Numer. Math. 51, 23-36 (1987)

Numerische Mathematik

A Practical Finite Element Approximation of a Semi-Definite Neumann Problem on a Curved Domain

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Summary. This paper considers the finite element approximation of the semi-definite Neumann problem: $-\nabla\cdot(\sigma\nabla u)=f$ in a curved domain $\Omega\subset\mathbb{R}^n$ ($n=2$ or 3), $\sigma\frac{\partial u}{\partial v}=g$ on $\partial\Omega$ and $\int_\Omega u\,dx=q$, a given constant, for data f and g satisfying the compatibility condition $\int_\Omega f\,dx+\int_{\partial\Omega} g\,ds=0$. Due to perturbation of domain errors ($\Omega\to\Omega^h$) the standard Galerkin approximation to the above problem may not have a solution. A remedy is to perturb the right hand side so that a discrete form of the compatibility condition holds. Using this approach we show that for a finite element space defined over D^h, a union of elements, with approximation power h^k in the L^2 norm and with $\operatorname{dist}(\Omega,\Omega^h)\leqq Ch^k$, one obtains optimal rates of convergence in the H^1 and L^2 norms whether Ω^h is fitted ($\Omega^h\equiv D^h$) or unfitted ($\Omega^h\subset D^h$) provided the numerical integration scheme has sufficient accuracy.

Subject Classifications: AMS(MOS): 65N30; CR: G1.8.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n=2$ or 3) with a smooth boundary $\partial\Omega$. Let σ be a sufficiently smooth function satisfying

$$\sigma_1\geqq\sigma(x)\geqq\sigma_0>0 \quad \text{a.e. in } \Omega. \tag{1.1}$$

Consider the numerical solution of the elliptic boundary-value problem

$$\begin{aligned} Au\equiv -\underline{\nabla}\cdot(\sigma\underline{\nabla}u)&=f \quad \text{in } \Omega,\\ \sigma\frac{\partial u}{\partial v}&=g \quad \text{on } \partial\Omega; \end{aligned} \tag{1.2a}$$

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* Partially supported by the National Science Foundation, Grant #DMS-8501397, the Air Force Office of Scientific Research and the Office of Naval Research

** *Present Address:* School of Mathematics & Physical Sciences, Univ. of Sussex, Brighton, BN1 9QH, UK where $\underline{\nu}$ denotes the outward pointing unit normal to $\partial\Omega$. For the existence of solutions to (1.2a) we require the data to be compatible

$$\int_{\Omega} f\,dx + \int_{\partial\Omega} g\,ds = 0 \tag{1.2b}$$

and for uniqueness we impose the condition

$$\int_{\Omega} u\,dx = q, \tag{1.2c}$$

where q is a given constant. This problem occurs in many practical situations; for example, solving the pressure equation in fluid flow calculations, see Ferris and Martin (1985).

A standard finite element (or finite difference) approximation of the system (1.2) may not have a solution as the discrete version of the compatibility condition (1.2b) may not be satisfied. One technique to overcome this problem is to study a penalised version of (1.2) by incorporating the term εu on the left hand side, as studied by Molchanov and Galba (1985). However, as we shall see in Sect. 2 the constraints required by Molchanov and Galba (1985) on the finite element approximation of the penalised problem in order to obtain the optimal rate of convergence in the H^1 norm are not satisfied by standard practical approximations, for example, isoparametric elements. Another approach is to adjust the data so that the discrete version of the compatibility condition is satisfied. Babuska and Aziz (1972) carry out this adjustment by introducing a Lagrange multiplier. A simpler technique is to carry out this adjustment a priori. This approach has been used in the finite difference context by Ferris and Martin (1985). In this paper we analyse a fully practical finite element approximation of (1.2) using this approach involving domain perturbation ($\Omega \to \Omega^h$) and numerical integration.

We show that for a finite element space defined over D^h, a union of elements, with approximation power h^k in the L^2 norm and with $\operatorname{dist}(\Omega, \Omega^h) \leqq Ch^k$, one obtains optimal rates of convergence in the H^1 and L^2 norms for this computationally simple a priori adjustment method whether Ω^h is fitted ($\Omega^h \equiv D^h$) or unfitted ($\Omega^h \subset D^h$). Our results are applicable to a $(k-1)$ regular family of fitted simplicial Lagrangian isoparametric elements as studied by Ciarlet and Raviart (1972) and Nedoma (1979); and to the unfitted piecewise linear approximation of Barrett and Elliott (1984a, 1985).

The outline of this paper is as follows: in the next section we define our finite element approximation to (1.2). In Sect. 3 we derive optimal rates of convergence in the H^1 and L^2 norms for the error in this approximation. Finally in Sect. 4 we present a numerical example using piecewise linears on an unfitted mesh.

We end this section by stating the notation we shall adopt throughout this paper. Given $m \in \mathbb{N}$ and a bounded domain G in $\mathbb{R}^n$,

$$W^{m,p}(G) = \{w \in L^p(G) : D^\eta w \in L^p(G) \text{ for all } |\eta| \leqq m\}$$

and

$$H^m(G) \equiv W^{m,2}(G)$$

denote, for $1 \leqq p \leqq \infty$, the standard Sobolev spaces, where we use the multi-index notation for the derivatives. We use the following norms and semi-norms on functions w defined by

$$\|w\|_{m,p,G} = [\sum_{|\eta| \leqq m} |D^\eta w|^p_{0,p,G}]^{1/p}, \qquad \|w\|_{m,G} \equiv \|w\|_{m,2,G},$$

$$|w|_{m,p,G} = [\sum_{|\eta| = m} |D^\eta w|^p_{0,p,G}]^{1/p}, \qquad |w|_{m,G} \equiv |w|_{m,2,G},$$

and

$$|w|_{0,p,G} = [\int_G |w|^p \, dx]^{1/p},$$

for $m \in \mathbb{N}$ and $1 \leqq p < \infty$ with the standard modification for $p = \infty$. For ∂G a section of the boundary of G we define $W^{m,p}(\partial G)$ and $\|\cdot\|_{m,p,\partial G}$ as above with G replaced by ∂G if ∂G is of class $C^{m,1}$, see Kufner et al. (1977), p. 305 and p. 327. If ∂G is only piecewise $C^{m,1}$, that is, $\partial G = \bigcup_{i=1}^{N} \partial_i G$ with $\partial_i G$ of class $C^{m,1}$, we define

$$\|w\|_{m,p,\partial G} = \left[\sum_{i=1}^{N} \|w\|^p_{m,p,\partial_i G} \right]^{1/p}.$$

