p) \geqq 2\delta I$ and $a^\delta_{ij}(p) \to \delta_{ij} - p_i p_j/|p|^2$ for $p \neq 0$. With this choice, we observe that

$$(\lim_{\delta\to 0})^\star_\star (a^\delta_{ij}(p)X_{ij}) = \left(\left(\delta_{ij} - \frac{p_i p_j}{|p|^2} X_{ij}\right)^\star_\star\right.$$

When passing to the limit as outlined in Lemma 6.1 of [5], this identity ensures that we obtain a proper viscosity solution of our equation. We obtain the initial-boundary-value problem

$$\begin{aligned} \psi_t - a^\delta_{ij}(\nabla\psi)\partial_{ij}\psi + f_\delta(\nabla\psi)(\psi - H\psi - q_\infty) = 0 \quad &\text{in } \Omega\times]0,T[, \\ \partial_\nu\psi = 0 \quad \text{on } \partial\Omega\times]0,T[, \quad \psi = \psi_{0,\delta} \quad &\text{in } \Omega\times\{0\}, \end{aligned} \tag{3}$$

where f_δ is smooth and satisfies

$$f_\delta(p) \to |p|, \quad f_\delta(p) \geqq \delta, \quad f_\delta(0) = \delta, \quad f_\delta(p) = |p| \quad \text{for } |p| \geqq 1.$$

The initial data are chosen to satisfy $\psi_{0,\delta} \in C^\infty(\overline{\Omega})$ with $\| \psi_{0,\delta} \|_{C^{0,1}(\overline{\Omega})} \leqq C(\Lambda)$ and $\partial_\nu\psi_{0,\delta} = 0$ on $\partial\Omega$, and $\psi_{0,\delta} \to \psi_0$ uniformly on $\overline{\Omega}$. As before, this problem admits a solution $\psi_\delta \in H^{2+\alpha,1+\alpha/2}(\Omega_T) \cap C^\infty_{\text{loc}}(\Omega_T)$. Again as before, one can establish

$$\| \psi_\delta \|_{L^\infty(\Omega_T)}, \; \| \nabla\psi_\delta \|_{L^\infty(\Omega_T)} \leqq C(\Lambda),$$

for $0 < \delta < c_0(\Lambda, T)$.

Since this approximation does not maintain the partial divergence form of our original equation, we have to proceed in a different way to estimate the time derivative of ψ. To this end, we differentiate (3) with respect to time, set $\gamma := \partial_t\psi \in H^{\alpha,\alpha/2}(\Omega_T) \cap C^\infty(\overline{\Omega_T})$ and obtain

$$\begin{aligned} &\gamma_t - a^\delta_{ij}(\nabla\psi)\partial_{ij}\gamma - \partial_l a^\delta_{ij}(\nabla\psi)\partial_{ij}\psi \, \partial_l\gamma \\ &\quad + \partial_l f_\delta(\nabla\psi)(\psi - H\psi - q_\infty)\partial_l\gamma + f_\delta(\nabla\psi)(\gamma - H\gamma) = 0 \quad \text{in } \Omega\times]0,T[, \\ &\qquad\qquad \partial_\nu\gamma = 0 \quad \text{on } \partial\Omega\times]0,T[. \end{aligned}$$

Certainly, this equation admits a maximum principle, and we conclude that

$$\| \partial_t \psi_\delta \|_{L^\infty(\Omega_T)} \leqq \| \partial_t \psi_\delta(.,0) \|_{L^\infty(\Omega)} \leqq C(\Lambda)(1+ \| \psi_{0,\delta} \|_{C^{1,1}(\overline{\Omega})}).$$

This yields existence of a solution for initial data $\psi_0 \in C^{1,1}(\overline{\Omega})$. Since according to Corollary 2.14 the solution operator is an L^∞ contraction, we obtain a solution for all $\psi_0 \in C^{0,1}(\overline{\Omega})$, concluding the proof. □

2.16. Proposition. *Let ψ be as in Theorem 2.9. Then $\psi(t)$ tends uniformly on $\overline{\Omega}$ to a stationary state.*

Proof. From 2.9(1), we see that $\{\psi(t)\}_{t \geqq 0}$ is uniformly bounded in $C^{0,1}(\overline{\Omega})$. Therefore, there exists a sequence $t_j \to \infty$, such that $\psi(t_j) \to \psi_\infty$ uniformly on $\overline{\Omega}$. We define $\psi_j(t) := \psi(t_j + t)$. These are solutions with initial data $\psi(t_j)$. Certainly, they satisfy the bounds 2.9(1). Hence they tend uniformly on $\overline{\Omega_T}$ for all $T > 0$ to the solution $\psi_\infty(.)$ with initial data ψ_∞. We obtain

$$\int_{\Omega_\infty} |\partial_t \psi_\infty|^2 \leqq \liminf_{j\to\infty} \int_{\Omega_\infty} |\partial_t \psi_j|^2 = \liminf_{j\to\infty} \int_{t_j}^\infty \int_\Omega |\partial_t \psi|^2 = 0;$$

hence $\psi_\infty(t) = \psi_\infty$ for all $t \geqq 0$, that is, ψ_∞ is a stationary state. Since by Corollary 2.14 the solution operator is an L^∞-contracting semi-group, we have that

$$\| \psi(t) - \psi_\infty \|_{L^\infty(\Omega)} \leqq \| \psi(t_j) - \psi_\infty \|_{L^\infty(\Omega)} \to 0 \quad \text{for } t > t_j,$$

and hence get the convergence of the whole family. □

2.17. Remark. Finally, the above results can be extended to the case of initial data which are only continuous. Existence of solutions for continuous initial data is immediate following Corollary 2.14.

