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A Finite-element Method for Solving Elliptic Equations with Neumann Data on a Curved Boundary Using Unfitted Meshes

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This paper considers a finite-element approximation of a Poisson equation in a region with a curved boundary on which a Neumann condition is prescribed. Piecewise linear and bilinear elements are used on *unfitted* meshes with the region of integration being replaced by a polygonal approximation. It is shown, despite the variational crimes, that the rate of convergence is still order (h) in the H^1 norm. Numerical examples show that the method is easy to implement and that the predicted rate of convergence is obtained.

1. Introduction

CONSIDER THE NUMERICAL SOLUTION of

$$-\nabla^2 u^{(k)} = f \tag{1.1a}$$

on a sequence of two-dimensional domains $\Omega^{(k)}$ having boundaries $\partial\Omega^{(k)} = \partial_1\Omega \cup \partial_2\Omega^{(k)}$, respectively, of the form depicted in Fig. 1, on which the following boundary conditions hold:

$$u^{(k)} = g_1 \text{ on } \partial_1\Omega, \qquad \frac{\partial u^{(k)}}{\partial n^{(k)}} = g_2 \text{ on } \partial_2\Omega^{(k)}, \tag{1.1b}$$

where $\mathbf{n}^{(k)}$ is the outward-pointing unit normal on $\partial_2\Omega^{(k)}$. Such a situation arises when solving either certain types of moving-boundary problems as in Barrett & Elliott (1982) or free-boundary problems by trial free boundary methods as in Cryer (1977).

The standard finite-element (or difference) approach would be to fit a mesh to each domain $\Omega^{(k)}$. However, for a Neumann condition on a curved boundary it is not necessary to fit the mesh to the boundary in order to retain the optimal rate of convergence in the Dirichlet norm. Consider a uniform partition of the interior of the closed curve $\partial_1\Omega$, taking no account of the position of $\partial_2\Omega^{(k)}$. Then one can define a finite-element approximation to $u^{(k)}$ by considering the associated variational form of (1.1) over a finite-element space based on this uniform partition. It is easy to show that the rate of convergence is optimal, see Babuška (1971). However, this method requires the evaluation of integrals over $\Omega^{(k)}$ and $\partial_2\Omega^{(k)}$ which in general cannot be performed exactly. Thus a practical approach is to perform the

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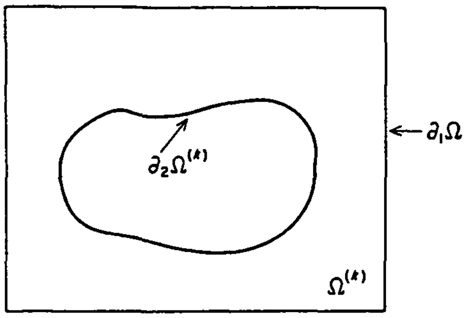

FIG. 1.

integrals over approximations $\Omega_h^{(k)}$ and $\partial_2\Omega_h^{(k)}$. It is the purpose of this paper to show that for the simplest trial spaces—piecewise linears on triangles and piecewise bilinears on rectangles—that by approximating the curved boundary by a straight line in each element the resulting approximation retains the optimal order of accuracy in the Dirichlet norm. Clearly more sophisticated trial spaces require a more elaborate approximation of the boundary to retain this optimality.

Although the effect of domain perturbation and numerical integration in the case of homogeneous Dirichlet boundary conditions is well understood (see, for example, Ciarlet & Raviart, 1972; Ciarlet, 1978; Wahlbin, 1978), very little work has appeared concerning the Neumann problem. Oganesyan (1966) and Strang & Fix (1973) consider the effect of domain perturbation for the Neumann problem when using a fitted triangular mesh. However, the present authors are unaware of any work which has appeared concerning the use of an unfitted mesh. Clearly the use of unfitted meshes has useful practical applications for free and moving-boundary problems as at each step one would only have to adjust the domain of integration and not the mesh—leading to a considerable saving in effort and computing time. The technique offers also a computationally simple approach to solving a single elliptic equation with a Neumann condition on a given curved boundary which occurs, for example, in exterior flow problems.

The outline of the paper is as follows: in the next section we describe our technique more precisely. In Section 3 we study a domain perturbation of a boundary value problem which plays an important role in the derivation of our error estimate. One should note that the analysis of Oganesyan (1966) and Strang & Fix (1973) for fitted meshes does not generalize in a straightforward manner to deal with the present technique. We piece together the various estimates to prove our main theorem in Section 4. Finally, in Section 5, we discuss the numerical implementation of the method, including the use of numerical integration, and report on some numerical computations.

