Numer. Math. 49, 343–366 (1986)

Numerische Mathematik

Finite Element Approximation of the Dirichlet Problem Using the Boundary Penalty Method

John W. Barrett and Charles M. Elliott

Department of Mathematics, Imperial College, London, S.W.7., Great Britain

Summary. This paper considers a finite element approximation of the Dirichlet problem for a second order self-adjoint elliptic equation, $Au=f$, in a region $\Omega\subset\mathbb{R}^n$ ($n=2$ or 3) by the boundary penalty method. If the finite element space defined over D^h, a union of elements, has approximation power h^K in the L^2 norm, then

(i) for $\Omega\equiv D^h$ convex polyhedral, we show that choosing the penalty parameter $\varepsilon\equiv h^\lambda$ with $\lambda\geqq K$ yields optimal H^1 and L^2 error bounds if $u\in H^{K+1}(\Omega)$;

(ii) for $\partial\Omega$ being smooth, an unfitted mesh ($\Omega\subseteq D^h$) and assuming $u\in H^{K+2}(\Omega)$ we improve on the error bounds given by Babuska [1]. As (ii) is not practical we analyse finally a fully practical piecewise linear approximation involving domain perturbation and numerical integration. We show that the choice $\lambda=2$ yields an optimal H^1 and interior L^2 rate of convergence for the error. A numerical example is presented confirming this analysis.

Subject Classifications: AMS(MOS): 65N30; CR: G1.8.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n=2$ or 3) with a Lipschitz boundary $\partial\Omega$. We assume either Ω is convex polyhedral or $\partial\Omega$ is smooth. Let σ and c be sufficiently smooth functions satisfying

$$\sigma_1\geqq\sigma(x)\geqq\sigma_0>0 \quad \text{a.e. in } \Omega \tag{1.1a}$$

and

$$c_1\geqq c(x)\geqq c_0\geqq 0 \quad \text{a.e. in } \Omega. \tag{1.1b}$$

Consider the elliptic boundary value problem of finding u such that

$$Au\equiv -\nabla\cdot(\sigma\nabla u)+cu=f \quad \text{in } \Omega, \tag{1.2a}$$

$$u=g \quad \text{on } \partial\Omega, \tag{1.2b}$$

for prescribed data $f \in L^2(\Omega)$ and $g \in H^2(\Omega)$. The variational form of (1.2) is: find $u \in H^1_E(\Omega) \equiv \{w \in H^1(\Omega): w = g \text{ on } \partial\Omega\}$ such that

$$a(u,v) = l(v) \quad \forall v \in H^1_0(\Omega) = \{w \in H^1(\Omega): w = 0 \text{ on } \partial\Omega\}; \tag{1.3}$$

where

$$a(w,v) \equiv (\sigma \nabla w, \nabla v)_\Omega + (c\,w, v)_\Omega, \tag{1.4}$$

$$l(v) = (f, v)_\Omega \tag{1.5}$$

and

$$(w,v)_G = \int_G w\,v\,dx.$$

It follows from the assumptions (1.1) that $a(\cdot,\cdot)$ is continuous and coercive over $H^1_0(\Omega)$ and hence (1.3) is a well-posed variational problem. From elliptic regularity theory it follows that $u \in H^2(\Omega)$.

The finite element approximation of (1.3) requires the construction of finite dimensional spaces approximating $H^1_E(\Omega)$ and $H^1_0(\Omega)$. This is achieved in practice by fitting a mesh to Ω. Either partitioning Ω into elements if Ω is polyhedral or approximating Ω by Ω^h, a union of isoparametric elements, if $\partial\Omega$ is curved. However, if $\partial\Omega$ is smooth and the imposed boundary condition is of the Neumann or Robin type; that is, (1.2b) is replaced by

$$\sigma \frac{\partial u}{\partial \nu} + \alpha u = g \quad \text{on } \partial\Omega, \tag{1.6}$$

where $\boldsymbol{\nu}$ denotes the outward pointing unit normal on $\partial\Omega$ and $\alpha(x) \geqq 0$; then it is not necessary to fit the elements to the boundary in order to retain the optimal rate of convergence, see Barrett and Elliott [6]. They show that if the finite element space defined over D^h, a union of elements, has approximation h^K in the L^2 norm and if the region of integration is approximated by Ω^h with $\operatorname{dist}(\Omega, \Omega^h) \leqq C h^K$ then the optimal rate of convergence for the error in the H^1 and L^2 norms can be retained whether Ω^h is fitted ($\Omega^h \equiv D^h$) or unfitted ($\Omega^h \subset D^h$). We note that unfitted meshes have useful practical applications to free and moving boundary problems as at each step only the domain of integration and not the mesh has to be adjusted; see Barrett and Elliott [3, 5] for example.

The need to fit the mesh, $\Omega^h \equiv D^h$, in the case of the finite element approximation of (1.3) is that the Dirichlet boundary condition is imposed essentially on the finite element trial space. To remove this constraint and be able to explore the possibility of using an unfitted mesh, $\Omega^h \subseteq D^h$, when $\partial\Omega$ is smooth for the Dirichlet problem the boundary condition has to be imposed weakly. One method of imposing the boundary condition weakly is the penalty method, as studied by Babuska [1].

The penalty method considers the penalised problem of finding u_ε such that

$$A u_\varepsilon = f \quad \text{in } \Omega, \tag{1.7a}$$

$$\sigma \frac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^{-1}(u_\varepsilon - g) = 0 \quad \text{on } \partial\Omega, \tag{1.7b}$$

where ε, a positive constant, is the penalty parameter. The variational form of (1.7) is: find $u_\varepsilon \in H^1(\Omega)$ such that

$$a_\varepsilon(u_\varepsilon, v) = l_\varepsilon(v) \qquad \forall v \in H^1(\Omega), \tag{1.8}$$

where

$$a_\varepsilon(w, v) \equiv a(w, v) + \varepsilon^{-1} \langle w, v \rangle_{\partial\Omega}, \tag{1.9}$$

$$l_\varepsilon(v) \equiv l(v) + \varepsilon^{-1} \langle g, v \rangle_{\partial G} \tag{1.10}$$

and

$$\langle w, v \rangle_{\partial G} = \int_{\partial G} w v \, ds.$$

Since $\varepsilon > 0$ it follows that $a_\varepsilon(\cdot,\cdot)$ is continuous and coercive over $H^1(\Omega)$ and hence (1.8) is a well-posed variational problem.

It is a simple matter, see Theorem 2.1 in the next section, to show that the solutions of (1.3) and (1.8) satisfy

$$\|u - u_\varepsilon\|_{0,\Omega} \leqq C\varepsilon \|u\|_{2,\Omega}, \tag{1.11}$$

where C is a constant independent of ε, and so $u_\varepsilon \to u$ as $\varepsilon \to 0$. As the boundary condition (1.7b) is of the Robin type the finite element approximation of (1.8) can be based on an unfitted mesh if $\partial\Omega$ is smooth. The important problem is how to choose the penalty parameter ε with respect to h and K so that u_ε^h, the finite element approximation of (1.8), converges to u at the optimal rate as $h \to 0$.

The outline of this paper is as follows: in the next section we define and analyse a finite element approximation to (1.8) in the absence of variational crimes; that is $\Omega^h \equiv \Omega \subseteq D^h$ and all integrations are performed exactly. In Subsect. 2.1 we consider the case of $\partial\Omega$ smooth and so (1.8) requires evaluating integrals over curved domains and hence is not practical. Babuska [1] analysed this approximation for Poisson's equation, $\sigma \equiv 1$ and $c \equiv 0$, with homogeneous boundary data, $g \equiv 0$. Setting $\varepsilon = h^\lambda$, $\lambda > 0$, Babuska [1] showed that u and u_ε^h satisfied the following error bounds for $\delta > 0$ arbitrary

$$\|u - u_\varepsilon^h\|_{1,\Omega} \leqq C h^{\mu_1 - \delta} \|u\|_{K,\Omega} \tag{1.12a}$$

and

$$\|u - u_\varepsilon^h\|_{0,\Omega} \leqq C h^{\mu_0 - \delta} \|u\|_{K,\Omega}, \tag{1.12b}$$

where

$$m = \max[1, \tfrac{1}{2}(\lambda + 1)], \tag{1.13a}$$

$$p = (K - m)/(K - 1), \tag{1.13b}$$

$$\mu_1 = \min[\lambda, K - 1, K - \tfrac{1}{2}(\lambda + 1), K + \tfrac{1}{2}(\lambda - 3)] \tag{1.14a}$$

and

$$\mu_0 = \min[\lambda, p + \tfrac{1}{2}\lambda, \mu_1 + \tfrac{1}{2}\lambda, \mu_1 + p]. \tag{1.14b}$$

Thus for piecewise linears, $K = 2$, the only choice of λ which yields an optimal H^1 error estimate, $\mu_1 = 1$, is $\lambda = 1$ and this leads to a suboptimal L^2 error estimate, $\mu_0 = 1$. For $K \geqq 3$ there is no choice of λ which yields an optimal H^1 estimate.

Assuming more smoothness on u, $u \in H^{K+2}(\Omega)$, we show in Subsect. 2.1 that these bounds can be improved so that (1.12) holds with $\delta=0$, $\mu_1=\mu_1^*$ and $\mu_0=\mu_0^*$ where

$$\mu_1^* = \min[\lambda, K-1, K-\tfrac{1}{2}\lambda] \tag{1.15a}$$

and

$$\mu_0^* = \min[\lambda, \mu_1^*+1, \mu_1^*+\tfrac{1}{2}\lambda, \mu_1^*-\tfrac{1}{2}(\lambda-3)]. \tag{1.15b}$$

Thus with $K=2$ an optimal H^1 error bound is obtained if λ is such that $1 \leqq \lambda \leqq 2$ which leads to $\mu_0^* = \min[\lambda, 1+\frac{1}{2}\lambda, \frac{1}{2}(5-\lambda)]$. Hence the best choice of λ is $\frac{5}{3}$ leading to $\mu_0^* = \frac{5}{3}$. This is an improvement over (1.14), but of course still not optimal. For $K=3$ the choice $\lambda=2$ yields an optimal H^1 error bound.

In Subsect. 2.2 we consider the case of Ω being convex polyhedral. We analyse a finite element approximation to (1.8) in the absence of variational crimes assuming $\Omega^h \equiv \Omega \equiv D^h$. This is a less interesting case as in practice it is straightforward to impose the boundary condition essentially using a finite element approximation of (1.3) as opposed to using the penalty method. However, Zhong-Ci Shi [14] has analysed this case to explain the numerical results of Utku and Carey [13]. Zhong-Ci Shi [14] shows that for piecewise linears choosing $\lambda=1$ one obtains an optimal H^1 error bound if $\frac{\partial u}{\partial \nu} \in H^{\frac{1}{2}}(\partial\Omega)$. We show for $K \geqq 2$ that if $u \in H^{K+1}(\Omega)$ then choosing $\lambda \geqq K$ one obtains optimal H^1 and L^2 error bounds.