Uniqueness and a comparison result are obtained as follows. For $\psi_{0,1} \leqq \psi_{0,2} \in C^0(\overline{\Omega})$, we choose $\psi_{0,i,\varepsilon} \in C^{0,1}(\overline{\Omega})$ with $\psi_{0,i,\varepsilon} \to \psi_{0,i}$ in $L^\infty(\Omega)$ for $i = 1, 2$, and without loss of generality we may assume that $\psi_{0,1,\varepsilon} \leqq \psi_{0,1} \leqq \psi_{0,2} \leqq \psi_{0,2,\varepsilon}$. For two solutions ψ_i with initial data $\psi_{0,i}$ for $i = 1, 2$, we conclude from Theorem 2.12 that $\psi_{1,\varepsilon} \leqq \psi_1$ and $\psi_2 \leqq \psi_{2,\varepsilon}$, where $\psi_{i,\varepsilon}$ are the unique solutions Lipschitz continuous in space obtained above. First, if $\psi_{0,1} = \psi_{0,2}$, we get from Corollary 2.14 that $\| \psi_1 - \psi_2 \|_{L^\infty(0,T;L^\infty(\Omega))} \leqq \| \psi_{1,\varepsilon} - \psi_{2,\varepsilon} \|_{L^\infty(0,T;L^\infty(\Omega))} \leqq \| \psi_{0,1,\varepsilon} - \psi_{0,2,\varepsilon} \|_{L^\infty(\Omega)} \to 0$, proving uniqueness. In any case, this implies that $\psi_{i,\varepsilon} \to \psi_i$ uniformly on Ω_T, hence $\psi_1 \leqq \psi_2$. Now the contraction principle extends immediately.

Lastly, let ψ be a solution with initial data ψ_0. Again we choose $\psi_{0,\varepsilon} \in C^{0,1}(\overline{\Omega})$ converging uniformly on $\overline{\Omega}$ to ψ_0 and consider the Lipschitz continuous in space solution ψ_ε with initial data $\psi_{0,\varepsilon}$. From Corollary 2.14, we get for any ε that $\| \psi(t) - \psi(s) \|_{L^\infty(\Omega)} \leqq 2 \| \psi_0 - \psi_{0,\varepsilon} \|_{L^\infty(\Omega)} + \| \psi_\varepsilon(t) - \psi_\varepsilon(s) \|_{L^\infty(\Omega)}$. Since $\psi_\varepsilon(t)$ is a Cauchy family for all ε, we conclude that $\psi(t)$ has a unique asymptotic limit as $t \to \infty$.

3. Applications to the Mean-Field Model

3.1. The Mean-Field Model. Throughout this section we assume in addition that Ω is simply-connected. In this case we study the relationship between the problem 2.1(1,2,3) and the two-dimensional mean-field model without curvature:

$$\begin{gathered}\partial_t \boldsymbol{\omega} - \nabla^{\perp}(|\boldsymbol{\omega}|\nabla^{\perp} \cdot \boldsymbol{H}) = 0 \quad \text{in } \Omega\times]0,T[,\\ \boldsymbol{\omega} \times \nu = 0 \quad \text{on } \partial\Omega\times]0,T[,\\ \boldsymbol{\omega} = \boldsymbol{\omega}_0 \quad \text{on } \Omega \times \{0\}\\ -\Delta \boldsymbol{H} + \boldsymbol{H} = \boldsymbol{\omega} \quad \text{in } \Omega, \quad \nabla^{\perp} \cdot \boldsymbol{H} = 0 \quad \text{in } \overline{\Omega}^c, \quad [\boldsymbol{H} \times \nu] = 0 \quad \text{on } \partial\Omega,\\ \nabla \cdot \boldsymbol{H} = 0 \quad \text{in } \mathbb{R}^2,\\ \boldsymbol{H} - \boldsymbol{H}_\infty \in \mathrm{L}^2(\mathbb{R}^2).\end{gathered} \tag{1}$$

This is a hyperbolic-elliptic system of equations. In this section, we approximate this system by introducing a viscosity and show that the limit is unique as the viscosity tends to zero. The key point in this procedure is that $\boldsymbol{\omega}$ and $\boldsymbol{H}$ can be represented as curls if Ω is simply-connected.

3.2. Definition. We say that $\boldsymbol{\omega} \in L^1(\Omega; \mathbb{R}^2)$ is *locally weakly divergence-free* ($\nabla \cdot \boldsymbol{\omega} = 0 \quad \text{in } \Omega$), if for all $\varphi \in C_0^1(\Omega)$,

$$\int_\Omega \boldsymbol{\omega} \cdot \nabla\varphi = 0.$$

3.3. Theorem. *Let $\boldsymbol{\omega}_0 \in \mathrm{L}^\infty(\Omega)$ be locally weakly divergence-free in Ω and let $\boldsymbol{H}_\infty$ be locally divergence-free in $\mathbb{R}^2$ and harmonic outside a ball. Then there exists a unique solution $\boldsymbol{\omega}_\delta \in \mathrm{L}^2(0,T; H^{1,2}(\Omega)) \cap \mathrm{L}^\infty(\Omega_T)$ and $\boldsymbol{H}_\delta \in \mathrm{L}^2(0,T; H^{1,2}_{\mathrm{loc}}(\mathbb{R}^2))$ of the system*

$$\begin{gathered}\partial_t \boldsymbol{\omega} - \delta\Delta\boldsymbol{\omega} - \nabla^{\perp}(|\boldsymbol{\omega}|\nabla^{\perp} \cdot \boldsymbol{H}) = 0 \quad \textit{in } \Omega\times]0,T[,\\ \boldsymbol{\omega} \times \nu = 0 \quad \textit{on } \partial\Omega\times]0,T[,\\ \boldsymbol{\omega} = \boldsymbol{\omega}_0 \quad \textit{in } \Omega \times \{0\},\\ -\Delta \boldsymbol{H} + \boldsymbol{H} = \boldsymbol{\omega} \quad \textit{in } \Omega, \quad \nabla^{\perp} \cdot \boldsymbol{H} = 0 \quad \textit{in } \overline{\Omega}^c,\\ [\boldsymbol{H} \times \nu] = 0 \quad \textit{on } \partial\Omega, \quad \nabla \cdot \boldsymbol{H} = 0 \quad \textit{in } \mathbb{R}^2,\\ \boldsymbol{H} - \boldsymbol{H}_\infty \in \mathrm{L}^2(\mathbb{R}^2).\end{gathered} \tag{1}$$

Moreover, $(\boldsymbol{\omega}_\delta, \boldsymbol{H}_\delta)_\delta$ has a unique weak* limit $(\boldsymbol{\omega}, \boldsymbol{H})$ in $\mathrm{L}^\infty(\Omega_T)$ as δ tends to 0.