Throughout this paper we adopt the following notation. With $\mathbb{N}$ the set of natural numbers and G a bounded open region in $\mathbb{R}^n$, setting

$$|\alpha| = \sum_{i=1}^{n} |\alpha_i|$$

for $\alpha \in \mathbb{N}^n$ we define the following norms and semi-norms for a function w defined on G:

$$\|w\|_{0,p,G} = \left[\int_G |w|^p \, d\mathbf{x}\right]^{1/p}, \qquad |w|_{m,p,G} = \left[\sum_{|\alpha|=m} \|D^\alpha w\|_{0,p,G}^p\right]^{1/p},$$

$$\|w\|_{m,p,G} = \left[\sum_{k=0}^{m} |w|_{k,p,G}^p\right]^{1/p}, \qquad \|w\|_{m,G} \equiv \|w\|_{m,2,G},$$

$$|w|_{m,G} = |w|_{m,2,G},$$

where $m \in \mathbb{N}$ and $p \geqslant 1$. If $p = \infty$, then we define

$$|w|_{m,\infty,G} = \max_{|\alpha|=m} \operatorname{ess\,sup}_{x \in G} |D^\alpha w|, \qquad \|w\|_{m,\infty,G} = \max_{|\alpha|\leqslant m} \operatorname{ess\,sup}_{x \in G} |D^\alpha w|.$$

The Banach spaces of functions associated with the norms $\|\cdot\|_{0,p,G}$, $\|\cdot\|_{m,p,G}$, $\|\cdot\|_{m,G}$ are then, respectively, $L^p(G)$, $W^{m,p}(G)$ and $H^m(G)$. The measure of a domain G is denoted by $\mathrm{m}(G)$ and C denotes a positive constant, independent of h, whose value may change in different relations.

2. The Technique

Let Ω be a bounded open region in $\mathbb{R}^2$ with a boundary $\partial\Omega$ such that $\partial\Omega = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega}$, where $\partial_1\Omega$ and $\partial_2\Omega$ are non-empty and disjoint. We assume that either $\partial_i\Omega$ are closed curves or that $\overline{\partial_1\Omega} \cap \overline{\partial_2\Omega} = \{\text{finite number of points } P_j\}$. We assume also that $\partial_2\Omega$ is smooth and that $\partial_1\Omega$ is polygonal. Let $f \in L^2(\Omega)$, $g_1 \in H^1(\Omega)$ and $g_2 \in L^2(\partial_2\Omega)$, then we shall approximate the problem: find $u \in H^1(\Omega)$ with $u - g_1 \in H^1_{E_0}(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega + \langle g_2, v\rangle_{\partial_2\Omega}, \quad \forall\ v \in H^1_{E_0}(\Omega), \tag{2.1}$$

where the following notation has been adopted for $G \subseteq \mathbb{R}^2$ and $\partial G = \overline{\partial_1 G} \cup \overline{\partial_2 G}$:

$$H^1_{E_0}(G) = \{w \in H^1(G): w = 0 \text{ on } \partial_1 G\}$$

and

$$(\mathbf{w}_1, \mathbf{w}_2)_G = \int_G \mathbf{w}_1 \cdot \mathbf{w}_2 \, d\mathbf{x}, \ \langle \mathbf{w}_1, \mathbf{w}_2\rangle_{\partial G} = \int_{\partial G} \mathbf{w}_1 \cdot \mathbf{w}_2 \, ds.$$

Equation (2.1) is the weak formulation of the mixed boundary-value problem:

$$-\nabla^2 u = f \text{ in } \Omega, \quad u = g_1 \text{ on } \partial_1\Omega, \quad \frac{\partial u}{\partial n} = g_2 \text{ on } \partial_2\Omega, \tag{2.2}$$

where $\mathbf{n}$ is the outward pointing unit normal to $\partial_2\Omega$.

Let $\mathscr{D}^*$ be a bounded set in $\mathbb{R}^2$ containing Ω which is the union of a collection of elements $\{e\}$ with disjoint interiors. The elements $\{e\}$, which we assume to be regular (see Ciarlet, 1978, p. 124), are either triangles or rectangles whose diameters are less than h in length. Thus the domain $\mathscr{D}^*$ is dependent on h, that is $\mathscr{D}^* \equiv \mathscr{D}^*(h)$. Then we define the domain $\mathscr{D} \equiv \mathscr{D}(h)$:

$$\mathscr{D} \equiv \bigcup_{e \in \beta} e, \quad \beta \equiv \{e \in \mathscr{D}^*: e \cap \Omega \neq \{\phi\}\}. \tag{2.3}$$

We shall assume that the elements e fit the boundary $\partial_1\Omega$, that is $\partial_1\mathscr{D} \equiv \partial_1\Omega$, and if

$\overline{\partial_1\Omega} \cap \overline{\partial_2\Omega} \neq \{\phi\}$ then each point of intersection is taken to be a vertex of an element. A polygonal domain Ω_h approximating Ω is constructed in the following way. If for an element e, $\partial_2\Omega \cap e \neq \{\phi\}$, then the arc of $\partial_2\Omega$ in e is approximated by its chord joining the points where it intersects the boundary of the element. If $\partial_2\Omega$ crosses the boundary of the element more than twice then the approximating chord is taken to be that which joins the first point of entry to the last point of exit. The resultant piecewise linear approximation to $\partial_2\Omega$ is denoted by $\partial_2\Omega_h$ and Ω_h is then the open bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega_h = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega_h}$. Examples of the construction of the boundary $\partial_2\Omega_h$ for rectangular elements are given in Fig. 2.