In Sect. 3 we return to the more important case of $\partial\Omega$ smooth and analyse a fully practical piecewise linear finite element approximation of (1.8) on an unfitted mesh involving domain perturbation and numerical integration. We show that with $\lambda=2$ the results (1.12) and (1.15) remain valid. Moreover, although the global L^2 error bound is only $O(h^{\frac{3}{2}})$ we obtain an optimal order interior L^2 error bound over a domain $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega^h$. Finally, in Sect. 4 we report on some numerical calculations with piecewise linear elements which confirm the error bounds derived above. The results show the superiority of choosing $\lambda=2$ as opposed to $\lambda=1$; which is generally quoted in the literature, see Utku and Carey [13] and King [8] for example, not only asymptotically but also for finite h.

We end this section by stating the notation we shall adopt throughout this paper. Given $m \in \mathbb{N}$ and a bounded domain G in $\mathbb{R}^n$,

$$W^{m,p}(G) = \{w \in L^p(G): D^\eta w \in L^p(G) \text{ for all } |\eta| \leqq m\}$$

and

$$H^m(G) \equiv W^{m,2}(G)$$

denote, for $1 \leqq p \leqq \infty$, the standard Sobolev spaces, where we use the multi-index notation for the derivatives. We use the following norms and semi-norms on functions w defined by

$$\|w\|_{m,p,G} = [\sum_{|\eta| \leqq m} |D^\eta w|_{0,p,G}^p]^{1/p}, \qquad \|w\|_{m,G} \equiv \|w\|_{m,2,G},$$

$$|w|_{m,p,G} = [\sum_{|\eta| = m} |D^\eta w|_{0,p,G}^p]^{1/p}, \qquad |w|_{m,G} \equiv |w|_{m,2,G}$$

and

$$|w|_{0,p,G}=[\int_G |w|^p dx]^{1/p}$$

for $m\in\mathbb{N}$ and $1\leqq p<\infty$ with the standard modification for $p=\infty$. For ∂G a section of the boundary of G we define $W^{m,p}(\partial G)$ and $\|\cdot\|_{m,p,\partial G}$ as above with G replaced by ∂G if ∂G is of class $\mathscr{C}^{m,1}$, see Kufner, John and Fucik ([10], p. 305). If ∂G is only piecewise $\mathscr{C}^{m,1}$, that is, $\partial G=\bigcup_{i=1}^{N}\partial_i G$ with $\partial_i G$ of class $\mathscr{C}^{m,1}$; we define

$$\|w\|_{m,p,\partial G}=\left[\sum_{i=1}^{N}\|w\|^p_{m,p,\partial_i G}\right]^{1/p}. \tag{1.16}$$

The measure of a domain G is denoted by $\mathbf{m}(G)$. Throughout C denotes a positive constant independent of h and the penalty parameter ε whose value may change in different relations. We require the trace inequalities: for ∂G of class $\mathscr{C}^{0,1}$

$$\|D^\eta w\|_{0,\partial G}\leqq C\|w\|_{|\eta|+1,G} \qquad \forall w\in H^{|\eta|+1}(G), \tag{1.17a}$$

and for ∂G of class $\mathscr{C}^{0,1}$ and piecewise $\mathscr{C}^{m-1,1}$ this implies that for $m\in\mathbb{N}$

$$\|w\|_{m-1,\partial G}\leqq C\|w\|_{m,G} \qquad \forall w\in H^m(G) \tag{1.17b}$$

and

$$\left\|\frac{\partial w}{\partial \nu}\right\|_{m-1,\partial G}\leqq C\|w\|_{m+1,G} \qquad \forall w\in H^{m+1}(G). \tag{1.17c}$$

We require also the fractional Sobolev space $H^{m-\frac{1}{2}}(\partial G)$, $m\in\mathbb{N}$, for ∂G of class $\mathscr{C}^{m-1,1}$ with norm defined by

$$\|w\|_{m-\frac{1}{2},\partial G}=\inf_{\substack{v\in H^m(G)\\ v\equiv w \text{ on } \partial G}}\|v\|_{m,G}.$$

The following trace inequalities hold

$$\|w\|_{m-\frac{1}{2},\partial G}\leqq\|w\|_{m,G} \qquad \forall w\in H^m(G) \tag{1.18a}$$

and for $m\geqq 2$

$$\left\|\frac{\partial w}{\partial \nu}\right\|_{m-\frac{3}{2},\partial G}\leqq C\|w\|_{m,G} \qquad \forall w\in H^m(G). \tag{1.18b}$$

In (1.17) and (1.18) above $\boldsymbol{\nu}$ is the outward pointing unit normal to ∂G and C is a constant independent of w; see Kufner, John and Fucik [10].

Finally, we require the Friedrichs' inequality for ∂G of class $\mathscr{C}^{0,1}$

$$\|w\|^2_{0,G}\leqq C\{|w|^2_{1,G}+\|w\|^2_{0,\partial G}\} \qquad \forall w\in H^1(G), \tag{1.19}$$

where C is a constant independent of w.

2. Error Bounds in the Absence of Variational Crimes

Before discussing the finite element approximation of the penalised problem (1.8) we first show that $u_\varepsilon\to u$ as $\varepsilon\to 0$.

Theorem 2.1. *The solutions u and u_ε of* (1.3) *and* (1.8) *satisfy:*

$$\|u-u_\varepsilon\|_{0,\Omega} \leqq C\varepsilon \|u\|_{2,\Omega}. \tag{2.1}$$

Proof. From (1.3) and (1.8) it follows that

$$a_\varepsilon(u-u_\varepsilon, v) = \left\langle \sigma \frac{\partial u}{\partial \nu}, v \right\rangle_{\partial\Omega} \qquad \forall v \in H^1(\Omega) \tag{2.2}$$

and so choosing $v = u - u_\varepsilon$ we obtain

$$\begin{aligned} \varepsilon^{-1} \|u-u_\varepsilon\|^2_{0,\partial\Omega} &\leqq a_\varepsilon(u-u_\varepsilon, u-u_\varepsilon) \\ &= \left\langle \sigma \frac{\partial u}{\partial \nu}, u-u_\varepsilon \right\rangle_{\partial\Omega}. \end{aligned}$$

From the trace inequalities (1.17) it follows that

$$\|u-u_\varepsilon\|_{0,\partial\Omega} \leqq C\varepsilon \|u\|_{2,\Omega}. \tag{2.3}$$

Observe that

$$\|u-u_\varepsilon\|_{0,\Omega} = \sup_{\eta \in L^2(\Omega)} \frac{|(u-u_\varepsilon, \eta)_\Omega|}{\|\eta\|_{0,\Omega}}. \tag{2.4}$$

For any $\eta \in L^2(\Omega)$ define z such that

$$Az = \eta \quad \text{in } \Omega \qquad z = 0 \quad \text{on } \partial\Omega. \tag{2.5}$$

It follows from elliptic regularity for either $\partial\Omega$ smooth or Ω convex polyhedral that

$$\|z\|_{2,\Omega} \leqq C \|\eta\|_{0,\Omega}. \tag{2.6}$$

Note that as $z \in H^1_0(\Omega)$ we have

$$\begin{aligned} |(u-u_\varepsilon, Az)_\Omega| &= \left| a(u-u_\varepsilon, z) - \left\langle u-u_\varepsilon, \sigma \frac{\partial z}{\partial \nu} \right\rangle_{\partial\Omega} \right| \\ &\leqq C\varepsilon \|u\|_{2,\Omega} \|z\|_{2,\Omega}, \end{aligned} \tag{2.7}$$

where we have noted (2.2) and applied the bound (2.3) and the trace inequality (1.17c). Combining (2.4), (2.5), (2.6) and (2.7) yields the desired result (2.1). □

Let D^* be a bounded domain in $\mathbb{R}^n$ containing Ω such that $\bar{D}^* \equiv \bigcup_{\tau \in T^*} \bar{\tau}$, where T^* is a collection of disjoint open regular simplicial elements τ, each with maximum diameter not exceeding h. We have in general as Ω is not a union of elements that

$$\bar{\Omega} \subseteq \bar{D} \subseteq \bar{D}^*(h) \subset \mathbb{R}^n, \tag{2.8a}$$

where

$$\bar{D} \equiv \bigcup_{\tau \in T} \bar{\tau} \tag{2.8b}$$

and

$$T \equiv \{\tau \in T^* : \tau \cap \Omega \neq \emptyset\}. \tag{2.8c}$$

In addition, we set

$$B \equiv \{\tau \in T: \mathbf{m}(\bar{\tau} \cap \partial\Omega) \neq 0 \text{ in } \mathbb{R}^{n-1}\}. \tag{2.8d}$$

Associated with T is a finite dimensional subspace S^h of $C^0(\bar{D})$, depending upon an integer $K \geqq 2$. Let π_h: $C^0(\bar{D}) \to S^h$ denote the interpolation operator. Sometimes we shall require a generalised interpolation operator π_h^g: $L^2(D) \to S^h$, see Clement [7] for example. We assume that the following approximation properties hold:

(A1) For integers K' and m satisfying $K' \geqq m \geqq 0$, $K \geqq K' \geqq 2$ and for $p \in [2, \infty]$

$$|w - \pi_h w|_{m,p,\tau} \leqq C_1 h^{K'-m} |w|_{K',p,\tau} \qquad \forall w \in W^{K',p}(\tau),\ \forall \tau \in T, \tag{2.9a}$$

and for integers K' and m satisfying $K' \geqq m \geqq 0$ and $K \geqq K' \geqq 0$

$$|w - \pi_h^g w|_{m,\tau} \leqq C_2 h^{K'-m} |w|_{K',\tau} \qquad \forall w \in H^{K'}(\tau),\ \forall \tau \in T; \tag{2.9b}$$

where C_1 and C_2 are constants independent of h and w. This assumption is satisfied by the standard piecewise polynomial spaces S^h used in practice.

As the mesh is unfitted, in order to define the interpolate (generalised interpolate) in S^h of a given function $w \in H^2(\Omega)$ ($L^2(\Omega)$) it is necessary to extend the domain of definition of w. It is convenient to introduce a domain $\tilde{\Omega} \subset \mathbb{R}^n$ with a smooth boundary such that

$$\Omega \subseteq D \subseteq \tilde{\Omega} \qquad \forall h < h_0. \tag{2.10}$$

For all integers $s \geqq 0$ there exists an extension operator E: $H^s(\Omega) \to H^s(\tilde{\Omega})$ such that

$$Ew = w \quad \text{on } \Omega \tag{2.11a}$$

and

$$\|Ew\|_{s,\tilde{\Omega}} \leqq C \|w\|_{s,\Omega}, \tag{2.11b}$$

where C is independent of w (see Kufner, John and Fucik [10]).

A finite element approximation to (1.8) is: find $u_\varepsilon^h \in S^h$ such that

$$a_\varepsilon(u_\varepsilon^h, \chi) = l_\varepsilon(\chi) \qquad \forall \chi \in S^h. \tag{2.12}$$

This approximation has been analysed by Babuska [1] for $\partial\Omega$ smooth and assuming $u \in H^K(\Omega)$. As (2.12) requires the exact evaluation of integrals over curved domains it is not a practical finite element approximation. In Subsect. 2.1 we also analyse this approximation in order to see what one can expect from the penalty method in an ideal situation. Assuming more smoothness on u, $u \in H^{K+2}(\Omega)$, we are able to improve on the rates of convergence in the H^1 and L^2 norms given by Babuska [1]. In Sect. 3 we see that these rates are retained by a fully practical piecewise linear finite element approximation of (1.8) involving domain perturbation and numerical integration. In Subsect. 2.2 we analyse the approximation (2.12) for the case of Ω being convex polyhedral with $\bar{\Omega} \equiv D$.