3.4. Remark. The essence of Theorem 3.3 is that the viscous approximation 3.3(1) has a unique solution and a unique limit $(\boldsymbol{\omega}, \boldsymbol{H})$. We obtain uniqueness of $\boldsymbol{\omega}$ in identifying it with the curl of the unique solution ψ of 2.1(1). In addition, we may conclude that $(\boldsymbol{\omega}, \boldsymbol{H})$ satisfies the differential equations of 3.1(1). On this basis we might define the unique zero-viscosity limit to be the solution of the hyperbolic-elliptic system 3.1(1). But this is unsatisfactory since we do not know in which sense the boundary conditions for $\boldsymbol{\omega}$ are met. In addition, we cannot prove uniqueness of weak solutions of 2.1(1), since this hyperbolic system only admits a small variety of entropies, as we will point out in the next subsection. Entropies were introduced in [11] to get unique entropy solutions for scalar hyperbolic equations.

3.5. Remark. We now examine the variety of entropies which are admitted by the hyperbolic system 3.1(1). With $\varphi := (-\boldsymbol{\omega}_2, \boldsymbol{\omega}_1)$, $3.1(1)_1$ takes the form

$$\partial_t \varphi + \nabla(|\varphi| V) = 0,$$

or likewise

$$\partial_t \varphi_i + \partial_{x_j} f_i^j(\varphi) = 0, \tag{1}$$

where $f_i^j(\varphi) = \delta_{ij}|\varphi| V$ for some V not depending on φ.

(η, q) is called an entropy pair (cf. [11]) if η is convex. Multiplying equation (1) by $\partial_i \eta(\varphi)$ yields $\eta(\varphi)_t + \partial_{x_j} q^j(\varphi) = g(\varphi)$ for some g. Hence, with $f(\varphi) = |\varphi| V$,

$$\partial_k q^j(\varphi) \partial_j \varphi_k = \partial_i \eta(\varphi) \partial_k f_i^j(\varphi) \partial_j \varphi_k = \partial_j \eta(\varphi) \partial_k f(\varphi) \partial_j \varphi_k \tag{2}$$

has to be satisfied.

If we require (2) to hold for any function φ, we get $\partial_k q^j(\varphi) = \partial_j \eta(\varphi) \partial_k f(\varphi)$. Differentiating (2) with respect to φ_l and observing that $f(\varphi) = |\varphi| V$, we infer that $\partial_{jl} \eta(\varphi) \varphi_k$ is symmetric in k and l, or likewise $\nabla \partial_j \eta(\varphi) \cdot (-\varphi_2, \varphi_1) = 0$, and hence $\nabla \eta$ is rotationally symmetric. Differentiating, we conclude that $\partial_{ij} \eta(\varphi) = \overline{\eta_j}(|\varphi|) \varphi_i$ is symmetric in i and j. This is impossible unless η is affine.

From $\nabla \cdot \boldsymbol{\omega} = 0$ now we get that $\nabla \varphi$ is symmetric, and (2) is satisfied for any solution if the symmetric parts of $\partial_j q^k$ and $\partial_j \eta \partial_k f$ coincide. For $a_{ij} := \partial_i \eta \partial_j f$, this gives

$$\partial_1 q_1 = a_{11}, \quad \partial_2 q_2 = a_{22}, \quad \partial_1 q_2 + \partial_2 q_1 = a_{12} + a_{21}.$$

Differentiating, these equations yields

$$0 = \partial_{221} q_1 + \partial_{112} q_2 - \partial_{12}(\partial_1 q_2 + \partial_2 q_1) = \partial_{22} a_{11} + \partial_{11} a_{22} - \partial_{12}(a_{12} + a_{21}).$$

Recalling that $a_{ij} = \partial_i \eta \partial_j f$, we get

$$\partial_{11} \eta \partial_{22} f + \partial_{22} \eta \partial_{11} f - 2 \partial_{12} \eta \partial_{12} f = 0.$$

Observing that $f(\varphi) = |\varphi| V$, we compute

$$\partial_{11} f = \frac{\varphi_2^2}{|\varphi|^3} V, \quad \partial_{12} f = -\frac{\varphi_1 \varphi_2}{|\varphi|^3} V, \quad \partial_{22} f = \frac{\varphi_1^2}{|\varphi|^3} V.$$

Combining these equations yields

$$0 = \varphi_1^2 \partial_{11} \eta + \varphi_2^2 \partial_{22} \eta + 2 \varphi_1 \varphi_2 \partial_{12} \eta = (\varphi_1, \varphi_2) \cdot D^2 \eta(\varphi_1, \varphi_2),$$

and thus, up to a constant, η is homogeneous of degree 1. Therefore the system 3.1(1)$_1$, 3 admits only a small variety of entropies. For example $\eta(\varphi) := |\varphi - k|$ is not part of an entropy pair, unless k is zero.