The measure of a domain G is denoted by $\underline{m}(G)$. Throughout C denotes a positive constant independent of h whose value may change in different relations. We require also the trace inequalities: for ∂G of class $C^{0,1}$

$$\|D^\eta w\|_{0,\partial G} \leqq C \|w\|_{|\eta|+1,G} \qquad \forall w \in H^{|\eta|+1}(G), \tag{1.3a}$$

and for ∂G of class $C^{0,1}$ and piecewise $C^{m,1}$ this implies that

$$\|w\|_{m,\partial G} \leqq C \|w\|_{m+1,G} \qquad \forall w \in H^{m+1}(G) \tag{1.3b}$$

and

$$\left\| \frac{\partial w}{\partial v} \right\|_{0,\partial G} \leqq C \|w\|_{2,G} \qquad \forall w \in H^2(G), \tag{1.3c}$$

where $\underline{v}$ is the outward pointing unit normal to ∂G and C is a constant independent of w, see Kufner et al. (1977).

We adopt the notation

$$(w, v)_G = \int_G w v \, dx \quad \text{and} \quad \langle w, v \rangle_{\partial G} = \int_{\partial G} w v \, ds.$$

Finally, we require the Poincaré inequality

$$\|w\|^2_{0,G} \leqq C\{|w|^2_{1,G} + (\int_G w \, dx)^2\} \qquad \forall w \in H^1(G), \tag{1.4}$$

where C is a constant independent of w.

2. Finite Element Approximation

Introducing

$$H^1(\Omega, q) = \{w \in H^1(\Omega) : \int_\Omega w\, dx = q\}; \tag{2.1}$$

the variational formulation of (1.2) is: find $u \in H^1(\Omega, q)$ such that

$$a(u, v) = l(v) \quad \forall\, v \in H^1(\Omega), \tag{2.2}$$

where

$$a(w, v) \equiv (\sigma \underline{\nabla} w, \underline{\nabla} v)_\Omega \tag{2.3a}$$

and

$$l(v) \equiv (f, v)_\Omega + \langle g, v \rangle_{\partial\Omega}. \tag{2.3b}$$

It follows from (1.1) and the Poincaré inequality (1.4) that $a(\cdot, \cdot)$ is continuous and coercive over $H^1(\Omega, 0)$ and hence (2.2) is a well-posed variational problem for data $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ satisfying $l(1) \equiv 0$.

Consider the finite element approximation of (2.2). The domain Ω and its boundary $\partial\Omega$ are approximated, respectively, by Ω^h and $\partial\Omega^h$. As we wish our analysis to cover the case of unfitted as well as fitted meshes, Ω^h is not necessarily a union of elements. Let D^* be a bounded domain in $\mathbb{R}^n$ containing Ω such that $\bar{D}^* = \bigcup_{\tau \in T^*} \bar{\tau}$, where T^* is a collection of disjoint open regular elements τ, each of maximum diameter bounded above by h. We have in general

$$\bar{\Omega}^h \subseteq \bar{D}^h \subseteq \bar{D}^*(h) \subset \mathbb{R}^n, \tag{2.4a}$$

where

$$\bar{D}^h \equiv \bigcup_{\tau \in T^h} \bar{\tau} \tag{2.4b}$$

and

$$T^h \equiv \{\tau \in D^* : \tau \cap \Omega^h \neq \emptyset\}. \tag{2.4c}$$

The domains Ω^h and D^h are identical in the case of a fitted mesh. In addition, we set

$$B^h \equiv \{\tau \in T^h : \underline{m}(\bar{\tau} \cap \partial\Omega^h) \neq 0 \text{ in } \mathbb{R}^{n-1}\}. \tag{2.5}$$

Associated with T^h is a finite dimensional subspace S^h of $C^0(\bar{D}^h)$, depending upon an integer $k \geqq 2$, such that $\chi|_\tau \in H^2(\tau)$ $\forall \chi \in S^h$ and $\forall \tau \in T^h$. Let $\pi_h : C^0(\bar{D}^h) \to S^h$ denote the interpolation operator. We assume that the following approximation property holds:

(A1) For integers k' and m satisfying $k' \geqq m \geqq 0$ and $k \geqq k' \geqq 2$

$$|w - \pi_h w|_{m,\tau} \leqq C h^{k'-m} \|w\|_{k',\tau} \quad \forall\, w \in H^{k'}(\tau),\ \forall \tau \in T^h, \tag{2.6}$$

where C is a constant independent of h and w.