In the results that follow we adopt the notation $\|\cdot\|_a^2 \equiv a(\cdot,\cdot)$. Lemmas 2.1 and 2.2 below derive abstract H^1 and L^2 error bounds for the Approximation (2.12).

Lemma 2.1. *The solutions u and u_ε^h of* (1.3) *and* (2.12) *satisfy:*

$$\|u-u_\varepsilon^h\|_a^2+\varepsilon^{-1}\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-u_\varepsilon^h\right\|_{0,\partial\Omega}^2 \leq \inf_{\chi\in S^h}\left[\|u-\chi\|_a^2+\varepsilon^{-1}\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-\chi\right\|_{0,\partial\Omega}^2\right]. \tag{2.13}$$

Proof. From (1.2) we have that

$$a(u,v)=l(v)+\left\langle\sigma\frac{\partial u}{\partial v},v\right\rangle_{\partial\Omega} \qquad \forall v\in H^1(\Omega). \tag{2.14}$$

Choosing $v=\chi$ and subtracting (2.12) we obtain

$$a(u-u_\varepsilon^h,\chi)+\varepsilon^{-1}\left\langle u-\varepsilon\sigma\frac{\partial u}{\partial v}-u_\varepsilon^h,\chi\right\rangle_{\partial\Omega}=0 \qquad \forall\chi\in S^h. \tag{2.15}$$

Therefore for all $\chi\in S^h$ it follows that

$$\begin{aligned}
&\|u-u_\varepsilon^h\|_a^2+\varepsilon^{-1}\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-u_\varepsilon^h\right\|_{0,\partial\Omega}^2\\
&=a(u-u_\varepsilon^h,u-\chi)+\varepsilon^{-1}\left\langle u-\varepsilon\sigma\frac{\partial u}{\partial v}-u_\varepsilon^h,u-\varepsilon\sigma\frac{\partial u}{\partial v}-\chi\right\rangle_{\partial\Omega}\\
&\leq\frac{1}{2}[\|u-u_\varepsilon^h\|_a^2+\|u-\chi\|_a^2]+\frac{\varepsilon^{-1}}{2}\left[\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-u_\varepsilon^h\right\|_{0,\partial\Omega}^2+\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-\chi\right\|_{0,\partial\Omega}^2\right].
\end{aligned}$$

Hence the desired result (2.13) holds. □

Lemma 2.2. *Assuming* (A1) *holds the solutions u and u_ε^h of* (1.3) *and* (2.12) *satisfy:*

$$\|u-u_\varepsilon^h\|_{0,\Omega}\leq C[h\|u-u_\varepsilon^h\|_a+\|u-u_\varepsilon^h\|_{0,\partial\Omega}] + C\varepsilon^{-1}\sup_{z\in H^2(\Omega)\cap H_0^1(\Omega)}\left[\frac{\|z-\pi_h\tilde{z}\|_{0,\partial\Omega}}{\|z\|_{2,\Omega}}\right]\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-u_\varepsilon^h\right\|_{0,\partial\Omega}, \tag{2.16}$$

where $\tilde{z}\equiv Ez$.

Proof. As

$$\|u-u_\varepsilon^h\|_{0,\Omega}=\sup_{\eta\in L^2(\Omega)}\frac{|(u-u_\varepsilon^h,\eta)_\Omega|}{\|\eta\|_{0,\Omega}},$$

defining z by (2.5) it follows from (2.15) that for all $\chi \in S^h$

$$|(u-u_\varepsilon^h, Az)_\Omega| = \left| a(u-u_\varepsilon^h, z) - \left\langle u-u_\varepsilon^h, \sigma \frac{\partial z}{\partial \nu} \right\rangle_{\partial\Omega} \right|$$
$$= \left| a(u-u_\varepsilon^h, z-\chi) + \varepsilon^{-1} \left\langle u - \varepsilon\sigma \frac{\partial u}{\partial \nu} - u_\varepsilon^h, z-\chi \right\rangle_{\partial\Omega} - \left\langle u-u_\varepsilon^h, \sigma \frac{\partial z}{\partial \nu} \right\rangle_{\partial\Omega} \right|$$
$$\leq \|u-u_\varepsilon^h\|_a \|z-\chi\|_a + \|u-u_\varepsilon^h\|_{0,\partial\Omega} \left\| \sigma \frac{\partial z}{\partial \nu} \right\|_{0,\partial\Omega}$$
$$+ \varepsilon^{-1} \left\| u - \varepsilon\sigma \frac{\partial u}{\partial \nu} - u_\varepsilon^h \right\|_{0,\partial\Omega} \|z-\chi\|_{0,\partial\Omega}.$$

Choosing $\chi = \pi_h \tilde{z}$, the desired result (2.16) follows from (2.9), (2.11b), (1.17c) and (2.6). □

From the abstract error bounds (2.13) and (2.16) we see the importance of the approximation property of S^h on $\partial\Omega$. From the proof of the trace theorem in Necas [11] p. 15 it follows that $\forall \delta > 0$ and $\forall w \in H^1(\Omega)$,

$$\|w\|_{0,\partial\Omega}^2 \leq \delta \|w\|_{0,\Omega}^2 + C\delta^{-1} \|w\|_{1,\Omega}^2, \tag{2.17}$$

where C is independent of δ and w. Applying this bound with $\delta = h^{-1}$ it follows from (A1) that for $1 \leq r \leq K$ and for all $w \in H^r(D)$ that

$$\|w-\pi_h^g w\|_{0,\partial\Omega}^2 \leq h^{-1} \|w-\pi_h^g w\|_{0,D}^2 + Ch \|w-\pi_h^g w\|_{1,D}^2$$
$$\leq Ch^{2r-1} |w|_{r,D}^2. \tag{2.18}$$

However, if we assume more smoothness on w it follows from (A1) that for $0 \leq r \leq K$ and for all $w \in H^{r+2}(D)$

$$\|w-\pi_h w\|_{0,\partial\Omega} \leq C \|w-\pi_h w\|_{0,\infty,\partial\Omega}$$
$$\leq C \|w-\pi_h w\|_{0,\infty,\Omega}$$
$$\leq Ch^r |w|_{r,\infty,D} \leq Ch^r \|w\|_{r+2,D}. \tag{2.19}$$

The application of the bound (2.19) instead of (2.18) leads to the improvement in the rate of convergence over that given by Babuska [1].

2.1. $\partial\Omega$ Smooth

We make the following regularity assumptions on the data:

(A2) $\partial\Omega$ is of class $\mathscr{C}^{K+1,1}$, $f \in H^K(\Omega)$, $g \in H^{K+\frac{3}{2}}(\partial\Omega)$; and σ, $c \in C^{K+1}(\Omega)$. It follows that the solution u of (1.3) is such that $u \in H^{K+2}(\Omega)$.

In order to obtain H^1 and L^2 error bounds we require the following approximation result.

Lemma 2.3. *Assume* (A1) *and* (A2) *hold. Setting* $\varepsilon=h^{\lambda}$, $\lambda>0$, *it follows that:*

$$\inf_{\chi\in S^h}\left[\|u-\chi\|_a^2+\varepsilon^{-1}\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-\chi\right\|_{0,\partial\Omega}^2\right] \leqq \begin{cases} Ch^{2\mu_1}\|u\|_{K,\Omega}^2 \\ Ch^{2\mu_1^*}\|u\|_{K+2,\Omega}^2 \end{cases}, \tag{2.20}$$

where

$$\mu_1=\min[\lambda, K-1, K-\tfrac{1}{2}(\lambda+1), K+\tfrac{1}{2}(\lambda-3)] \tag{2.21 a}$$

and

$$\mu_1^*=\min[\lambda, K-1, K-\tfrac{1}{2}\lambda]. \tag{2.21 b}$$

Proof. If $u\in H^r(\Omega)$ then $\frac{\partial u}{\partial v}\in H^{r-\frac{3}{2}}(\partial\Omega)$ for $2\leqq r\leqq K+2$. Hence there exists a harmonic function $v\in H^{r-1}(\Omega)$ such that $v=\sigma\frac{\partial u}{\partial v}$ on $\partial\Omega$ and from elliptic regularity theory and the trace inequality (1.18 b) it follows that

$$\|v\|_{r-1,\Omega}\leqq C\left\|\sigma\frac{\partial u}{\partial v}\right\|_{r-\frac{3}{2},\partial\Omega}\leqq C\|u\|_{r,\Omega}. \tag{2.22}$$

Choose $\chi=\pi_u^g\tilde{u}-\varepsilon\pi_h^g\tilde{v}$ in (2.20), where $\tilde{u}=Eu$ and $\tilde{v}=Ev$. The approximation assumption (A1), (2.11 b) and (2.22) yield

$$\begin{aligned}\|u-\pi_h^g\tilde{u}+\varepsilon\pi_h^g\tilde{v}\|_a^2 &\leqq 2[\|u-\pi_h^g\tilde{u}\|_a^2+\varepsilon^2\|\pi_h^g\tilde{v}\|_a^2] \\ &\leqq C[h^{2(K-1)}|\tilde{u}|_{K,D}^2+\varepsilon^2\|\tilde{v}\|_{1,D}^2] \\ &\leqq C[h^{2(K-1)}+h^{2\lambda}]\|u\|_{K,\Omega}^2. \end{aligned}\tag{2.23}$$

The approximation result (2.18), (2.11 b) and (2.22) yield

$$\begin{aligned}&\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-\pi_h^g\tilde{u}+\varepsilon\pi_h^g\tilde{v}\right\|_{0,\partial\Omega}^2 \\ &\leqq 2[\|u-\pi_h^g\tilde{u}\|_{0,\partial\Omega}^2+\varepsilon^2\|v-\pi_h^g\tilde{v}\|_{0,\partial\Omega}^2] \\ &\leqq C[h^{2K-1}|\tilde{u}|_{K,D}^2+h^{2\lambda+2K-3}|\tilde{v}|_{K-1,D}^2] \\ &\leqq C[h^{2K-1}+h^{2\lambda+2K-3}]\|u\|_{K,\Omega}^2. \end{aligned}\tag{2.24}$$

Combining (2.23) and (2.24) yields the desired result (2.20) with (2.21 a).

Assuming $u\in H^{K+2}(\Omega)$ and choosing $\chi=\pi_h\tilde{u}-\varepsilon\pi_h\tilde{v}$ in (2.20), it follows that

$$\|u-\pi_h\tilde{u}+\varepsilon\pi_h\tilde{v}\|_a^2\leqq C[h^{2(K-1)}+h^{2\lambda}]\|u\|_{K+1,\Omega}^2 \tag{2.25}$$

and from (2.19), (2.11 b) and (2.22) that

$$\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-\pi_h\tilde{u}+\varepsilon\pi_h\tilde{v}\right\|_{0,\partial\Omega}^2\leqq C[h^{2K}+h^{2\lambda+2K-2}]\|u\|_{K+2,\Omega}^2. \tag{2.26}$$

Combining (2.26) with (2.25) yields the desired result (2.20) with (2.21 b). □

Combining the above lemmas we obtain the H^1 and L^2 error bounds.