3.6. Proof of Theorem 3.3. First, we represent $\boldsymbol{H}_\infty = \nabla^\perp q_\infty$, and since $\boldsymbol{H}_\infty$ is only relevant near infinity, we assume that q_∞ satisfies 2.3(2). Since Ω is simply-connected, we can as well choose $\psi_0 \in C^{0,1}(\overline{\Omega})$ with $\nabla^\perp \psi_0 = \boldsymbol{\omega}_0$. Then the system

$$\begin{gathered} \psi_t - \delta \Delta \psi - |\nabla \psi|(q - \psi) = 0 \quad \text{in } \Omega \times]0, T[, \\ \partial_\nu \psi = 0 \quad \text{on } \partial\Omega \times]0, T[, \quad \psi = \psi_0 \quad \text{in } \Omega \times \{0\}, \\ -\Delta q + \chi_\Omega q = \chi_\Omega \psi \quad \text{in } \mathbb{R}^2, \\ \nabla(q - q_\infty) \in \mathrm{L}^2(\mathbb{R}^2) \end{gathered} \tag{1}$$

has a solution. This can easily be seen by approximating ψ_0 and $|\cdot|$ smoothly and passing to the limit. Moreover, as in the proof of Theorem 2.9, we can establish the bounds

$$\| \psi_\delta \|_{\mathrm{L}^\infty(\Omega_T)}, \ \| \nabla \psi_\delta \|_{\mathrm{L}^\infty(\Omega_T)}, \| \partial_t \psi_\delta \|_{\mathrm{L}^2(\Omega_T)}, \ \| \psi_\delta \|_{H^{\alpha,\frac{\alpha}{2}}(\overline{\Omega_T})} \leqq C, \tag{2}$$

where C is independent of δ. The proof is even simpler, since now $a^\delta(p) = \delta I$ is independent of p. In defining $\boldsymbol{\omega}_\delta := \nabla^\perp \psi_\delta$ and $\boldsymbol{H}_\delta := \nabla^\perp q_\delta$, we obtain a solution of the system 3.3(1).

To prove uniqueness of this solution, we assume that $\boldsymbol{\omega}_i \in \mathrm{L}^2(0, T; H^{1,2}(\Omega)) \cap \mathrm{L}^\infty(\Omega_T)$ are two solutions of 3.3(1) for $i = 1, 2$. We define $\boldsymbol{\omega} := \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1$ and $\boldsymbol{H} := \boldsymbol{H}_2 - \boldsymbol{H}_1$. We want to use $\boldsymbol{\omega}$ as a test function in 3.3(1). To this end we approximate $\boldsymbol{\omega}$ by $\boldsymbol{\eta} = \boldsymbol{\eta}_n$ with $\boldsymbol{\eta} \in C_0^2(\Omega)$. Since $\boldsymbol{\omega} \times \nu = 0$ on $\partial\Omega$, we may choose $\boldsymbol{\eta}$ with the property that $\nabla^\perp \cdot \boldsymbol{\eta} \to \nabla^\perp \cdot \boldsymbol{\omega}$ strongly in $\mathrm{L}^2(\Omega)$. Since $\boldsymbol{\omega}$ is weakly divergence-free in Ω, we find that

$$\int_\Omega \nabla \boldsymbol{\omega} \cdot \nabla \boldsymbol{\eta} = -\int_\Omega \boldsymbol{\omega} \Delta \boldsymbol{\eta} = -\int_\Omega \boldsymbol{\omega}(\nabla^\perp \nabla^\perp \cdot \boldsymbol{\eta} + \nabla(\nabla \cdot \boldsymbol{\eta})) = \int_\Omega \nabla^\perp \cdot \boldsymbol{\omega} \, \nabla^\perp \cdot \boldsymbol{\eta}.$$

Thus, if we use $\boldsymbol{\eta}$ as a test function and then pass to the limit, we obtain

$$\begin{aligned} \tfrac{1}{2} \int_\Omega |\boldsymbol{\omega}(t_0)|^2 &+ \delta \iint_{\Omega_{t_0}} |\nabla^\perp \cdot \boldsymbol{\omega}|^2 \\ &= -\iint_{\Omega_{t_0}} \Big(|\boldsymbol{\omega}_2| \nabla^\perp \cdot \boldsymbol{H}_2 - |\boldsymbol{\omega}_1| \nabla^\perp \cdot \boldsymbol{H}_1 \Big) \nabla^\perp \cdot \boldsymbol{\omega} \\ &\leqq C \Big(\iint_{\Omega_{t_0}} |\boldsymbol{\omega}| |\nabla^\perp \cdot \boldsymbol{\omega}| + \iint_{\Omega_{t_0}} |\nabla^\perp \cdot \boldsymbol{H}| |\nabla^\perp \cdot \boldsymbol{\omega}| \Big) \\ &\leqq \tfrac{1}{2} \delta \iint_{\Omega_{t_0}} |\nabla^\perp \cdot \boldsymbol{\omega}|^2 + C(\delta) \iint_{\Omega_{t_0}} (|\boldsymbol{\omega}|^2 + |\nabla^\perp \cdot \boldsymbol{H}|^2), \end{aligned} \tag{3}$$

since $\boldsymbol{\omega}_i$ and consequently $\boldsymbol{H}_i$ are in $\mathrm{L}^\infty(\Omega_T)$.