The boundary $\partial\Omega^h$ is constructed with the following approximation properties. For each element $\tau \in B^h$ there exists a local co-ordinate system (X_τ, Y_τ) such that $X_\tau \in \Delta_\tau$ and $Y_\tau \in \mathbb{R}$, where Δ_τ is either an interval ($n=2$) or a triangle ($n=3$). The surface $\partial\Omega^h_\tau \equiv \partial\Omega^h \cap \bar{\tau}$ is locally described by $Y_\tau = \psi^h_\tau(X_\tau)$. The surface $\partial\Omega$ is locally described by $Y_\tau = \psi_\tau(X_\tau)$. We denote this section of $\partial\Omega$ by $\partial\Omega_\tau$. It is assumed that $\psi_\tau, \psi_\tau^h \in C^{1,1}(\Delta_\tau)$ and that they vanish at the vertices of Δ_τ. This immediately implies that

$$\|\underline{\nabla}\psi_\tau\|_{0,\infty,\Delta_\tau} \leqq Ch \quad \text{and} \quad \|\underline{\nabla}\psi_\tau^h\|_{0,\infty,\Delta_\tau} \leqq Ch \qquad \forall \tau \in B^h, \tag{2.7}$$

where C is a constant independent of h. We make the following assumption on this boundary approximation:

(A2) For $\psi_\tau(\cdot) \in C^k(\Delta_\tau)$

$$\|\psi_\tau - \psi_\tau^h\|_{m,\infty,\Delta_\tau} \leqq Ch^{k-m} \qquad 0 \leqq m \leqq k,\ \forall \tau \in B^h, \tag{2.8}$$

where C is independent of h.

Remark 2.1. The assumptions (A1) and (A2) are satisfied for a $k-1$ regular family of simplicial Lagrangian isoparametric elements as introduced by Ciarlet and Raviart (1972). In this case $\bar{\Omega}^h \equiv \bar{D}^h$ and each element has at most one face on $\partial\Omega^h$ and only faces on $\partial\Omega^h$ are allowed to be curved. In addition ψ_τ^h agrees with ψ_τ at all nodes lying on $\partial\Omega_\tau^h$.

The assumptions are also satisfied by the unfitted mesh approximation introduced by Barrett and Elliott (1984a, 1985) and in this case $\Omega^h \subseteq D^h \subset \mathbb{R}^n$ where $\partial\Omega^h$ is chosen in each element τ with $\underline{m}(\bar{\tau} \cap \partial\Omega) \neq 0$ in $\mathbb{R}^{n-1}$ to be a $k-1$ degree polynomial interpolating $\partial\Omega$ such that n of the interpolation points occur where $\partial\Omega$ crosses either the element sides ($n=2$) or edges ($n=3$). □

Because in general, $\Omega^h \nsubseteq \Omega$ it is necessary to extend the data. It is convenient to introduce a domain $\tilde{\Omega} \subset \mathbb{R}^n$ with a smooth boundary such that

$$\Omega^h \subseteq D^h \subseteq \tilde{\Omega} \qquad \forall h < h_0 \quad \text{and} \quad \Omega \subseteq \tilde{\Omega}. \tag{2.9}$$

For all integers $s \geqq 0$ there exists an extension operator $E: H^s(\Omega) \to H^s(\tilde{\Omega})$ such that

$$Ew = w \qquad \text{on } \Omega \tag{2.10a}$$

and

$$\|Ew\|_{s,\tilde{\Omega}} \leqq C\|w\|_{s,\Omega}, \tag{2.10b}$$

where C is independent of w. (See Kufner et al. (1977).)

Remark 2.2. We note from (2.9) and the proof of the trace theorem in Nečas (1967) p. 15 that the constants C are independent of h for the trace inequalities (1.3) for $0 \leqq m \leqq k$ with $G \equiv \Omega^h$. □

We make the following regularity assumptions on the data:

$$f \in H^k(\Omega), \tag{2.11a}$$

σ and g are the restrictions to Ω and $\partial\Omega$, respectively, of functions

$$\tilde{\sigma} \in C^{k+1}(\tilde{\Omega}) \quad \text{and} \quad \tilde{g} \in H^{k+1}(\tilde{\Omega}), \tag{2.11b}$$

where

$$\tilde{\sigma}_1 \geqq \tilde{\sigma}(x) \geqq \tilde{\sigma}_0 > 0 \qquad \forall x \in \tilde{\Omega}. \tag{2.11c}$$

In addition we assume that $\partial\Omega$ is of class $C^{k+1,1}$ and then it follows from elliptic regularity theory that the solution u of (2.2) is such that

$$u \in H^{k+2}(\Omega). \tag{2.12}$$

We define

$$\tilde{A} w \equiv -\underline{\nabla} \cdot (\tilde{\sigma} \underline{\nabla} w) \tag{2.13}$$

and

$$\tilde{a}^h(w, v) \equiv (\tilde{\sigma} \underline{\nabla} w, \underline{\nabla} v)_{\Omega^h} \tag{2.14a}$$

$$= (\tilde{A} w, v)_{\Omega^h} + \left\langle \tilde{\sigma} \frac{\partial w}{\partial \nu^h}, v \right\rangle_{\partial\Omega^h}, \tag{2.14b}$$

where $\underline{\nu}^h$ is the outward pointing unit normal to $\partial\Omega^h$. We define

$$\tilde{l}^h(v) = (\tilde{f}, v)_{\Omega^h} + \langle \tilde{g}, v \rangle_{\partial\Omega^h}, \tag{2.15}$$

where $\tilde{f}$ is an extension of f. It follows from (2.9) that there exists a constant C such that for all $w, v \in H^1(\Omega^h)$

$$|\tilde{a}^h(w, v)| \leq C \|w\|_{1,\Omega^h} \|v\|_{1,\Omega^h}. \tag{2.16}$$

Introducing

$$S^h(q) = \{\chi \in S^h : \int_{\Omega^h} \chi \, dx = q\}; \tag{2.17}$$

a Galerkin finite element approximation to (2.2) is: find $\tilde{u}^h \in S^h(q)$ such that

$$\tilde{a}^h(\tilde{u}^h, \chi) = \tilde{l}^h(\chi) \qquad \forall \chi \in S^h. \tag{2.18}$$

However, for a solution to exist to this approximation we require $\tilde{l}^h(1) = 0$ and this will not be true in general due to the perturbation of domain.