Theorem 2.2. *Assume* (A1) *and* (A2) *hold. Setting* $\varepsilon=h^\lambda$, $\lambda>0$, *the solutions* u *and* u_ε^h *of* (1.3) *and* (2.12) *satisfy:*

$$\|u-u_\varepsilon^h\|_{0,\partial\Omega}\leqq\begin{cases}Ch^{\mu_2}\|u\|_{K,\Omega}\\ Ch^{\mu_2^*}\|u\|_{K+2,\Omega}\end{cases} \tag{2.27}$$

and for $i=0$ *and* 1

$$|u-u_\varepsilon^h|_{i,\Omega}\leqq\begin{cases}Ch^{\mu_i}\|u\|_{K,\Omega}\\ Ch^{\mu_i^*}\|u\|_{K+2,\Omega}\end{cases}; \tag{2.28}$$

where μ_1 *and* μ_1^* *are as defined in* (2.21),

$$\mu_2^{(*)}=\min[\lambda,\mu_1^{(*)}+\tfrac{1}{2}\lambda] \tag{2.29}$$

and

$$\mu_0^{(*)}=\min[\mu_1^{(*)}+1,\mu_1^{(*)}+\tfrac{1}{2}(3-\lambda),\mu_2^{(*)}]. \tag{2.30}$$

Proof. From (2.13) and (2.20) we obtain

$$\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-u_\varepsilon^h\right\|_{0,\partial\Omega}\leqq\begin{cases}Ch^{\mu_1+\frac{1}{2}\lambda}\|u\|_{K,\Omega}\\ Ch^{\mu_1^*+\frac{1}{2}\lambda}\|u\|_{K+2,\Omega}\end{cases}. \tag{2.31}$$

As

$$\|u-u_\varepsilon^h\|_{0,\partial\Omega}\leqq\left\|u-\varepsilon\sigma\frac{\partial u}{\partial v}-u_\varepsilon^h\right\|_{0,\partial\Omega}+\varepsilon\left\|\sigma\frac{\partial u}{\partial v}\right\|_{0,\partial\Omega}, \tag{2.32}$$

applying the trace inequality (1.17c) and the result (2.31) we obtain the desired results (2.27) and (2.29).

As $|\cdot|_{1,\Omega}\leqq C\|\cdot\|_a$ the H^1 error bound (2.28) follows directly from (2.13) and (2.20). The L^2 error bound (2.28) follows directly from (2.16), (2.18), (2.11b), (2.13), (2.20), (2.27) and (2.31). □

The value of μ_1 is exactly that obtained by Babuska [1]. The value of μ_0 is a slight improvement to that given by Babuska [1], see (1.14). Clearly there is a substantial improvement in the rate of convergence by assuming more smoothness on u and this is usually not a restriction in practice. As stated previously an optimal H^1 error bound is now obtained for piecewise linears, $K=2$, by choosing λ such that $1\leqq\lambda\leqq 2$, but this does not lead to an optimal L^2 error bound. For $K=3$ the choice $\lambda=2$ yields an optimal H^1 error bound, but once again a suboptimal L^2 bound. For $K\geqq 4$ no choice of λ leads even to an optimal H^1 error bound. Therefore in practice the penalty formulation for $\partial\Omega$ smooth and $\Omega\subseteq D$ should only be used for low order finite element spaces. In Sect. 3 we analyse a fully practical piecewise linear approximation to (1.8) and show that the bounds (2.27) and (2.28) remain valid if λ is chosen to be 2. Moreover, although the global L^2 error bound is only $O(h^{\frac{3}{2}})$ we obtain an optimal order interior L^2 error bound over a domain $\Omega_0\subset\subset\Omega_1\subset\subset\Omega^h$.

Before discussing the case of Ω being a convex polyhedral we end this subsection by mentioning the extrapolation method of King [8]. Let u_ε^h solve

$$a_\varepsilon(u_\varepsilon^h,\chi)=l_\varepsilon(\chi)\qquad\forall\chi\in S^h \tag{2.33}$$

where $\varepsilon = 2^{-j}\gamma h$, $j = 0, 1, 2, \dots$ and define

$$v_{2^{-j}h} \equiv u^h_{2^{-j}\gamma h}$$

where h and γ are fixed. Using Richardson's extrapolation with respect to j,

$$u_h^{(1)} = 2v_{h/2} - v_h \tag{2.34a}$$

$$u_h^{(j)} = \frac{2^j u_{h/2}^{(j-1)} - u_h^{(j-1)}}{2^{j-1}} \qquad j \geqq 2; \tag{2.34b}$$

one obtains the following error bounds for Poisson's equation,

$$\|u - u_h^{(K-2)}\|_{1,\Omega} \leqq C h^{K-1} \|u\|_{K,\Omega} \tag{2.35a}$$

$$\|u - u_h^{(K-1)}\|_{0,\Omega} \leqq C h^{K} \|u\|_{K+\frac{1}{2}+\delta,\Omega} \tag{2.35b}$$

where $\delta > 0$ is arbitrary. Once again this analysis is for smooth $\partial\Omega$ and in that case (2.33) is an impractical method. Numerical results for this approach when $\bar{\Omega} \equiv \bar{D}$ is *a* polygon are given in King and Serbin [9].

2.2. Ω Convex Polyhedral

Zhong-Ci Shi [14] analysed the approximation (2.12) for Poisson's equation assuming that Ω is polygonal and $\bar{\Omega} \equiv \bar{D}$. As stated previously, this is a less interesting case as now the Dirichlet boundary condition can be imposed essentially in practice using a standard finite element approximation of (1.3). However, this analysis was undertaken to explain the numerical results of Utku and Carey [13]. They solved the problem

$$-\nabla^2 u = 0 \quad \text{in } \Omega \equiv [0,1] \times [0,1] \qquad u = g \quad \text{on } \partial\Omega, \tag{2.36}$$

where g was chosen so that the solution $u = x + y - 2xy$. They found in practice using piecewise linear elements with $\lambda = 1$ that the approximation (2.12) converged at the optimal rate in H^1, although their suboptimal analysis predicted $O(h^{\frac{1}{2}})$. Zhong-Ci Shi [14] showed that for $\bar{\Omega} \equiv \bar{D}$ polygonal, $\varepsilon = h^\lambda$ and assuming that $\frac{\partial u}{\partial \nu} \in H^{\frac{1}{2}}(\partial\Omega)$ the solutions of (1.3) and (2.12) for $K = 2$ satisfy

$$\|u - u_\varepsilon^h\|_{1,\Omega} \leqq C h^{\mu_1} \left[\|u\|_{2,\Omega}^2 + \left\| \frac{\partial u}{\partial \nu} \right\|_{\frac{1}{2},\partial\Omega}^2 \right]^{\frac{1}{2}}, \tag{2.37}$$

where

$$\mu_1 = \min[\lambda, 1, \tfrac{1}{2}(3-\lambda), \tfrac{1}{2}(\lambda+1)]. \tag{2.38}$$

The proof of (2.37) and (2.38) follow the same lines as for $\partial\Omega$ smooth, except that in the proof of Lemma 2.3 $u \in H^2(\Omega)$ does not imply that $\frac{\partial u}{\partial \nu} \in H^{\frac{1}{2}}(\partial\Omega)$ so one needs to make it an assumption. Hence the value of μ_1 agrees exactly with the analysis of Babuska [1] for $\partial\Omega$ smooth, (2.28) and (2.21a) with $K=2$. Clearly the result (2.37) with (2.38) applies to the problem (2.36) as $\frac{\partial u}{\partial v}\in H^{\frac{1}{2}}(\partial\Omega)$.

We now present an alternative analysis. For $\bar{\Omega}\equiv\bar{D}$ polyhedral we define ∂S^h where $S^h\equiv S_0^h\oplus\partial S^h$ and

$$S_0^h\equiv\{\chi\in S^h: \chi=0 \text{ on } \partial\Omega\}. \tag{2.39}$$

We introduce $P_h g$, the $L^2(\partial\Omega)$ projection of the boundary data g onto ∂S^h, defined by: $P_h g\in\partial S^h$ such that

$$\langle g-P_h g, \chi\rangle_{\partial\Omega}=0 \qquad \forall\chi\in\partial S^h \tag{2.40}$$

and set

$$S_E^h\equiv\{\chi\in S^h: \chi=P_h g \text{ on } \partial\Omega\}. \tag{2.41}$$

Then a finite element approximation of (1.3) is: find $u^h\in S_E^h$ such that

$$a(u^h,\chi)=l(\chi) \qquad \forall\chi\in S_0^h. \tag{2.42}$$

As $\bar{\Omega}\equiv\bar{D}$ if we let $\varepsilon\to 0$ for fixed h, $u_\varepsilon^h\to u^h$. It is well known that $u^h\to u$ at an optimal rate in the H^1 and L^2 norms and so we would expect the convergence rate of u_ε^h to u for fixed K to tend to the optimal rate as λ increases. This is not reflected in the analysis of Zhong-Ci Shi [14] due to the presence of the term $\frac{1}{2}(3-\lambda)$ in (2.38). The reason for this is the analysis of Zhong-Ci Shi [14] does not make use of the fact that $\bar{\Omega}\equiv\bar{D}$. This we do below.

Firstly, we require some assumptions. Note that the interpolation operator π_h induces an interpolation operator on $C^0(\overline{\partial\Omega})$ into ∂S^h, which for notational convenience is still denoted by π_h. We assume the following approximation property holds:

(A3) For an integer K' with $K\geqq K'\geqq 2$

$$\|w-\pi_h w\|_{0,\partial\Omega_\tau}\leqq C_3 h^{K'}\|w\|_{K',\partial\Omega_\tau} \quad \forall w\in H^{K'}(\partial\Omega_\tau)\cap C^0(\partial\Omega_\tau),\ \forall\tau\in B, \tag{2.43}$$

where $\partial\Omega_\tau\equiv\partial\Omega\cap\bar{\tau}$ and C_3 is a constant independent of h and w. In addition we assume the inverse inequalities:

(A4) For all $\chi\in S^h$

$$\|\chi\|_{1,\tau}\leqq C_4 h^{-1}\|\chi\|_{0,\tau} \qquad \forall\tau\in T \tag{2.44a}$$

and

$$\|\chi\|_{0,\infty,\partial\Omega_\tau}\leqq C_5 h^{-(n-1)/2}\|\chi\|_{0,\partial\Omega_\tau} \qquad \forall\tau\in B, \tag{2.44b}$$

where C_4 and C_5 are constants independent of h and χ. Once again these assumptions are satisfied by the standard piecewise polynomial spaces S^h.