Since $\boldsymbol{H}$ and $\boldsymbol{\omega}$ are divergence-free in $\mathbb{R}^2$ and Ω, respectively, we can write $\boldsymbol{H} = \nabla^\perp q$ and $\boldsymbol{\omega} = \nabla^\perp \psi$, with $\nabla q \in \mathrm{L}^2(\mathbb{R}^2)$ since $\boldsymbol{H} \in \mathrm{L}^2(\mathbb{R}^2)$. After adding appropriate constants, we get $-\Delta q + \chi_\Omega q = \chi_\Omega \psi$ in $\mathbb{R}^2$ and $\int_\Omega \psi = 0$. Multiplying (3) by $-\Delta q$ and integrating the product yields

$$\int_{\mathbb{R}^2} |\Delta q|^2 + \int_{\mathbb{R}^2} |\nabla q|^2 = -\int_\Omega \psi \Delta q.$$

After applying Cauchy's inequality, we obtain

$$\int_\Omega |\nabla^\perp \cdot H|^2 = \int_\Omega |\Delta q|^2 \leqq \int_\Omega |\psi|^2 \leqq C \int_\Omega |\boldsymbol{\omega}|^2.$$

Using this estimate in (3) yields

$$\int_\Omega |\boldsymbol{\omega}(t_0)|^2 \leqq C(\delta) \iint_{\Omega_{t_0}} |\boldsymbol{\omega}|^2;$$

hence Gronwall's Lemma implies that $\boldsymbol{\omega} = 0$, thereby implying uniqueness.

Certainly, $(\boldsymbol{\omega}_\delta, \boldsymbol{H}_\delta)_\delta$ has a subsequence converging in the weak* topology of $\mathrm{L}^\infty(\Omega_T)$, and the limit $(\boldsymbol{\omega}, \boldsymbol{H})$ is uniquely determined by $(\boldsymbol{\omega}, \boldsymbol{H}) = (\nabla^\perp \psi, \nabla^\perp q)$, where (ψ, q) is both the limit of $(\psi_\delta, q_\delta)_\delta$ and the unique viscosity solution of 2.1(1,2,3) for $\sigma = 0$. □

4. Special Solutions of the Stationary Problem

4.1. The Stationary Problem. In this section we assume that $\sigma = 0$ and that Ω is bounded. We construct special solutions to the stationary problem corresponding to 2.1(1),(3), i.e.,

$$\begin{aligned} |\nabla \psi| \, (q - \psi) = 0 \quad & \text{in } \Omega, \\ -\Delta q + \chi_\Omega q = \chi_\Omega \psi \quad & \text{in } \mathbb{R}^2, \\ \nabla (q - q_\infty) \in \mathrm{L}^2(\mathbb{R}^2). & \end{aligned} \tag{1}$$

In view of the regularity of the time-dependent solutions, we will look for a Lipschitz continuous ψ.

The domain Ω is decomposed into sets, where either $q = \psi$ or $\nabla \psi = 0$. We set $C_0 := \{q = \psi\} \cap \Omega$. This set is relatively closed in Ω. The complement $\Omega \backslash C_0$ is thus open, and since $q \neq \psi$ there, we find $\nabla \psi = 0$ in this set. Thus on any connected component Ω_i of $\Omega \backslash C_0$ we find that ψ has a constant value c_i. We know that $\overline{\Omega_i} \cap \Omega_j = \emptyset$ if $i \neq j$. Now, $\overline{\Omega_i} \subset \Omega$ leads to a contradiction: We observe that $-\Delta q + (q - c_i) = 0$ in Ω_i, and since $\partial \Omega_i \subset C_0 \cap \overline{\Omega_i}$, we find that $q = \psi = c_i$ on $\partial \Omega_i$. This implies that $q = c_i = \psi$ in Ω_i, which contradicts the definition of Ω_i. Hence we have shown that $\partial \Omega_i \cap \partial \Omega \neq \emptyset$.

We now want to construct special solutions that admit a functional dependence of ψ on q, which is consistent with (1), i.e., we seek solutions that satisfy

$$\begin{aligned} \psi &= q - f(q) \quad \text{in } \Omega, \\ -\Delta q + f(q)\chi_\Omega &= 0 \quad \text{in } \mathbb{R}^2, \\ \nabla(q - q_\infty) &\in \mathrm{L}^2(\mathbb{R}^2). \end{aligned} \tag{2}$$

For such a solution $(1)_1$ reads $(f'(q) - 1)f(q)|\nabla q| = 0$. We therefore set

$$f(q) := f_{\alpha\beta}(q) := (q-\alpha)_- + (q-\beta)_+$$

with $-\infty \leqq \alpha < \beta \leqq +\infty$. For a solution q of (2) we define the superconducting phases

$$\Omega_- := \{q < \alpha\} \cap \Omega, \quad \Omega_+ := \{q > \beta\} \cap \Omega.$$

Then $\psi = \alpha$ in Ω_-, $\psi = \beta$ in Ω_+ and $\alpha \leqq q \leqq \beta$ in $\Omega \backslash (\Omega_- \cup \Omega_+)$.

We finally point out that there are solutions of (1) which do not satisfy (2): since $\Delta q = 0$ outside Ω, we may vary Ω without changing the solution as long as we do not change $\partial\Omega \cap \overline{\Omega_\pm}$. But, since $\partial\Omega \cap \overline{\Omega_\pm} \neq \partial\Omega$, this allows us to enlarge Ω so that $\alpha \leqq q \leqq \beta$ in $\Omega \backslash (\Omega_- \cup \Omega_+)$ is violated.

We now prove existence of such solutions. In particular, we show that for certain values of α and β there exist solutions which have one or two nonempty phases. The properties of the solution depend on the applied field q_∞, which we assume to be normalized as in 2.3(a) (cf. 2.4 as well).