An alternative approach is to approximate the penalised problem

$$A_\varepsilon u_\varepsilon \equiv (A + \varepsilon) u_\varepsilon = f + \varepsilon[q/\underline{m}(\Omega)] \quad \text{in } \Omega, \tag{2.19a}$$

$$\sigma \frac{\partial u_\varepsilon}{\partial \nu} = g \quad \text{on } \partial\Omega; \tag{2.19b}$$

where ε, a positive constant, is the penalty parameter. Note that for all data $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ there exists a unique solution u_ε to (2.19) satisfying

$$\int_\Omega u_\varepsilon \, dx = q + \varepsilon^{-1}\left[\int_\Omega f \, dx + \int_{\partial\Omega} g \, ds\right] \equiv q + \varepsilon^{-1} l(1). \tag{2.20}$$

Thus if the data satisfies (1.2b) we have $\int_\Omega u_\varepsilon \, dx = q$.

The Galerkin finite element approximation to (2.19) involving domain perturbation is: find $\tilde{u}^h_\varepsilon \in S^h(q)$ such that

$$\tilde{a}^h_\varepsilon(\tilde{u}^h_\varepsilon, \chi) = \tilde{l}^h_\varepsilon(\chi) \qquad \forall \chi \in S^h; \tag{2.21}$$

where

$$\tilde{a}^h_\varepsilon(w, v) \equiv \tilde{a}^h(w, v) + \varepsilon(w, v)_{\Omega^h} \tag{2.22a}$$

and

$$\tilde{l}^h_\varepsilon(v) \equiv \tilde{l}^h(v) + \varepsilon[(q, v)_{\Omega^h}/\underline{m}(\Omega^h)]. \tag{2.22b}$$

Putting $\chi\equiv 1$ in (2.21) yields the analogue of (2.20):

$$\int_{\Omega^h} \tilde{u}_\varepsilon^h\, dx = q + \varepsilon^{-1}\,\tilde{l}^h(1). \tag{2.23}$$

The important problem is how to choose ε with respect to h and k so that $\tilde{u}_\varepsilon^h$ converges to u at the optimal rate as $h\to 0$. Since from (1.2) and (2.19) we have

$$A_\varepsilon(u-u_\varepsilon)=\varepsilon\{u-[q/\underline{m}(\Omega)]\}, \tag{2.24a}$$

$$\sigma\frac{\partial}{\partial v}(u-u_\varepsilon)=0; \tag{2.24b}$$

elliptic regularity theory yields that

$$\|u-u_\varepsilon\|_{2,\Omega}\leqq C\varepsilon\|u\|_{0,\Omega}, \tag{2.25}$$

where C is a constant independent of u and ε. From (2.25) it follows that a necessary condition for $\tilde{u}_\varepsilon^h$ to converge to u at an optimal rate in $H^1(L^2)$ is that $\varepsilon=O(h^\lambda)$ with $\lambda\geqq k-1(\lambda\geqq k)$. However, this is not a sufficient condition, since (1.2b), (2.20) and (2.23) yield that

$$\int_\Omega u\, dx-\int_{\Omega^h}\tilde{u}_\varepsilon^h\, dx=-\varepsilon^{-1}\,\tilde{l}^h(1). \tag{2.26}$$

For approximations satisfying (A1) and (A2), such as isoparametrics, it can be shown, see Lemma 3.4, that in general

$$|\tilde{l}^h(1)|\leqq Ch^k[\|\tilde{f}\|_{2,\tilde{\Omega}}+\|\tilde{g}\|_{2,\tilde{\Omega}}] \tag{2.27}$$

and so the choice $\varepsilon=O(h^\lambda)$ with $\lambda\geqq k$ gives rise to an approximation $\tilde{u}_\varepsilon^h$ which does not even converge to u as $h\to 0$ and with $\lambda<k$ $\tilde{u}_\varepsilon^h$ may converge to u but at a rate well below the optimal.

The approximation of the penalised problem (2.19) is advocated by Molchanov and Galba (1985) and their analysis shows that the optimal rate of convergence in the H^1 norm is achieved if $\varepsilon=O(h^{k-1})$ and any variational crimes present in the method are such that

$$|\tilde{l}^h(1)|\leqq Ch^{2(k-1)}[\|\tilde{f}\|_{2,\Omega}+\|\tilde{g}\|_{2,\Omega}].$$

However, as we have seen this condition is not satisfied by standard practical approximations; for example, isoparametric elements.

Below we present a technique which is easily implemented and achieves the optimal rate of convergence in the H^1 and L^2 norms for approximations satisfying (A1) and (A2). In addition, we take into account the use of numerical integration. The technique is to perturb the right hand side of (2.18), the Galerkin approximation to (2.2), so that the existence of an approximation is guaranteed.

Our approximation to (2.2) is: find $u^h\in S^h(q)$ such that

$$a^h(u^h,\chi)=l^h(\chi)-\frac{l^h(1)}{\underline{m}(\Omega^h)}\int_{\Omega^h}\chi\, dx\qquad\forall\,\chi\in S^h, \tag{2.28}$$

where $a^h(\cdot,\cdot)$ and $l^h(\cdot)$ are approximations to $\tilde{a}^h(\cdot,\cdot)$ and $\tilde{l}^h(\cdot)$ satisfying:

(A3) There exists a constant C independent of h and χ such that

$$a^h(\chi,\chi) \geqq C|\chi|^2_{1,\Omega^h} \qquad \forall \chi \in S^h. \tag{2.29}$$