Lemma 2.4. *Assume* $\bar{\Omega}\equiv\bar{D}$ *polyhedral* (A1) *and* (A4) *hold. The solutions* u *and* u_ε^h *of* (1.3) *and* (2.12) *satisfy:*

$$\|u-u_\varepsilon^h\|_a\leqq\|u-\pi_h u\|_a+Ch^{-\frac{1}{2}}[\|u-\pi_h u\|_{0,\partial\Omega}+\|u-u_\varepsilon^h\|_{0,\partial\Omega}]. \tag{2.45}$$

Proof. From (2.12) and (1.3) we obtain for all $\chi \in S_0^h$ that

$$a(u_\varepsilon^h, \chi) = a_\varepsilon(u_\varepsilon^h, \chi) = l_\varepsilon(\chi) = l(\chi) = a(u, \chi). \tag{2.46}$$

Let $r^h \in \partial S^h$ be such that $r^h = \pi_h u - u_\varepsilon^h$ on $\partial\Omega$ and so from (2.46) it follows that

$$\begin{aligned}\|u - u_\varepsilon^h\|_a^2 &= a(u - u_\varepsilon^h, u - \pi_h u + r^h) \\ &\leqq \|u - u_\varepsilon^h\|_a [\|u - \pi_h u\|_a + \|r^h\|_a].\end{aligned} \tag{2.47}$$

Let B^* denote those elements $\tau \in T$ with a node on $\partial\Omega$. It follows from the inverse inequalities (2.44) that

$$\begin{aligned}\|r^h\|_{1,\Omega}^2 &= \sum_{\tau \in B^*} \|r^h\|_{1,\tau}^2 \leqq C h^{-2} \sum_{\tau \in B^*} \|r^h\|_{0,\tau}^2 \\ &\leqq C h^{n-2} \sum_{\tau \in B^*} \|r^h\|_{0,\infty,\tau}^2 \\ &\leqq C h^{n-2} \sum_{\tau \in B} \|r^h\|_{0,\infty,\partial\Omega_\tau}^2 \\ &\leqq C h^{-1} \sum_{\tau \in B} \|r^h\|_{0,\partial\Omega_\tau}^2 \\ &= C h^{-1} \|r^h\|_{0,\partial\Omega}^2.\end{aligned} \tag{2.48}$$

As $\|\cdot\|_a \leqq C\|\cdot\|_1$ the desired result (2.45) follows from (2.47) and (2.48). □

Theorem 2.3. *Assume* $\bar{\Omega} \equiv \bar{D}$ *convex polyhedral* (A1), (A3) *and* (A4) *hold. Setting* $\varepsilon = h^\lambda$, $\lambda > 0$, *the solutions* u *and* u_ε^h *of* (1.3) *and* (2.12) *satisfy:*

$$\|u - u_\varepsilon^h\|_{0,\partial\Omega} \leqq \begin{cases} C h^{\hat{\mu}_2} \|u\|_{K,\Omega} \\ C h^{\hat{\mu}_2^*} \|u\|_{K+1,\Omega} \end{cases}, \tag{2.49}$$

and for $i = 0$ *and* 1

$$|u - u_\varepsilon^h|_{i,\Omega} \leqq \begin{cases} C h^{\hat{\mu}_i} \|u\|_{K,\Omega} \\ C h^{\hat{\mu}_i^*} \|u\|_{K+1,\Omega} \end{cases}, \tag{2.50}$$

where

$$\hat{\mu}_2 = \min[\lambda, K - \tfrac{1}{2}, K - 1 + \tfrac{1}{2}\lambda], \tag{2.51 a}$$

$$\hat{\mu}_2^* = \min[\lambda, K, K - 1 + \tfrac{1}{2}\lambda], \tag{2.51 b}$$

$$\hat{\mu}_1^{(*)} = \min[K - 1, \hat{\mu}_2^{(*)} - \tfrac{1}{2}] \tag{2.52}$$

and

$$\hat{\mu}_0^{(*)} = \min[\hat{\mu}_1^{(*)} + 1, \hat{\mu}_2^{(*)}]; \tag{2.53}$$

assuming u *is sufficiently regular.*

Proof. From (2.13) it follows that

$$\begin{aligned}\|u - u_\varepsilon^h\|_{0,\partial\Omega}^2 &\leqq 2\left[\left\|u - \varepsilon\sigma\frac{\partial u}{\partial v} - u_\varepsilon^h\right\|_{0,\partial\Omega}^2 + \varepsilon^2\left\|\sigma\frac{\partial u}{\partial v}\right\|_{0,\partial\Omega}^2\right] \\ &\leqq \inf_{\chi \in S^h}[2\varepsilon\|u - \chi\|_a^2 + 4\|u - \chi\|_{0,\partial\Omega}^2] + 6\varepsilon^2\left\|\sigma\frac{\partial u}{\partial v}\right\|_{0,\partial\Omega}^2.\end{aligned} \tag{2.54}$$

Choosing $\chi=\pi_h u$, applying the approximation results (2.9a), (2.18) and the trace inequality (1.17c) yields the desired result (2.49) with (2.51a). Applying (2.43) and the trace inequality (1.17b) instead of (2.18) yields the desired result (2.49) with (2.51b).

The result (2.50) with (2.52) follows from (2.45), (2.9a), (2.18) and (2.49). The result (2.50) with (2.53) follows from (2.16), (2.49) and (2.52). □

Hence it follows for a given $K \geqq 2$, if $u \in H^K(\Omega)$ choosing $\lambda \geqq K-\frac{1}{2}$ yields an optimal H^1 error bound and if $u \in H^{K+1}(\Omega)$ choosing $\lambda \geqq K$ yields optimal H^1 and L^2 error bounds.

3. Error Bounds in the Presence of Variational Crimes

As stated previously the approximation (2.12) is not practical for $\partial\Omega$ smooth because it requires the exact evaluation of integrals over curved domains. We now study a fully practical approximation using piecewise linears, $K=2$, under the assumption (A2). The domain Ω and its boundary $\partial\Omega$ are approximated by Ω^h and $\partial\Omega^h$. We make the following assumption on this boundary approximation:

(A5) $$\operatorname{dist}(\Omega, \Omega^h) \leqq C h^2, \tag{3.1}$$

which is achieved by choosing $\partial\Omega^h$ in each element $\tau \in B$ to be the linear polynomial interpolating $\partial\Omega$. The interpolation points being where $\partial\Omega$ crosses either the elements sides ($n=2$) or edges ($n=3$) yielding a polyhedral domain Ω^h.

We now have

$$\bar{\Omega}^h \subseteq \bar{D}^h \subseteq \bar{D}^*(h) \subset \mathbb{R}^n, \tag{3.2a}$$

where

$$\bar{D}^h \equiv \bigcup_{\tau \in T^h} \bar{\tau}, \tag{3.2b}$$

$$T^h \equiv \{\tau \in T^* : \tau \cap \Omega^h \neq \emptyset\} \tag{3.2c}$$

and

$$B^h \equiv \{\tau \in T^h : \mathbf{m}(\bar{\tau} \cap \partial\Omega^h) \neq 0 \text{ in } \mathbb{R}^{n-1}\}. \tag{3.2d}$$

We require the approximation property (2.9) to hold now for all $\tau \in T^h$ and $\tilde{\Omega}$ in (2.10) to be such that

$$\Omega^h \subseteq D^h \subseteq \tilde{\Omega} \quad \forall h < h_0 \quad \text{and} \quad \Omega \subseteq \tilde{\Omega}. \tag{3.3}$$

It follows immediately from (3.3) and the proof of the trace theorem in Necas ([11], p. 15) that the constants C are independent of h for the trace inequalities (1.17) with $G \equiv \Omega^h$. Similarly (1.19) and (2.18) hold with $\partial\Omega$ and Ω replaced by $\partial\Omega^h$ and Ω^h with C independent h and so for $1 \leqq r \leqq 2$

$$\|w-\pi_h^g w\|_{0,\partial\Omega^h} \leqq C h^{r-\frac{1}{2}} |w|_{r,D^h}. \tag{3.4a}$$

Assuming more smoothness, as in (2.19), we have for $0 \leqq r \leqq 2$

$$\|w-\pi_h w\|_{0,\partial\Omega^h} \leqq C h^r \|w\|_{r+2,D^h}. \tag{3.4b}$$

In addition there exists an extension operator E^h: $H^s(\Omega^h)\to H^s(\tilde{\Omega})$, in analogue to (2.11), for all integers $s\geqq 0$ such that

$$E^h w = w \quad \text{on } \Omega^h \tag{3.5a}$$

and

$$\|E^h w\|_{s,\tilde{\Omega}} \leqq C\|w\|_{s,\Omega^h} \tag{3.5b}$$

where C is independent of w and h as it depends only on the Lipschitz constants of $\partial\Omega^h$, which are clearly independent of h, by its construction, see Babuska and Aziz ([2], p. 30).

Because, in general, $\Omega^h \not\subseteq \Omega$ it is necessary to extend the data. We assume that f, g, σ and c are restrictions to Ω of $\tilde{f}\in H^2(\tilde{\Omega})$, $\tilde{g}\in H^4(\tilde{\Omega})$, $\tilde{\sigma}$ and $\tilde{c}\in C^3(\tilde{\Omega})$ and in addition that

$$\tilde{\sigma}_1 \geqq \tilde{\sigma}(x) \geqq \tilde{\sigma}_0 > 0 \qquad \forall x\in\tilde{\Omega},$$

$$\tilde{c}_1 \geqq \tilde{c}(x) \geqq \tilde{c}_0 \geqq 0 \qquad \forall x\in\tilde{\Omega}.$$

We define

$$\tilde{A}w \equiv -\nabla\cdot(\tilde{\sigma}\nabla w) + \tilde{c}w \tag{3.6}$$

and

$$\tilde{a}^h(w,v) \equiv (\tilde{\sigma}\nabla w, \nabla v)_{\Omega^h} + (\tilde{c}w, v)_{\Omega^h} \tag{3.7a}$$

$$= (\tilde{A}w, v)_{\Omega^h} + \left\langle \tilde{\sigma}\frac{\partial w}{\partial \nu^h}, v\right\rangle_{\partial\Omega^h}, \tag{3.7b}$$

where ν^h is the outward pointing unit normal to $\partial\Omega^h$. We define also

$$\tilde{a}^h_\varepsilon(w,v) = \tilde{a}^h(w,v) + \varepsilon^{-1}\langle w, v\rangle_{\partial\Omega^h}, \tag{3.8a}$$

$$\tilde{l}^h(v) = (\tilde{f}, v)_{\Omega^h} \tag{3.8b}$$

and

$$\tilde{l}^h_\varepsilon(v) = \tilde{l}^h(v) + \varepsilon^{-1}\langle \tilde{g}, v\rangle_{\partial\Omega^h}. \tag{3.8c}$$

It follows from (3.2a) that for all w, $v\in H^1(\Omega^h)$

$$|\tilde{a}^h(w,v)| \leqq C\|w\|_{1,\Omega^h}\|v\|_{1,\Omega^h}, \tag{3.9}$$

where C is independent of h and ε.

A fully practical piecewise linear finite element approximation to (1.8) is: find $u^h_\varepsilon\in S^h$ such that

$$a^h_\varepsilon(u^h_\varepsilon, \chi) = l^h_\varepsilon(\chi) \qquad \forall \chi\in S^h, \tag{3.10}$$

where $a^h_\varepsilon(\cdot,\cdot)$ and $l^h_\varepsilon(\cdot)$ are approximations to $\tilde{a}^h_\varepsilon(\cdot,\cdot)$ and $\tilde{l}^h_\varepsilon(\cdot)$.