4.2. Proposition. (a) *If $\int_{\mathbb{R}^2} \Delta q_\infty > 0$, then for all $-\infty \leqq \alpha < \beta < +\infty$, there exists a unique solution $q \in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ of (4.1(2)). $q \in H^{3,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{H}^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $1 \leqq p < \infty$ and $\psi \in C^{0,1}(\Omega)$. Moreover $\Omega_+ \neq \emptyset$.*

If $\{q_\infty < 0\} \cap \Omega \neq \emptyset$, then there exists a positive constant $c_0(q_\infty, \Omega)$, such that

$$\Omega_- \neq \emptyset \quad \textit{for all } \beta - \alpha < c_0, \quad \Omega_- = \emptyset \quad \textit{for all } \beta - \alpha \geqq c_0.$$

If $q_\infty \geqq 0$ in Ω, then

$$\Omega_- = \emptyset \quad \textit{for all } \alpha < \beta.$$

(b) *If $\int_{\mathbb{R}^2} \Delta q_\infty < 0$, then similar statements hold for $-\infty < \alpha < \beta \leqq +\infty$.*

(c) *If $\int_{\mathbb{R}^2} \Delta q_\infty = 0$, then for all $-\infty < \alpha < \beta < +\infty$ there exists a solution $q \in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ of (4.1(2)). This solution satisfies the same regularity properties as in (a). In addition, $\beta - \alpha < \operatorname{osc}_\Omega \hat{q}_\infty$ implies that $\Omega_-, \Omega_+ \neq \emptyset$ and implies the uniqueness of q, whereas $\beta - \alpha \geqq \operatorname{osc}_\Omega \hat{q}_\infty$ implies that $q = \hat{q}_\infty + \mathrm{const}$, where $\hat{q}_\infty$ is harmonic in $\mathbb{R}^2$ with $\nabla(\hat{q}_\infty - q_\infty) \in \mathrm{L}^2(\mathbb{R}^2)$ (cf. Remark 2.4).*

Proof. Assume first that $-\infty < \alpha < \beta < \infty$. We solve 4.1(2) by minimizing

$$E_{\alpha,\beta}(q) := \tfrac{1}{2} \int_{\mathbb{R}^2} |\nabla(q - q_\infty)|^2 + \int_\Omega F_{\alpha\beta}(q)$$

over $q \in q_\infty + X$, where X is as in Lemma 2.2 and where $F_{\alpha\beta}(q) := \frac{1}{2}(q-\alpha)_-^2 + \frac{1}{2}(q-\beta)_+^2 - q \cdot q_\infty$. This functional is bounded below by $-C(\Omega, q_\infty)$ and, since q_∞ is admissible with $E_{\alpha,\beta}(q_\infty) < \infty$, there exists a minimizing sequence q_n. Obviously for a subsequence $\nabla(q_n - q_\infty) \to \nabla(q - q_\infty)$ weakly in $\mathrm{L}^2(\mathbb{R}^2)$ and $(q_n - q_\infty) \to (q - q_\infty)$ strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$. Thus passage to the limit implies that q is a minimizer and consequently satisfies 4.1(2). In deriving the Euler-Lagrange equation we used the normalization $-\Delta q_\infty + q_\infty \chi_\Omega = 0$. Elliptic regularity theory then implies that $q \in H^{3,p}_{\mathrm{loc}}(\Omega) \cap \mathrm{H}^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$. Integration of 4.1(2) gives

$$\int_{\Omega_-} (q-\alpha) + \int_{\Omega_+} (q-\beta) = \int_{\mathbb{R}^2} \Delta q_\infty. \tag{1}$$

Proof of (a). Now assume that $\int_{\mathbb{R}^2} \Delta q_\infty > 0$ and $-\infty < \alpha < \beta < +\infty$. Then (1) implies $|\Omega_+| > 0$. Uniqueness of the solution follows, since $f_{\alpha\beta}$ is monotone and since $|\Omega_+| \neq 0$.

In order to obtain a solution for $-\infty = \alpha$ we study the limit $\alpha \to -\infty$, and in order to understand the existence of the Ω_--phase, we study the limit $\alpha \to \beta$. Thus we need several a priori estimates of the solution which are independent of α.

First we observe that by 2.3(5) and 2.3(4),

$$q_\infty + \alpha \leqq q = q_\infty + H(q - f_{\alpha,\beta}(q)) \leqq q_\infty + \beta. \tag{2}$$

Next we multiply 4.1(2) by $q - \tilde{q}_\infty$, where $\tilde{q}_\infty$ is a solution of $-\Delta \tilde{q}_\infty = \big(-\frac{1}{|\Omega|}\int_{\mathbb{R}^2} \Delta q_\infty\big)\chi_\Omega$ with $\nabla(\tilde{q}_\infty - q_\infty) \in \mathrm{L}^2(\mathbb{R}^2)$ (cf. Remark 2.4). Evaluating all terms, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla(q-\tilde{q}_\infty)|^2 + \int_{q\leqq\alpha, q\leqq\tilde{q}_\infty} (q-\alpha)(q-\tilde{q}_\infty) + \int_{\tilde{q}_\infty\leqq q\leqq\alpha} (q-\alpha)(q-\tilde{q}_\infty) \\
&\quad + \int_{\tilde{q}_\infty\geqq q\geqq\beta} (q-\beta)(q-\tilde{q}_\infty) + \int_{q\geqq\beta, q\geqq\tilde{q}_\infty} (q-\beta)(q-\tilde{q}_\infty) \\
&= \frac{1}{|\Omega|}\left(\int_{\mathbb{R}^2} \Delta q_\infty\right)\left(\int_{q\geqq\tilde{q}_\infty} (q-\tilde{q}_\infty) + \int_{q<\tilde{q}_\infty} (q-\tilde{q}_\infty)\right).
\end{aligned}$$

We now observe that we may estimate the modulus of the third and the forth terms of the left-hand side by $C(\Omega, q_\infty, \beta)$, by using that $|\{\tilde{q}_\infty \leqq q \leqq \alpha\}| = 0$ when $\alpha < \min \tilde{q}_\infty$. The first term of the right-hand side admits an estimate of the same type, by using (2). All the other terms have the proper sign. We eventually find that

$$\int_{\mathbb{R}^2} |\nabla(q-\tilde{q}_\infty)|^2 \leqq C(\Omega, q_\infty, \beta). \tag{3}$$

We now use a subscript to indicate dependence on α. Then the derivative $q_{\alpha,\alpha} \in X \cap H^{2,p}(\Omega)$ and we may differentiate 4.1(2) with respect to α to obtain

$$-\Delta q_{\alpha,\alpha} + (q_{\alpha,\alpha} - 1)\chi_{\Omega_-^\alpha} + q_{\alpha,\alpha}\chi_{\Omega_+^\alpha} = 0.$$

By the weak maximum principle we obtain $q_{\alpha,\alpha} \geqq 0$ and $q_{\alpha,\alpha} - 1 \leqq 0$.