(A4) For $h<h_0$ it is assumed that for all $w \in H^k(\tilde{\Omega})$ and for all w^h and $\chi \in S^h$

$$|\tilde{a}^h(w^h,\chi)-a^h(w^h,\chi)| \leqq C\{h^{k-1}\|w\|_{k,\Omega^h}+\|w-w^h\|_{1,\Omega^h}\}\|\chi\|_{1,\Omega^h}, \tag{2.30a}$$

$$|\tilde{a}^h(w^h,\chi)-a^h(w^h,\chi)| \leqq Ch\{h^{k-1}\|w\|_{k,\Omega^h}+\|w-w^h\|_{1,\Omega^h}\}\Big(\sum_{\tau\in T^h}\|\chi\|^2_{2,\tau}\Big)^{\frac{1}{2}}, \tag{2.30b}$$

$$|\tilde{l}^h(\chi)-l^h(\chi)| \leqq Ch^{k-1}\{\|\tilde{f}\|_{k,\Omega^h}+\|\tilde{g}\|_{k+1,\Omega^h}\}\|\chi\|_{1,\Omega^h}, \tag{2.30c}$$

$$|\tilde{l}^h(\chi)-l^h(\chi)| \leqq Ch^{k}\{\|\tilde{f}\|_{k,\Omega^h}+\|\tilde{g}\|_{k+1,\Omega^h}\}\Big(\sum_{\tau\in T^h}\|\chi\|^2_{2,\tau}\Big)^{\frac{1}{2}}. \tag{2.30d}$$

In addition, we make the following assumption:

(A5) $a^h(\cdot,\cdot)$ and $l^h(\cdot)$ are assumed to depend on the evaluation of
(i) $\tilde{\sigma}$ and $\tilde{f}$ at points in $\bar{\Omega}$,
(ii) $\tilde{g}$ at points on $\partial\Omega$.

Remark 2.3. The assumptions (A3)→(A5) are satisfied for a $k-1$ regular family of simplicial Lagrangian isoparametric elements when using isoparametric integration, see Barrett and Elliott (1985), and for the quadrature rule for unfitted linear elements defined in Barrett and Elliott (1985). □

Note that putting $\chi\equiv 1$ in (2.28) both sides vanish and so (A3) guarantees the existence of a unique solution u^h. A convenient method for obtaining the approximation u^h, as suggested by Forsythe and Wasow (1960; §25.9), is to solve the indefinite system: obtain a $\hat{u}^h \in S^h$ such that

$$a^h(\hat{u}^h,\chi)=l^h(\chi)-\frac{l^h(1)}{\underline{m}(\Omega^h)}\int_{\Omega^h}\chi\,dx \qquad \forall\chi\in S^h; \tag{2.31}$$

by an iterative method, such as S.O.R., and then adjust

$$u^h=\hat{u}^h+\frac{1}{\underline{m}(\Omega^h)}\Big[q-\int_{\Omega^h}\hat{u}^h\,dx\Big]. \tag{2.32}$$

It follows from (A5) (i) that the approximation is independent of the extensions of σ and f. For the analysis in Sect. 3 it is convenient to choose

$$\tilde{f}+\tilde{A}\tilde{u}, \tag{2.33}$$

where $\tilde{u}=Eu$. It follows from (2.12), (2.10) and (2.33) that

$$\|\tilde{f}\|_{k,\Omega^h}\leqq C\|u\|_{k+2,\Omega}. \tag{2.34}$$

It follows from (A5) (ii) that the approximation is independent of the extension of g. However, this is not necessary for the error analysis and practical schemes dependent on computable extensions of g can also be derived.

In the next section we derive the following optimal rates of convergence in the H^1 and L^2 norms for the error in the approximation (2.28) under the assumptions (A1)→(A5):

$$\|\tilde{u}-u^h\|_{m,\Omega^h} \leqq Ch^{k-m}[\|u\|_{k+2,\Omega}+\|\tilde{g}\|_{k+1,\Omega^h}] \qquad m=0 \text{ and } 1.$$

The fact that $\|u\|_{k+2,\Omega}$ appears on the right hand side rather than $\|u\|_{k,\Omega}$ is due to the extra smoothness required for f in order to take account of numerical integration. If exact integration was employed then we take $f \in H^{k-2}(\Omega)$ and on the right hand side we have $\|u\|_{k,\Omega}$.

3. Error Bounds

In deriving H^1 and L^2 error bounds for the approximation (2.28) the following results are useful.

Lemma 3.1. *Let* (A2) *hold. If*

$$\frac{\partial w}{\partial \nu}=\rho \qquad on\ \partial\Omega,$$

then

$$\left\|\frac{\partial w}{\partial \nu^h}-\rho\right\|_{0,\partial\Omega^h} \leqq \begin{cases} Ch^{k/2}[\|w\|_{2,\tilde{\Omega}}+\|\rho\|_{1,\tilde{\Omega}}] & (3.1\text{a}) \\ Ch^{k-1}[\|w\|_{2,\infty,\tilde{\Omega}}+\|\rho\|_{2,\infty,\tilde{\Omega}}]. & (3.1\text{b}) \end{cases}$$

Proof. The proof is given in Lemma 3.1 of Barrett and Elliott (1985). □

Lemma 3.2. *Let* (A2) *hold. It follows that for all* $w \in H^1(\tilde{\Omega})$

$$\|w\|_{0,\Omega^h\setminus\Omega} \leqq C\{h^k|w|_{1,\Omega^h\setminus\Omega}+h^{k/2}\|w\|_{0,\partial\Omega^h}\} \tag{3.2a}$$

and

$$\|w\|_{0,\Omega\setminus\Omega^h} \leqq C\{h^k|w|_{1,\Omega\setminus\Omega^h}+h^{k/2}\|w\|_{0,\partial\Omega}\}. \tag{3.2b}$$

Proof. The proof of (3.2a) is given in Lemma 3.2 of Barrett and Elliott (1984b). The proof of (3.2b) follows in a similar manner. □

Lemma 3.3. *Let* (A1) *and* (A2) *hold. Then for all* $w \in H^2(\tilde{\Omega})$ *it follows that*

$$|\tilde{a}^h(\tilde{u},\pi_h\tilde{w})-\tilde{l}^h(\pi_h\tilde{w})| \leqq Ch^k[\|u\|_{4,\Omega}+\|\tilde{g}\|_{3,\tilde{\Omega}}]\,\|w\|_{2,\tilde{\Omega}}, \tag{3.3}$$

where $\tilde{u}=Eu$.