As the mesh is unfitted we have to approximate integrals over the subregions $\Omega^h\cap\tau$ and $\partial\Omega^h\cap\bar{\tau}$ for those elements $\tau\in B^h$. The subregion $\Omega^h\cap\tau$ is either a simplex or a union of 2 (3) simplices for $n=2$ (3) so that

$$\Omega^h\cap\tau = \bigcup t \qquad \forall\tau\in T^h \tag{3.11a}$$

and

$$\partial\Omega^h\cap\bar{\tau} = \gamma \qquad \forall\tau\in B^h, \tag{3.11b}$$

where t is a simplex in $\mathbb{R}^n$ and γ is a simplex in $\mathbb{R}^{n-1}$ being the boundary face of a $t\subset\Omega^h\cap\tau$. We now choose integration rules over t and γ. With $\{a_i\}_{i=1}^{n+1}$

being the vertices of t and $\{a_i\}_{i=1}^n$ the vertices of γ we define quadrature rules $I_t(v)$ and $I_\gamma(v)$ approximating the integrals $\int_t v(x)\,dx$ and $\int_\gamma v(x)\,dx$ by

$$I_t(v)=\frac{\mathbf{m}(t)}{n+1}\sum_{i=1}^{n+1} v(a_i) \tag{3.12a}$$

and

$$I_\gamma(v)=\begin{cases}\dfrac{\mathbf{m}(\gamma)}{6}(v(a_1)+4v(\tfrac{1}{2}(a_1+a_2))+v(a_2)) & \text{if } n=2\\[2ex] \dfrac{\mathbf{m}(\gamma)}{3}\displaystyle\sum_{i\neq j}\sum v(\tfrac{1}{2}(a_i+a_j)) & \text{if } n=3.\end{cases} \tag{3.12b}$$

We denote by $(\cdot,\cdot)^h$ and $\langle\cdot,\cdot\rangle^h$ the approximations of $(\cdot,\cdot)_{\Omega^h}$ and $\langle\cdot,\cdot\rangle_{\partial\Omega^h}$ by the quadrature formula (3.12a) and (3.12b), respectively. Then we set

$$a_\varepsilon^h(\cdot,\cdot)\equiv(\tilde{\sigma}\nabla\cdot,\nabla\cdot)^h+(\tilde{c}\cdot,\cdot)^h+\varepsilon^{-1}\langle\cdot,\cdot\rangle^h \tag{3.13a}$$

and

$$l_\varepsilon^h(\cdot)\equiv(\tilde{f},\cdot)^h+\varepsilon^{-1}\langle\tilde{g},\cdot\rangle^h. \tag{3.13b}$$

Lemma 3.1. *With $a_\varepsilon^h(\cdot,\cdot)$ and $l_\varepsilon^h(\cdot)$ defined by* (3.13) *it follows that*

(i) $a_\varepsilon^h(\chi,\chi)\geq C\tilde{a}_\varepsilon^h(\chi,\chi)\quad\forall\chi\in S^h$, (3.14)

(ii) *for all $w\in H^2(\tilde{\Omega})$ and for all $w^h,\chi\in S^h$,*

$$|\tilde{a}_\varepsilon^h(w^h,\chi)-a_\varepsilon^h(w^h,\chi)| \leq Ch\{h\|w\|_{2,\Omega^h}+\|w-w^h\|_{1,\Omega^h}\}\|\chi\|_{1,\Omega^h} \tag{3.15a}$$

and

$$|\tilde{l}_\varepsilon^h(\chi)-l_\varepsilon^h(\chi)| \leq Ch^2\{\|\tilde{f}\|_{2,\Omega^h}\|\chi\|_{1,\Omega^h}+\varepsilon^{-1}\|\tilde{g}\|_{3,\Omega^h}\|\chi\|_{0,\partial\Omega^h}\}. \tag{3.15b}$$

Proof. As (3.12b) integrates quadratics exactly it follows that

$$\langle w^h,\chi\rangle^h\equiv\langle w^h,\chi\rangle_{\partial\Omega^h}\qquad\forall w^h,\chi\in S^h. \tag{3.16}$$

Therefore the result (3.14) is a direct consequence of Lemma 5.10 in Barrett and Elliott [6]. Likewise the results (3.15) are a direct consequence of Lemmas 5.8 and 5.9 in Barrett and Elliott [6] and the trace inequality (1.17b). □

Remark 3.1. The result (3.14) implies the well-posedness of the approximation (3.10). We note that the sampling points of $(\cdot,\cdot)^h$ lie in $\bar{\Omega}$ so that the approximation u_ε^h is independent of the extensions $\tilde{f}$, $\tilde{\sigma}$ and $\tilde{c}$. □

In the results that follow we adopt the notation

$$\|\ \|_{\tilde{a}^h}^2\equiv\tilde{a}^h(\cdot,\cdot)\quad\text{and}\quad\|\cdot\|_{\tilde{a}_\varepsilon^h}^2\equiv\tilde{a}_\varepsilon^h(\cdot,\cdot).$$

Since the approximation u_ε^h is independent of the extension f, $\tilde{f}$ may be chosen for the convenience of the analysis. A suitable choice is

$$\tilde{f}\equiv\tilde{A}\tilde{u}, \tag{3.17}$$

where $\tilde{u}=Eu$. Clearly $\tilde{f}\equiv\tilde{A}u=f$ in Ω, which implies, together with (A2) and (2.11) that

$$\|\tilde{f}\|_{2,\Omega^h}\equiv\|\tilde{A}\tilde{u}\|_{2,\Omega^h}\leqq C\|\tilde{u}\|_{4,\Omega^h} \leqq C\|u\|_{4,\Omega}. \tag{3.18}$$

The following lemma derives abstract H^1 and H^{-r}, $r\geqq 0$, error estimates. The negative norm estimate is required later in obtaining an interior L^2 error bound.

Lemma 3.2. *Let* (A2) *hold. The solutions u and u_ε^h of* (1.3) *and* (3.10) *satisfy:*

$$\|\tilde{u}-u_\varepsilon^h\|_{\tilde{a}_\varepsilon^h}\leqq C\inf_{\xi\in S^h}\left[\|\tilde{u}-\xi\|_{\tilde{a}_\varepsilon^h}+\sup_{\chi\in S^h}\frac{|\tilde{a}_\varepsilon^h(\xi,\chi)-a_\varepsilon^h(\xi,\chi)|}{\|\chi\|_{\tilde{a}_\varepsilon^h}}\right] + \sup_{\chi\in S^h}\left[\frac{|\tilde{l}_\varepsilon^h(\chi)-l_\varepsilon^h(\chi)|+|\tilde{a}_\varepsilon^h(\tilde{u},\chi)-\tilde{l}_\varepsilon^h(\chi)|}{\|\chi\|_{\tilde{a}_\varepsilon^h}}\right] \tag{3.19}$$

and for $r\geqq 0$

$$\begin{aligned}\|\tilde{u}-u_\varepsilon^h\|_{-r,\Omega^h}\leqq C\{&\|\tilde{u}-u_\varepsilon^h\|_{0,\Omega^h-\Omega}+\|\tilde{u}-u_\varepsilon^h\|_{0,\partial\Omega^h}\\ &+\sup_{z\in H^{r+2}(\Omega)\cap H_0^1(\Omega)}\|z\|_{r+2,\Omega}^{-1}[\|\tilde{z}-\pi_h\tilde{z}\|_{1,\Omega^h}\|\tilde{u}-u_\varepsilon^h\|_{1,\Omega^h}\\ &+\varepsilon^{-1}\|\pi_h\tilde{z}\|_{0,\partial\Omega^h}\|\tilde{u}-u_\varepsilon^h\|_{0,\partial\Omega^h}+|\tilde{a}_\varepsilon^h(u_\varepsilon^h,\pi_h\tilde{z})-a_\varepsilon^h(u_\varepsilon^h,\pi_h\tilde{z})|\\ &+|\tilde{l}_\varepsilon^h(\pi_h\tilde{z})-l_\varepsilon^h(\pi_h\tilde{z})|+|\tilde{a}_\varepsilon^h(\tilde{u},\pi_h\tilde{z})-\tilde{l}_\varepsilon^h(\pi_h\tilde{z})|]\},\end{aligned} \tag{3.20}$$

where $\tilde{u}=Eu$ and $\tilde{z}=Ez$.

Proof. Evidently the inequality

$$\|\tilde{u}-u_\varepsilon^h\|_{\tilde{a}_\varepsilon^h}\leqq\|\tilde{u}-\xi\|_{\tilde{a}_\varepsilon^h}+\|\xi-u_\varepsilon^h\|_{\tilde{a}_\varepsilon^h} \tag{3.21}$$

holds for all $\xi\in S^h$. Setting $\chi=u_\varepsilon^h-\xi$ it follows from (3.14) that

$$\|\chi\|_{\tilde{a}_\varepsilon^h}^2\leqq C^{-1}a_\varepsilon^h(\chi,\chi) \tag{3.22}$$

and

$$a_\varepsilon^h(\chi,\chi)=\tilde{a}_\varepsilon^h(\tilde{u}-\xi,\chi)+[\tilde{a}_\varepsilon^h(\xi,\chi)-a_\varepsilon^h(\xi,\chi)] +[l_\varepsilon^h(\chi)-\tilde{l}_\varepsilon^h(\chi)]+[\tilde{l}_\varepsilon^h(\chi)-\tilde{a}_\varepsilon^h(\tilde{u},\chi)]. \tag{3.23}$$

Combining (3.21), (3.22) and (3.23) yields the desired result (3.19).

We have for $r\geqq 0$

$$\|\tilde{u}-u_\varepsilon^h\|_{-r,\Omega^h}=\sup_{\eta\in H^r(\Omega^h)}\frac{|(\tilde{u}-u_\varepsilon^h,\eta)_{\Omega^h}|}{\|\eta\|_{r,\Omega^h}}. \tag{3.24}$$

Extend η to $\tilde{\Omega}$ by setting $\tilde{\eta}=E^h\eta$ and define z such that

$$Az=\tilde{\eta}\quad\text{in }\Omega\qquad z=0\quad\text{on }\partial\Omega. \tag{3.25}$$

From (A2) and elliptic regularity theory and (3.5b) it follows that

$$\|z\|_{r+2,\Omega}\leqq C\|\tilde{\eta}\|_{r,\Omega}\leqq C\|\eta\|_{r,\Omega^h}. \tag{3.26}$$

It follows from (3.7b) and (3.10) that

$$\begin{aligned}(\tilde{u}-u_\varepsilon^h,\eta)_{\Omega^h} &= (\tilde{u}-u_\varepsilon^h,\tilde{A}\tilde{z})_{\Omega^h}+(\tilde{u}-u_\varepsilon^h,\eta-\tilde{A}\tilde{z})_{\Omega^h-\Omega}\\ &= \tilde{a}^h(\tilde{u}-u_\varepsilon^h,\tilde{z}-\pi_h\tilde{z})-\left\langle \tilde{u}-u_\varepsilon^h,\tilde{\sigma}\frac{\partial\tilde{z}}{\partial v^h}+\varepsilon^{-1}\pi_h\tilde{z}\right\rangle_{\partial\Omega^h}\\ &\quad+[a_\varepsilon^h(u_\varepsilon^h,\pi_h\tilde{z})-\tilde{a}_\varepsilon^h(u_\varepsilon^h,\pi_h\tilde{z})]+[\tilde{l}_\varepsilon^h(\pi_h\tilde{z})-l_\varepsilon^h(\pi_h\tilde{z})]\\ &\quad+[\tilde{a}_\varepsilon^h(\tilde{u},\pi_h\tilde{z})-\tilde{l}_\varepsilon^h(\pi_h\tilde{z})]+(\tilde{u}-u_\varepsilon^h,\eta-\tilde{A}\tilde{z})_{\Omega^h-\Omega}. \end{aligned} \tag{3.27}$$

Bounding the right hand side of (3.27) using (3.9), noting that

$$\|\tilde{A}\tilde{z}\|_{0,\Omega^h-\Omega}\leqq C\|\tilde{z}\|_{2,\Omega^h}\leqq C\|z\|_{2,\Omega} \tag{3.28}$$

and applying the trace inequality (1.17b) the desired result (3.20) follows from combining this bound with (3.24) and (3.26). □

The following estimates are required in obtaining the error bounds.