We may as well show the monotonicity of q_α and $q_\alpha - \alpha$ by the following direct argument, avoiding differentiation with respect to α. Let $q := q_{\alpha'} - q_\alpha$ with $\alpha' > \alpha$. Then

$$0 = -\Delta q + \big(f_{\alpha',\beta}(q_{\alpha'}) - f_{\alpha,\beta}(q_\alpha)\big)\,\chi_\Omega \leqq -\Delta q + \big(f_{\alpha,\beta}(q_{\alpha'}) - f_{\alpha,\beta}(q_\alpha)\big)\,\chi_\Omega.$$

Since $f_{\alpha,\beta}$ is monotone, multiplication by q_- and integration implies that q_- is a nonpositive constant. But if this constant is different from 0, then q itself is a negative constant. Again using the equation for q, we obtain a contradiction since $\Omega_+^\alpha \neq \emptyset$. Thus

$$q_{\alpha'} \geqq q_\alpha \quad \text{for all } \alpha' \geqq \alpha. \tag{4}$$

Similarly we find that

$$q_{\alpha'} - \alpha' \leqq q_\alpha - \alpha \quad \text{for all } \alpha' \geqq \alpha. \tag{5}$$

We now first consider $\alpha \to -\infty$.

Due to (4) we have $\Omega_+^\alpha \subset \Omega_+^{\alpha'}$ if $\alpha \leqq \alpha'$, and thus $\lim_{\alpha\to-\infty} \Omega_+^\alpha =: \Omega_+^{-\infty}$ exists. Using (1) and (2) we obtain $\int_{\Omega_+^{-\infty}} q_\infty \geqq \int_{\mathbb{R}^2} \Delta q_\infty$. Thus $|\Omega_+^{-\infty}| \neq 0$. We conclude that $\beta \leqq q_\alpha \leqq q_\infty + \beta$ in $\Omega_+^{-\infty}$, and combining this with (3) we find a subsequence $\alpha \to -\infty$ with $\nabla(q_\alpha - q_\infty) \to \nabla(q_{-\infty} - q_\infty)$ weakly in $L^2(\mathbb{R}^2)$ and $q_\alpha \to q_{-\infty}$ strongly in any $L^p(\Omega)$ and almost everywhere. We conclude that $f_{\alpha,\beta}(q_\alpha) \to f_{-\infty,\beta}(q_{-\infty})$. Thus this approximation procedure leads to a solution of 4.1(2) for $\alpha = -\infty$. This limit is unique since $|\Omega_+^{-\infty}| \neq 0$.

Next we study $\alpha \to \beta$.

By (2), q_α converges uniformly in $\mathbb{R}^2$ to a solution $\overline{q}$ of 4.1(2) with $\alpha = \beta$. By Remark 2.3(5), $\overline{q} = q_\infty + H\beta = q_\infty + \beta$.

Now, if $\{q_\infty < 0\} \cap \Omega = \{\overline{q} - \beta < 0\} \neq \emptyset$, then for some positive $\tilde{c}_0(q_\infty, \Omega)$ we have $\{q_\alpha - \beta < -\tilde{c}_0\} \cap \Omega \neq \emptyset$ for all $\beta - \alpha \leqq \tilde{c}_0$. Since $q_\alpha - \alpha < \beta - \alpha - \tilde{c}_0 \leqq 0$ in $\{q_\alpha - \beta < -\tilde{c}_0\} \cap \Omega$ for all $\beta - \alpha \leqq \tilde{c}_0$, we have shown that $\Omega_-^\alpha \neq \emptyset$. Since $\Omega_-^\alpha \subset \Omega_-^{\alpha'}$ for $\alpha' \geqq \alpha$ by (5), there thus exists a positive critical number $c_0(q_\infty, \Omega)$, such that $\Omega_-^\alpha \neq \emptyset$ for $\beta - \alpha < c_0$ and $\Omega_-^\alpha = \emptyset$ for $\beta - \alpha \geqq c_0$.

If, to the contrary, $q_\infty \geqq 0$ in Ω, then $\overline{q} - \beta \geqq 0$. Due to (5), we have $q_\alpha - \alpha \geqq \overline{q} - \beta \geqq 0$ and we find that $\Omega_-^\alpha = \emptyset$.

This finishes the proof of (a). For (b) we argue in a similar way.

Proof of (c). Now, let $\int_{\mathbb{R}^2} \Delta q_\infty = 0$ and $-\infty < \alpha < \beta < +\infty$. Then by Remark 2.4 a function $\hat{q}_\infty$ as in (c) exists and is unique up to the addition of a constant. If now $\beta - \alpha < \operatorname{osc}_\Omega \hat{q}_\infty$, then $|\Omega_-|, |\Omega_+| = 0$ implies that $q = \hat{q}_\infty + \text{const.}$ and $\operatorname{osc}_\Omega q \leqq \beta - \alpha$. Thus $\operatorname{osc}_\Omega \hat{q}_\infty = \operatorname{osc}_\Omega q \leqq \beta - \alpha < \operatorname{osc}_\Omega \hat{q}_\infty$. This is a contradiction. We conclude that either $|\Omega_-|$ or $|\Omega_+|$ is different from 0. But in this case, (1) implies that both sets have positive measure. If $\beta - \alpha \geqq \operatorname{osc}_\Omega \hat{q}_\infty$, then $q = \hat{q}_\infty + \text{const.}$

4.3. Remark. The free boundary of the solution q of 4.1(1) has the following regularity property: the set $\{x \in \mathbb{R}^2 : \; q(x) = \alpha, \; \nabla q(x) = 0\}$ locally consists of finitely many points. The same applies for the free boundary $q = \beta$.