Proof. The proof is given in Lemma 4.2 of Barrett and Elliott (1985). □

Lemma 3.4. *Let* (A2) *hold. If* $l(1)=0$ *then*

$$|\tilde{l}^h(1)| \leqq Ch^k[\|\tilde{f}\|_{2,\tilde{\Omega}}+\|\tilde{g}\|_{2,\tilde{\Omega}}]. \tag{3.4}$$

Proof. We have

$$|\tilde{l}^h(1)| \leqq |\int_{\Omega^h \setminus \Omega} \tilde{f}\,dx - \int_{\Omega \setminus \Omega^h} f\,dx| + |\int_{\partial\Omega^h} \tilde{g}\,ds^h - \int_{\partial\Omega} g\,ds|$$

and

$$|\int_{\Omega^h \setminus \Omega} \tilde{f}\,dx - \int_{\Omega \setminus \Omega^h} f\,dx| \leqq [\underline{m}(\Omega^h \setminus \Omega) + \underline{m}(\Omega \setminus \Omega^h)]\,\|\tilde{f}\|_{0,\infty,\tilde{\Omega}}$$
$$\leqq Ch^k\,\|\tilde{f}\|_{2,\tilde{\Omega}},$$

where we have used (A2) and Sobolev's embedding Theorem. It can be shown, see the proof of Lemma 4.2 in Barrett and Elliott (1985), that

$$|\int_{\partial\Omega^h} \tilde{g}\,ds^h - \int_{\partial\Omega} g\,ds| \leqq Ch^k\,\|\tilde{g}\|_{2,\tilde{\Omega}}.$$

Hence the desired result (3.4) is obtained. □

Lemma 3.5. *For any $\chi \in S^h$ there exists a constant C (independent of h and χ) such that*

$$\|\chi\|^2_{0,\Omega^h} \leqq C\{|\chi|^2_{1,\Omega^h} + (\int_{\Omega^h} \chi\,dx)^2\}. \tag{3.5}$$

Proof. This result follows from the proof of Poincaré's inequality in Nečas (1967, Ch. 1, p. 16). □

Lemma 3.6. *Let* (A2) *hold. If $\int_\Omega w\,dx = q$ and $w \in H^2(\tilde{\Omega})$ then*

(i) $$|\int_{\Omega^h} (\pi_h w)\,dx - q| \leqq C[\|w - \pi_h w\|_{0,\Omega^h} + h^k\,\|w\|_{2,\tilde{\Omega}}] \tag{3.6a}$$

and

(ii) *setting*

$$\hat{w}^h_I = \pi_h w + \frac{1}{\underline{m}(\Omega^h)}\,[q - \int_{\Omega^h} (\pi_h w)\,dx] \tag{3.6b}$$

it follows that $\hat{w}^h_I \in S^h(q)$ and for $m = 0$ and 1,

$$\|w - \hat{w}^h_I\|_{m,\Omega^h} \leqq C[\|w - \pi_h w\|_{m,\Omega^h} + h^k\,\|w\|_{2,\tilde{\Omega}}]. \tag{3.6c}$$

Proof. It follows from (A2) and Sobolev's embedding theorem that

$$|\int_\Omega w\,dx - \int_{\Omega^h} (\pi_h w)\,dx| = |\int_{\Omega^h} (w - \pi_h w)\,dx + \int_{\Omega \setminus \Omega^h} w\,dx - \int_{\Omega^h \setminus \Omega} w\,dx|$$
$$\leqq [\underline{m}(\Omega^h)]^{\frac{1}{2}}\,\|w - \pi_h w\|_{0,\Omega^h} + \underline{m}(\Omega \setminus \Omega^h)\,\|w\|_{0,\infty,\Omega}$$
$$\leqq \underline{m}(\Omega^h \setminus \Omega)\,\|w\|_{0,\infty,\Omega^h}$$
$$\leqq C[\|w - \pi_h w\|_{0,\Omega^h} + h^k\,\|w\|_{2,\tilde{\Omega}}],$$

yielding the desired result (3.6a). Then (3.6c) follows directly from (3.6a, b). □

We now prove the H^1 and L^2 error bounds.

Theorem 3.1. *Let* (A1)→(A5) *hold. The solutions u and u^h of* (2.2) *and* (2.28) *satisfy:*

$$\|\tilde{u}-u^h\|_{1,\Omega^h} \leqq Ch^{k-1}[\|u\|_{k+2,\Omega}+\|\tilde{g}\|_{k+1,\Omega^h}], \tag{3.7}$$

where $\tilde{u}=Eu$.

Proof. Setting

$$\hat{u}_I^h=\pi_h\tilde{u}+\frac{1}{\underline{m}(\Omega^h)}\left[q-\int_{\Omega^h}(\pi_h\tilde{u})\,dx\right], \tag{3.8}$$

and $\xi=u^h-\hat{u}_I^h\in S^h$ it follows that

$$\int_{\Omega^h}\xi\,dx=0. \tag{3.9}$$

Since

$$\|\tilde{u}-u^h\|_{1,\Omega^h} \leqq \|\tilde{u}-\hat{u}_I^h\|_{1,\Omega^h}+\|\xi\|_{1,\Omega^h} \tag{3.10}$$

and it follows from (3.6c), (A2) and (2.10) that

$$\|\tilde{u}-\hat{u}_I^h\|_{1,\Omega^h} \leqq Ch^{k-1}\|u\|_{k,\Omega}, \tag{3.11}$$

the desired result (3.7) holds if we can bound $\|\xi\|_{1,\Omega^h}$ by the right hand side of (3.7).