Lemma 3.3. *Assuming* (A5) *we have for all* $w\in H^1(\tilde{\Omega})$:

$$\|w\|_{0,\Omega^h-\Omega}\leqq C\{h^2|w|_{1,\Omega^h-\Omega}+h\|w\|_{0,\partial\Omega^h}\} \tag{3.29a}$$

and

$$\|w\|_{0,\Omega-\Omega^h}\leqq C\{h^2|w|_{1,\Omega-\Omega^h}+h\|w\|_{0,\partial\Omega}\}. \tag{3.29b}$$

Proof. The proof of (3.29a) is given in Lemma 3.2 in Barrett and Elliott [4]. The proof of (3.29b) follows in a similar manner. □

Lemma 3.4. *Assume* (A5) *holds. If* $w\in H^2(\tilde{\Omega})$ *and* $w=0$ *on* $\partial\Omega$ *then*

$$\|w\|_{0,\partial\Omega^h}\leqq Ch^2\|w\|_{2,\tilde{\Omega}}. \tag{3.30}$$

Proof. For each element $\tau\in B^h$ there exists a local co-ordinate system (X_τ, Y_τ) such that $X_\tau\in\Delta_\tau$ and $Y_\tau\in\mathbb{R}$, where Δ_τ is either an interval ($n=2$) or a triangle ($n=3$). The surface $\partial\Omega_\tau^h\equiv\partial\Omega^h\cap\bar{\tau}$ is locally described by $Y_\tau=\psi_\tau^h(X_\tau)$. The surface $\partial\Omega$ is locally described by $Y_\tau=\psi_\tau(X_\tau)$. Assuming (A5) it follows that

$$\|\psi_\tau-\psi_\tau^h\|_{0,\infty,\Delta_\tau}\leqq Ch^2\qquad\forall\tau\in B^h, \tag{3.31}$$

as ψ is sufficiently smooth by (A2).

Adopting the notation $w\equiv w(X,\psi_\tau(X))$ and $w(h)\equiv w(X,\psi_\tau^h(X))$ we have

$$\|w\|_{0,\partial\Omega^h}^2=\sum_{\tau\in B^h}\|w\|_{0,\partial\Omega_\tau^h}^2$$

and

$$\begin{aligned}\|w\|_{0,\partial\Omega_\tau^h}^2 &= \int_{\Delta_\tau}(w(h))^2(1+|\nabla\psi_\tau^h|^2)^{\frac12}\,dX_\tau\\ &= \int_{\Delta_\tau}[w(h)-w]^2(1+|\nabla\psi_\tau^h|^2)^{\frac12}\,dX_\tau\\ &= \int_{\Delta_\tau}\left[\int_{\psi_\tau(X)}^{\psi_\tau^h(X)}\frac{\partial w}{\partial Y_\tau}dY_\tau\right](1+|\nabla\psi_\tau^h|^2)^{\frac12}\,dX_\tau\\ &\leqq \|(1+|\nabla\psi_\tau^h|^2)^{\frac12}(\psi_\tau^h-\psi_\tau)\|_{0,\infty,\Delta_\tau}\int_{\Delta_\tau}\int_{\psi(X)}^{\psi^h(X)}\left[\frac{\partial w}{\partial Y_\tau}\right]^2 dY_\tau\,dX_\tau.\end{aligned}$$

Therefore we obtain

$$\|w\|^2_{0,\partial\Omega^h} \leqq Ch^2[|w|^2_{1,\Omega-\Omega^h} + |w|^2_{1,\Omega^h-\Omega}], \tag{3.32}$$

applying the bound (3.31) above. From the trace inequality (1.17b) and (3.29) we have for all $w \in H^1(\tilde{\Omega})$ that

$$\|w\|_{0,\Omega^h-\Omega} \leqq Ch\|w\|_{1,\Omega^h} \tag{3.33a}$$

and

$$\|w\|_{0,\Omega-\Omega^h} \leqq Ch\|w\|_{1,\Omega}, \tag{3.33b}$$

which imply for all $w \in H^2(\tilde{\Omega})$ that

$$|w|^2_{1,\Omega^h-\Omega} + |w|^2_{1,\Omega-\Omega^h} \leqq Ch^2\|w\|^2_{2,\tilde{\Omega}}. \tag{3.34}$$

Combining (3.32) and (3.34) yields the desired result (3.30). □

Combining the above lemmas we obtain the H^1 error bound.

Theorem 3.1. *Let the assumptions* (A1), (A2) *and* (A5) *hold. Then the solutions* u *and* u^h_ε *of* (1.3) *and* (3.10) *satisfy:*

$$\|\tilde{u} - u^h_\varepsilon\|_{1,\Omega^h} \leqq C\{[h + \varepsilon^{-\frac{1}{2}}h^2 + \varepsilon^{\frac{1}{2}}]\|u\|_{4,\Omega} + \varepsilon^{-\frac{1}{2}}h^2\|\tilde{g}\|_{3,\Omega^h}\} \tag{3.35}$$

and

$$\|\tilde{u} - u^h_\varepsilon\|_{0,\partial\Omega^h} \leqq C\{[\varepsilon^{\frac{1}{2}}h + h^2 + \varepsilon]\|u\|_{4,\Omega} + h^2\|\tilde{g}\|_{3,\Omega^h}\}, \tag{3.36}$$

where $\tilde{u} = Eu$.

Proof. Choosing $\xi = \pi_h\tilde{u}$ in (3.19) together with (3.15) and (3.18) yields

$$\|\tilde{u} - u^h_\varepsilon\|_{\tilde{a}^h_\varepsilon} \leqq C\left\{\|\tilde{u} - \pi_h\tilde{u}\|_{\tilde{a}^h_\varepsilon} + h^2\|u\|_{4,\Omega} + \varepsilon^{-\frac{1}{2}}h^2\|\tilde{g}\|_{3,\Omega^h} + \sup_{\chi\in S^h}\frac{|\tilde{a}^h_\varepsilon(\tilde{u},\chi) - \tilde{l}^h_\varepsilon(\chi)|}{\|\chi\|_{\tilde{a}^h_\varepsilon}}\right\}. \tag{3.37}$$

From (3.7), (3.8) and (3.17) it follows that

$$\begin{aligned}|\tilde{a}^h_\varepsilon(\tilde{u},\chi) - \tilde{l}^h_\varepsilon(\chi)| &= \varepsilon^{-1}\left|\left\langle \varepsilon\tilde{\sigma}\frac{\partial\tilde{u}}{\partial v^h} + \tilde{u} - \tilde{g}, \chi\right\rangle_{\partial\Omega^h}\right| \\ &\leqq C\left[\left\|\frac{\partial\tilde{u}}{\partial v^h}\right\|_{0,\partial\Omega^h} + \varepsilon^{-1}\|\tilde{u} - \tilde{g}\|_{0,\partial\Omega^h}\right]\|\chi\|_{0,\partial\Omega^h} \\ &\leqq C[(1 + \varepsilon^{-1}h^2)\|u\|_{2,\Omega} + \varepsilon^{-1}h^2\|\tilde{g}\|_{2,\Omega^h}]\|\chi\|_{0,\partial\Omega^h},\end{aligned} \tag{3.38}$$

where we have applied the trace inequality (1.17c), (2.11), the bound (3.30) as $u = g$ on $\partial\Omega$ and noted that $\|\chi\|_{0,\partial\Omega^h} \leqq C\varepsilon^{\frac{1}{2}}\|\chi\|_{\tilde{a}^h_\varepsilon}$.

From (A1), (3.4b) and (2.11) we obtain

$$\|\tilde{u} - \pi_h\tilde{u}\|_{\tilde{a}^h_\varepsilon} \leqq C[h^2 + \varepsilon^{-1}h^4]^{\frac{1}{2}}\|u\|_{4,\Omega}. \tag{3.39}$$

From (1.19) it follows that $\|\cdot\|_{1,\Omega^h} \leqq C\|\cdot\|_{\tilde{a}^h_\varepsilon}$ and so combining (3.37), (3.38) and (3.39) yields the desired results (3.35) and (3.36). □

We see from (3.35) and (3.36) that optimal rates of confergence are obtained if and only if $\varepsilon=O(h^2)$.

Theorem 3.2. *Let the assumptions* (A1), (A2) *and* (A5) *hold and* $\varepsilon=h^2$. *Then the solutions* u *and* u_ε^h *of* (1.3) *and* (3.10) *satisfy:*

$$\|\tilde{u}-u_\varepsilon^h\|_{0,\Omega^h} \leqq Ch^{\frac{3}{2}}[\|u\|_{4,\Omega}+\|\tilde{g}\|_{3,\Omega^h}] \tag{3.40}$$

and

$$\|\tilde{u}-u_\varepsilon^h\|_{-2,\Omega^h} \leqq Ch^2[\|u\|_{4,\Omega}+\|\tilde{g}\|_{3,\Omega^h}], \tag{3.41}$$

where $\tilde{u}=Eu$.

Proof. From (3.20) we see that an important term to estimate is $\|\pi_h\tilde{z}\|_{0,\partial\Omega^h}$. If $r=0$, so $z\in H^2(\Omega)\cap H_0^1(\Omega)$, then from (3.4a) we have

$$\|\pi_h\tilde{z}\|_{0,\partial\Omega^h} \leqq Ch^{\frac{3}{2}}\|z\|_{2,\Omega}, \tag{3.42a}$$

where $\tilde{z}=Ez$. However if $r=2$, so $z\in H^4(\Omega)\cap H_0^1(\Omega)$, then from (3.4b) we have

$$\|\pi_h\tilde{z}\|_{0,\partial\Omega^h} \leqq Ch^2\|z\|_{4,\Omega}. \tag{3.42b}$$

Therefore combining (3.20) with $r=0(2)$, (3.29a), (3.35), (3.36), (A1), (3.42a(b)), (3.15) and (3.38) with $\chi\equiv\pi_h\tilde{z}$ yields the desired results (3.40) and (3.41). □

Although we have only been able to prove that the approximation (3.10) with $\varepsilon=h^2$ converges at the rate $O(h^{\frac{3}{2}})$ globally in L^2, we now show that it converges at the optimal rate in L^2 over a domain $\Omega_0\subset\subset\Omega_1\subset\subset\Omega^h$ using the techniques of Nitsche and Schatz [12] for obtaining interior estimates.