This result is due to CAFFARELLI & FRIEDMAN [1]. They show that if u has a bounded gradient in $B_1(0) \subset \mathbb{R}^2$ and satisfies $|\Delta u| \leqq C_1|u| + C_2|\nabla u|$ in $B_1(0)$, then $\{x \in B_{1-\eta}(0) : u(x) = 0, \ \nabla u(x) = 0\}$ is finite for any $0 < \eta < 1$.

This result applies in our case, since $\Delta q = f_{\alpha,\beta}(q)\chi_\Omega$, and $|f_{\alpha,\beta}(q)| \leqq |q-\alpha|$ in some neighbourhood of any point x_0 with $q(x_0) = \alpha$. The latter observation is true because $q \in \mathrm{H}^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ and thus q is continuous. In addition, ∇q is locally bounded.

4.4. Remark. We remark that it is possible to construct solutions of 4.1(1) which have more than two constant phases. To this end let D_1 and D_2 be two nonempty, bounded domains in $\mathbb{R}^2$ with disjoint closures, and let $-\infty < \alpha_1 < \beta_1 < +\infty$ and $-\infty < \alpha_2 < \beta_2 < +\infty$ be two choices of threshold values. Then there exists $q \in \mathrm{H}^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$, unique up to a constant, with $\nabla(q - q_\infty) \in \mathrm{L}^2(\mathbb{R}^2)$ and

$$-\Delta q + f_{\alpha_1,\beta_1}(q)\chi_{D_1} + f_{\alpha_2,\beta_2}(q)\chi_{D_2} = 0. \tag{1}$$

The proof follows the same lines as the existence proof of Proposition 4.2.

Now assume that $\int_{\mathbb{R}^2} \Delta q_\infty = 0$. We fix D_1 and choose $\beta_1 - \alpha_1 < \operatorname{osc}_{D_1} \hat{q}_\infty$, where $\hat{q}_\infty$ is as in Proposition 4.2. We define q_1 to be the solution of 4.1(2) with $\Omega = D_1$. Then by Proposition 4.2 we find that $\{x \in D_1 : q_1 < \alpha_1\}$ and $\{x \in D_1 : q_1 > \beta_1\}$ are nonempty.

Next we choose D_2 with $\overline{D_1} \cap \overline{D_2} = \emptyset$. We set $\alpha_2 := \inf_{D_2} q_1$ and we choose $\beta_2 < \sup_{D_2} q_1$. Then necessarily either $\Omega^2_- := \{x \in D_2 : q < \alpha_2\}$ or $\Omega^2_+ := \{x \in D_2 : q > \beta_2\}$ is nonempty, since otherwise $\alpha_2 \leqq q \leqq \beta_2$ and thus $f_{\alpha_2,\beta_2}(q)\chi_{D_2} = 0$ and $q = q_1$, which is impossible by the choice of β.

Now, if $\beta_2 \to \sup_{D_2} q_1$, then $q \to q_1$ locally uniformly. This follows by similar arguments as in the last part of the proof of Proposition 4.2. Since q_1 has two nonempty phases in D_1, this then implies that $\Omega^1_- := \{x \in D_1 : q < \alpha_1\}$ and $\Omega^1_+ := \{x \in D_1 : q > \beta_1\}$ are nonempty, if only β_2 is close enough to $\sup_{D_2} q_1$.

In consequence, we have constructed a solution of 4.1(1) which has at least three superconducting phases, i.e., domains where ψ takes constant values.

At the moment the domain $D_1 \cup D_2$ occupied by the superconductor is not connected. But we may even construct a connected Ω, for which q as above is a solution of 4.1(1). The idea is that since $-\Delta q = 0$ in $D_i \backslash (\Omega^i_- \cup \Omega^i_+)$ $(i = 1, 2)$ as well as in the complement of $\overline{D_1 \cup D_2}$, we may change the domain without changing the solution. Since both $\partial D_i \cap (\overline{\Omega^i_-} \cup \overline{\Omega^i_+})$ $(i = 1, 2)$, contain parts of a graph, we may connect $D_1 \cup D_2$ by a handle and q is also a solution of 4.1(1) in the enlarged domain $D_1 \cup D_2 \cup$ handle.

5. Conclusion

We have studied a degenerate parabolic-elliptic system arising in the mean-field theory of superconducting vortices. The model is two-dimensional with all vortices perpendicular to a given direction and is formulated in terms of a scalar magnetic potential q and a scalar stream function ψ. The equation for ψ is similar to the level-set formulation of the mean-curvature flow, though in the right-hand side the coupling to the magnetic field occurs in the form of a nonlocal operator in terms of ψ. Nevertheless it was possible to construct unique viscosity solutions in the case of both vanishing and non-vanishing curvature coefficient σ.

For the fully three-dimensional equations this approach does not seem to be appropriate, since then the magnetic potential and the stream function are vector-valued.

Next, in the case of vanishing curvature coefficient σ, we related the model formulated in terms of q and ψ to the original mean-field model formulated in terms of the magnetic field $\boldsymbol{H}$ and the vorticity $\boldsymbol{\omega}$. We showed that this hyperbolic-elliptic system admits a unique "viscous" solution (i.e., a solution obtained by a viscous approximation). However, due to the lack of entropies, we cannot prove in general the uniqueness of any weak solution.

Finally, we constructed special solutions of the corresponding stationary problem, assuming a functional dependence of ψ on q.

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