(A3), (3.5) and (3.9) yield that

$$\begin{aligned} C\|\xi\|_{1,\Omega^h}^2 \leqq C|\xi|_{1,\Omega^h}^2 &\leqq a^h(\xi,\xi)=\tilde{a}^h(\tilde{u}-\hat{u}_I^h,\xi) \\ &+[\tilde{a}^h(\hat{u}_I^h,\xi)-a^h(\hat{u}_I^h,\xi)]+[l^h(\xi)-\tilde{l}^h(\xi)] \\ &+[\tilde{l}^h(\xi)-\tilde{a}^h(\tilde{u},\xi)]. \end{aligned} \tag{3.12}$$

From (2.14), (2.33), (3.1b), (1.3b), Sobolev's embedding theorem and (2.10) it follows that

$$|\tilde{l}^h(\xi)-\tilde{a}^h(\tilde{u},\xi)| \leqq Ch^{k-1}[\|u\|_{4,\Omega}+\|\tilde{g}\|_{3,\tilde{\Omega}}]\,\|\xi\|_{1,\Omega^h}. \tag{3.13}$$

Applying the numerical assumptions (2.30a, c), (A1), the results (3.11), (3.13) and (2.34) to (3.12) yields the desired bound (3.7). □

Theorem 3.2. *Let* (A1)→(A5) *hold. The solutions* u *and* u^h *of* (2.2) *and* (2.28) *satisfy:*

$$\|\tilde{u}-u^h\|_{0,\Omega^h} \leqq Ch^k[\|u\|_{k+2,\Omega}+\|\tilde{g}\|_{k+1,\Omega^h}], \tag{3.14}$$

where $\tilde{u}=Eu$.

Proof. Define $\tilde{\eta}\in L^2(\tilde{\Omega})$ to be

$$\tilde{\eta}=\begin{cases}\tilde{u}-u^h & \text{on } \Omega^h\\ 0 & \text{on } \tilde{\Omega}\backslash\Omega^h\end{cases} \tag{3.15}$$

and set

$$\delta=\frac{1}{\underline{m}(\Omega)}\int_{\Omega}\tilde{\eta}\,dx. \tag{3.16}$$

Define z to be the solution of

$$Az=\tilde{\eta}-\delta \quad \text{in } \Omega \tag{3.17a}$$

$$\sigma\frac{\partial z}{\partial \nu}=0 \quad \text{and} \quad \int_\Omega z\,dx=0. \tag{3.17b}$$

The fact that $\int_\Omega(\tilde{\eta}-\delta)\,dx=0$ implies the existence and uniqueness of z and from elliptic regularity theory it follows that

$$\|z\|_{2,\Omega}\leqq C\,\|\tilde{\eta}-\delta\|_{0,\Omega}\leqq C\,\|\tilde{u}-u^h\|_{0,\Omega^h}. \tag{3.18}$$

Observe that

$$\begin{aligned}\|\tilde{u}-u^h\|^2_{0,\Omega^h}&=(\tilde{u}-u^h,\tilde{\eta})_{\Omega^h}\\&=(\tilde{u}-u^h,\tilde{\eta}-\tilde{A}\tilde{z})_{\Omega^h\cap\Omega}+(\tilde{u}-u^h,\tilde{\eta}-\tilde{A}\tilde{z})_{\Omega^h\cap(\Omega}+(\tilde{u}-u^h,\tilde{A}\tilde{z})_{\Omega^h},\\&=\sum_{j=1}^{3}I_j,\end{aligned} \tag{3.19}$$

where $\tilde{z}=Ez$.

Now

$$|I_1|\equiv|(\tilde{u}-u^h,\delta)_{\Omega^h\cap\Omega}|\leqq C\,\|\tilde{u}-u^h\|_{0,\Omega^h}\left|\int_{\Omega^h\cap\Omega}(\tilde{u}-u^h)\,dx\right| \tag{3.20}$$

and as $u\in H^1(q)$ and $u^h\in S^h(q)$ it follows that

$$\begin{aligned}\left|\int_{\Omega^h\cap\Omega}(\tilde{u}-u^h)\,dx\right|&\leqq\left|\int_{\Omega\setminus\Omega^h}u\,dx-\int_{\Omega^h\setminus\Omega}u^h\,dx\right|\\&\leqq[\underline{m}(\Omega\backslash\Omega^h)]^{\frac{1}{2}}\,\|u\|_{0,\Omega\setminus\Omega^h}+[\underline{m}(\Omega^h\backslash\Omega)]^{\frac{1}{2}}\,\|u^h\|_{0,\Omega^h\setminus\Omega}\\&\leqq Ch^k[\|u\|_{1,\Omega}+\|u^h\|_{1,\Omega^h}],\end{aligned} \tag{3.21}$$

where we have used (A2), the results (3.2) and (1.3b). Therefore combining (3.20), (3.21) and (3.7) it follows that

$$|I_1|\leqq Ch^k[\|u\|_{k+2,\Omega}+\|\tilde{g}\|_{k+1,\Omega^h}]\,\|\tilde{u}-u^h\|_{0,\Omega^h}. \tag{3.22}$$

The term I_2 is easily bounded using (2.10), (3.18), (3.2a), (1.3b) and (3.7):

$$\begin{aligned}|I_2|&\leqq C\,\|\tilde{u}-u^h\|_{0,\Omega^h\cap(\Omega}\,\|\tilde{\eta}\|_{0,\Omega^h}\\&\leqq Ch^{k/2}\,\|\tilde{u}-u^h\|_{1,\Omega^h}\,\|\tilde{\eta}\|_{0,\Omega^h}\\&\leqq Ch^k[\|u\|_{k+2,\Omega}+\|\tilde{g}\|_{k+1,\Omega^h}]\,\|\tilde{u}-u^h\|_{0,\Omega^h}.\end{aligned} \tag{3.23}$$