Lemma 3.5. *Let the assumptions* (A1), (A2) *and* (A5) *hold. Then the solutions* u *and* u_ε^h *of* (1.3) *and* (3.10) *satisfy for* $i=0$ *and* 1:

$$\|\tilde{u}-u_\varepsilon^h\|_{-i,G_i} \leqq C[\|\tilde{u}-u_\varepsilon^h\|_{-(i+1),G_{i+1}}+h^2\|u\|_{4,\Omega}], \tag{3.43}$$

where G_0 *and* G_1 *are concentric spheres,* $G_i\subset\subset G_{i+1}\subset\subset\Omega^h$, *and* $\tilde{u}=Eu$.

Proof. For $i=0$ and 1, let $\omega_i\in C_0^\infty(G_i')$ be such that

$$\begin{aligned} &\omega_i\equiv 1 \quad \text{on } G_i, \qquad \omega_i\geqq 0 \quad \text{on } G_i' \quad \text{and} \\ &\omega_i\equiv 0 \quad \text{on } \Omega\setminus G_i' \end{aligned} \tag{3.44}$$

where G_i' is a sphere satisfying $G_i\subset\subset G_i'\subset\subset G_{i+1}$. Setting $e=\tilde{u}-u_\varepsilon^h$ and so it follows that

$$\begin{aligned} \|e\|_{-i,G_i} &\leqq \|\omega_i e\|_{-i,G_{i+1}} \\ &= \sup_{\eta_i\in H^i(G_{i+1})} \frac{|(\omega_i e,\eta_i)_{G_{i+1}}|}{\|\eta_i\|_{i,G_{i+1}}}. \end{aligned} \tag{3.45}$$

Introduce z_i such that

$$Az_i=\eta_i \quad \text{in } G_{i+1} \qquad z_i=0 \quad \text{on } \partial G_{i+1}. \tag{3.46}$$

From (A2) and elliptic regularity theory it follows that

$$\|z_i\|_{i+2,G_{i+1}} \leqq C\|\eta_i\|_{i,G_{i+1}}. \tag{3.47}$$

As $z_i \in H^{i+2}(G_{i+1}) \cap H_0^1(G_{i+1})$ and $\omega_i \in C_0^\infty(G_i')$ it follows that

$$\begin{aligned}(\omega_i e, \eta_i)_{G_{i+1}} &= a(z_i, \omega_i e)_{G_{i+1}} \\ &= a(e, \omega_i z_i) + (e, \sigma \nabla z_i \cdot \nabla \omega_i)_{G_{i+1}} \\ &\quad + (e, \nabla \cdot (\sigma z_i \nabla \omega_i))_{G_{i+1}}\end{aligned}$$

and so from (3.45) we obtain

$$\begin{aligned}\|e\|_{-i,G_i} &\leqq C\,\|e\|_{-(i+1),G_{i+1}} \\ &\quad + \sup_{z_i \in H^{i+2}(G_{i+1}) \cap H_0^1(G_{i+1})} \frac{|a(e, \omega_i z_i)|}{\|z_i\|_{i+2,G_{i+1}}}.\end{aligned} \tag{3.48}$$

In addition, it follows that

$$a(e, \omega_i z_i) = a(e, \omega_i z_i - \pi_h(\omega_i \tilde{z}_i)) + a(e, \pi_h(\omega_i \tilde{z}_i)) \tag{3.49}$$

and that

$$\begin{aligned}a(e, \pi_h(\omega_i \tilde{z}_i)) &= \tilde{a}^h(e, \pi_h(\omega_i \tilde{z}_i)) \\ &= [a^h(u_\varepsilon^h, \pi_h(\omega_i \tilde{z}_i)) - \tilde{a}^h(u_\varepsilon^h, \pi_h(\omega_i \tilde{z}_i))] \\ &\quad + [\tilde{l}^h(\pi_h(\omega_i \tilde{z}_i)) - l^h(\pi_h(\omega_i \tilde{z}_i))].\end{aligned} \tag{3.50}$$

Hence the desired result (3.43) follows from combining (3.48), (3.49) and (3.50) with (A1) and the bounds (3.15) and (3.18). □

Theorem 3.3. *Let the assumptions* (A1), (A2) *and* (A5) *hold and* $\varepsilon = h^2$. *Then the solutions* u *and* u_ε^h *of* (1.3) *and* (3.10) *satisfy:*

$$\|\tilde{u} - u_\varepsilon^h\|_{0,\Omega_0} \leqq C h^2 [\|u\|_{4,\Omega} + \|\tilde{g}\|_{3,\Omega^h}], \tag{3.51}$$

where $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega^h$ *and* $\tilde{u} = Eu$.

Proof. Setting $e = \tilde{u} - u_\varepsilon^h$ we have from (3.43) for $\Omega_0 \subseteqq G_0$ that

$$\begin{aligned}\|e\|_{0,\Omega_0} &\leqq C[\|e\|_{-1,G_1} + h^2\,\|u\|_{4,\Omega}] \\ &\leqq C[\|e\|_{-2,G_2} + h^2\,\|u\|_{4,\Omega}].\end{aligned}$$

The desired result (3.51) then follows from (3.41). □

4. Numerical Example

We now report on a numerical example. The problem chosen was

$$-\nabla^2 u = 0 \quad \text{in } \Omega \equiv x^2 + y^2 \leqq 1 \qquad u = g \quad \text{on } \partial\Omega,$$

where g was chosen so that the solution $u = x^2 - y^2$. Due to symmetry the problem was solved in a single quadrant. For our trial space we took piecewise linears on uniform right-angled triangles, resulting from a uniform partition of the complete square $[0,1] \times [0,1]$ into squares with sides of length $h = 1/J$ and then into triangles by joining the SW to the NE vertex. The computational

Table 1

$h=1/J$ J	$\varepsilon=h^\lambda$ λ	$\lvert u-u_\varepsilon^h\rvert_{1,\Omega^h}$	$\lVert u-u_\varepsilon^h\rVert_{0,\Omega^h}$	$\lVert u-u_\varepsilon^h\rVert_{0,\Omega_0}$	$\max_{\text{nodes } x_j\in\Omega}\lvert u(x_j)-u_\varepsilon^h(x_j)\rvert$
4	1	1.75819	0.47962	0.07414	0.33396
8	1	1.04683	0.28778	0.04238	0.19930
16	1	0.58026	0.16018	0.02338	0.11074
32	1	0.30699	0.08494	0.01238	0.05870
4	2	0.85301	0.15715	0.03332	0.10556
8	2	0.38118	0.04211	0.00864	0.02804
16	2	0.18326	0.01066	0.00218	0.00712
32	2	0.09072	0.00268	0.00055	0.00179
4	3	0.71392	0.05188	0.02570	0.02114
8	3	0.36119	0.01125	0.00638	0.00316
16	3	0.18173	0.00312	0.00161	0.00145
32	3	0.09407	0.00092	0.00040	0.00117
4	4	0.72100	0.04537	0.02593	0.01185
8	4	0.38135	0.01554	0.00664	0.00934
16	4	0.24053	0.01441	0.00278	0.01307

domain Ω^h was obtained by replacing $\partial\Omega$ by its chord in each triangle it intersects as described in Sect. 3. Choosing $\tilde{g}\equiv x^2-y^2$ as the extension for g, we obtained approximations u_ε^h from (3.10) to u for different values of h and λ with $\varepsilon=h^\lambda$. The results are presented in Table 1. For the interior estimate the domain Ω_0 was chosen to be the square $[-\frac{1}{2},\frac{1}{2}]\times[-\frac{1}{2},\frac{1}{2}]$.

From Table 1 we see that for $\lambda=2$ the H^1, global L^2 and interior L^2 errors are converging at the optimal rate. The H^1 and interior L^2 rates confirm the analysis of Sect. 3, whilst the global L^2 rate is better than that predicted. Overall we see the vast superiority of choosing $\lambda=2$ as opposed to $\lambda=1$, which is the choice often quoted in the literature. Although, it appears that the H^1 error is converging at the optimal rate for $\lambda=1$, better than that predicted. Moreover, the choice $\lambda=3$ yields optimal rates in the H^1 and the interior L^2 norms. Once again this is not predicted by the analysis. However, we see that for $\lambda=4$ these optimal rates are lost. In fact in this case decreasing h can lead to an increase in the error.

To conclude, we see that this example certainly confirms the optimal H^1 and interior L^2 rate predicted for the choice $\lambda=2$ and the superiority of this choice over choosing $\lambda=1$. However, it would appear that the rates predicted for other choices of λ could be improved.

References

1. Babuska, I.: The finite element method with penalty. Math. Comput. **27**, 221–228 (1973)
2. Babuska, I., Aziz, A.K.: Survey lectures on the mathematical foundations of the finite element method. In: The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A.K. Aziz, ed.), pp. 3–363. New York: Academic Press 1972

3. Barrett, J.W., Elliott, C.M.: A finite element method on a fixed mesh for the Stefan problem in a saturated porous medium. In: Numerical Methods for Fluid Dynamics (K.W. Morton and M.J. Baines, eds.), pp. 389-409. London: Academic Press 1982
4. Barrett, J.W., Elliott, C.M.: Total flux estimates for a finite element approximation of elliptic equations. (Submitted for publication.)
5. Barrett, J.W., Elliott, C.M.: Fixed mesh finite element approximations to a free boundary problem for an elliptic equation with an oblique derivative boundary condition. Comput. Math. Appl. **11**, 335-345 (1985)
6. Barrett, J.W., Elliott, C.M.: Finite element approximation of elliptic equations with a Neumann or Robin condition on a curved boundary. (Submitted for publication)
7. Clement, Ph.: Approximation by finite element functions using local regularization. RAIRO Anal. Numer. **9**, 77-84 (1975)
8. King, J.T.: New error bounds for the penalty method and extrapolation. Numer. Math. **23**, 153-165 (1974)
9. King, J.T., Serbin, S.M.: Computational experiments and techniques for the penalty method with extrapolation. Math. Comput. **32**, 111-126 (1978)
10. Kufner, A., John, O., Fucik, S.: Function Spaces. Leyden: Nordhoff 1977
11. Necas, J.: Les Méthodes Directes en Théorie des Equations Elliptiques. Paris: Masson 1967
12. Nitsche, J.A., Schatz, A.H.: Interior estimates for Ritz-Galerkin methods. Math. Comput. **28**, 937-958 (1974)
13. Utku, M., Carey, G.F.: Boundary penalty techniques. Comput. Methods Appl. Mech. Eng. **30**, 103-118 (1982)
14. Zhong-Ci Shi: On the convergence rate of the boundary penalty method. Int. J. Numer. Methods Eng. **20**, 2027-2032 (1984)

Received September 5, 1985/January 16, 1986