The final term I_3 can be bounded as follows:

$$I_3=\tilde{a}^h(\tilde{u}-u^h,\tilde{z})-\left\langle\tilde{u}-u^h,\tilde{\sigma}\frac{\partial\tilde{z}}{\partial\nu^h}\right\rangle_{\partial\Omega^h} \tag{3.24}$$

and

$$\begin{aligned}\left|\left\langle\tilde{u}-u^h,\tilde{\sigma}\frac{\partial\tilde{z}}{\partial\nu^h}\right\rangle_{\partial\Omega^h}\right|&\leqq\|\tilde{u}-u^h\|_{0,\partial\Omega^h}\left\|\tilde{\sigma}\frac{\partial\tilde{z}}{\partial\nu^h}\right\|_{0,\partial\Omega^h}\\&\leqq Ch^{k/2}\,\|\tilde{u}-u^h\|_{1,\Omega^h}\,\|\tilde{u}-u^h\|_{0,\Omega^h},\end{aligned} \tag{3.25}$$

where we have used (1.3b), (3.1a), (2.10) and (3.18). Setting

$$\hat{z}_I^h = \pi_h \tilde{z} - \frac{1}{\underline{m}(\Omega^h)} \int_{\Omega^h} (\pi_h \tilde{z})\, dx$$

it follows that $\hat{z}_I^h \in S^h(0)$ and from (3.6c), (A1) and (2.10) that

$$\|\tilde{z} - \hat{z}_I^h\|_{1,\Omega^h} \leqq Ch \|z\|_{2,\Omega}. \tag{3.26}$$

Observe that

$$\begin{aligned}\tilde{a}^h(\tilde{u} - u^h, \tilde{z}) = &\tilde{a}^h(\tilde{u} - u^h, \tilde{z} - \hat{z}_I^h) + [a^h(u^h, \hat{z}_I^h) - \tilde{a}^h(u^h, \hat{z}_I^h)] \\ &+ [\tilde{l}^h(\hat{z}_I^h) - l^h(\hat{z}_I^h)] + [\tilde{a}^h(\tilde{u}, \hat{z}_I^h) - \tilde{l}^h(\hat{z}_I^h)].\end{aligned} \tag{3.27}$$

Next note that

$$\begin{aligned}|\tilde{a}^h(\tilde{u}, \hat{z}_I^h) - \tilde{l}^h(\hat{z}_I^h)| &\leqq |\tilde{a}^h(\tilde{u}, \pi_h \tilde{z}) - \tilde{l}^h(\pi_h \tilde{z})| \\ &\quad + \frac{1}{\underline{m}(\Omega^h)} \left|\int_{\Omega^h} (\pi_h \tilde{z})\, dx\right| |\tilde{l}^h(I)| \\ &\leqq Ch^k [\|u\|_{4,\Omega} + \|\tilde{g}\|_{3,\Omega^h}] \|z\|_{2,\Omega}\end{aligned} \tag{3.28}$$

where we have used the results (3.3), (3.4), (3.6a), (A1), (2.34) and (2.10). Finally note that

$$\begin{aligned}\sum_{\tau \in T^h} \|\hat{z}_I^h\|_{2,\tau}^2 &\leqq C\Big[\sum_{\tau \in T^h} \|\pi_h \tilde{z}\|_{2,\tau}^2 + \Big|\int_{\Omega^h} (\pi_h z)\, dx\Big|^2\Big] \\ &\leqq C \|z\|_{2,\Omega}^2\end{aligned} \tag{3.29}$$

where we have used the results (3.6a), (A1) and (2.10). Applying the numerical assumptions (2.30b, d), (A1), the bounds (3.26), (3.28), (3.7), (3.18), (3.29) and (2.34) to (3.27) yields

$$|\tilde{a}^h(\tilde{u} - u^h, \tilde{z})| \leqq Ch^k [\|u\|_{k+2,\Omega} + \|\tilde{g}\|_{k+1,\Omega^h}] \|\tilde{u} - u^h\|_{0,\Omega^h}. \tag{3.30}$$

Combining (3.30), (3.25) and (3.7) yields

$$|I_3| \leqq Ch^k [\|u\|_{k+2,\Omega} + \|\tilde{g}\|_{k+1,\Omega^h}] \|\tilde{u} - u^h\|_{0,\Omega^h}. \tag{3.31}$$

Combining (3.19), (3.22), (3.23) and (3.31) yields the desired result (3.14). □

4. Numerical Example

We now report on a numerical example using an unfitted mesh. The problem chosen was

$$\nabla^2 u = 4 \quad \text{in } \Omega \equiv x^2 + y^2 \leqq 1,$$

$$\frac{\partial u}{\partial \nu} = 2 \quad \text{on } \partial\Omega \quad \text{and} \quad \int_\Omega u\, dx = 0.$$

This has the solution $u = x^2 + y^2 - \frac{1}{2}$. Due to symmetry the problem was solved in a single quadrant. For our trial space we took piecewise linears on uniform right-angled triangles, resulting from a uniform partition of the complete square $[0, 1] \times [0, 1]$ into squares with sides of length $h = 1/J$ and then into triangles by joining the SW to NE vertex. The computational domain Ω^h was obtained by replacing $\partial\Omega$ by its chord in each triangle it intersects as described in Sect. 2. Setting $\tilde{g} \equiv 2$ the results obtained from the approximation (2.28) are presented in Table 1. Clearly the analysis of Sect. 3 is confirmed.

Table 1

$h = 1/J$ J	$\lvert u - u^h \rvert_{1, \Omega^h}$	$\lVert u - u^h \rVert_{0, \Omega^h}$	$\max_{\text{nodes } x_j \in \Omega} \lvert (u - u^h)(x_j) \rvert$
4	0.164497	0.008390	0.047387
8	0.086794	0.002357	0.014682
12	0.058763	0.001075	0.007209
16	0.044393	0.000609	0.004337
24	0.029796	0.000274	0.002081

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Received January 22/November 24, 1986