The approximation to $u \in H^1(\Omega)$ will be a function U whose domain of definition is $\mathscr{D}_h$, where

$$\mathscr{D}_h \equiv \bigcup_{e \in \beta_h} e, \quad \beta_h \equiv \{e \in \mathscr{D}^* : e \cap \Omega_h \neq \{\phi\}\}, \tag{2.4}$$

and in general $\beta_h \not\equiv \beta$ because of the possibility that a boundary element may have an edge with two vertices on a convex arc of $\partial_2\Omega$. Also U will belong to a finite-dimensional space $S^h_E(\mathscr{D}_h)$, where either

$$S^h(\mathscr{D}_h) = \{W \in C(\mathscr{D}_h) : W \text{ is linear on each triangle}\}$$

or

$$S^h(\mathscr{D}_h) = \{W \in C(\mathscr{D}_h) : W \text{ is bilinear on each rectangle}\}, \tag{2.5a}$$

$$S^h_0(\mathscr{D}_h) = \{W \in S^h(\mathscr{D}_h) : W = 0 \text{ on } \partial_1\Omega\} \tag{2.5b}$$

and

$$S^h_E(\mathscr{D}_h) = \{W \in S^h(\mathscr{D}_h) : W(x_i) = g_1(x_i) \text{ for each vertex } x_i \text{ on } \partial_1\Omega\}. \tag{2.5c}$$

Thus we have $S^h(\mathscr{D}_h) \subset H^1(\mathscr{D}_h)$ and $S^h_0(\mathscr{D}_h) \subset H^1_{E_0}(\mathscr{D}_h)$. The space $S^h(\mathscr{D}_h)$, in either case, has the following approximation property: for $w - g_1 \in H^1_{E_0}(\mathscr{D}_h) \cap H^2(\mathscr{D}_h)$ there exists an interpolate $r_h w \in S^h_E(\mathscr{D}_h)$ such that

$$|w - r_h w|_{0,\mathscr{D}_h} + h|w - r_h w|_{1,\mathscr{D}_h} \leqslant Ch^2|w|_{2,\mathscr{D}_h}, \tag{2.6}$$

where C is a constant independent of w and h (see Ciarlet, 1978, p. 124).

Assuming $\{f, g_2\}$ are the restrictions on $\bar{\Omega}$ of functions $\{\hat{f}, \hat{g}_2\}$ defined on $\mathbb{R}^2$ which are smooth in a neighbourhood of Ω, containing Ω_h, the finite-element approximation to (2.1) we wish to present and analyse is: find $U \in S^h_E(\mathscr{D}_h)$ such that

$$(\nabla U, \nabla V)_{\Omega_h} = (\hat{f}, V)_{\Omega_h} + \langle \hat{g}_2, V \rangle_{\partial_2\Omega_h} \quad \forall\, V \in S^h_0(\mathscr{D}_h). \tag{2.7}$$

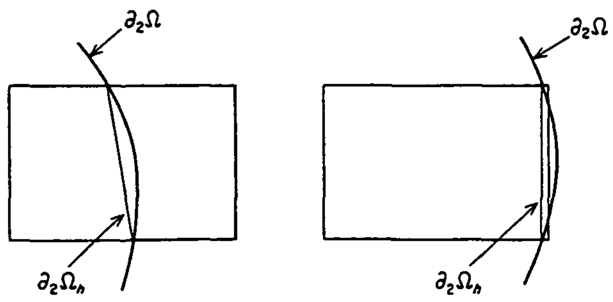

FIG. 2.

The reason for considering (2.7) rather than: find $U \in S_E^h(\mathscr{D})$ such that

$$(\nabla U, \nabla V)_\Omega = (f, V)_\Omega + \langle g_2, V\rangle_{\partial_2\Omega}, \ \forall\ V \in S_0^h(\mathscr{D}), \tag{2.8}$$

where $S_0^h(\mathscr{D})$ and $S_E^h(\mathscr{D})$ are the obvious generalizations of (2.5), is that the integrals in (2.8) are over regions with curved boundaries and thus being difficult to evaluate (2.8) is not a practical method. The use of (2.7) in place of (2.8) is a so-called "variational crime". The method (2.7) is based on the use of an "unfitted" mesh as the approximation U is defined outside the region Ω_h which is *not* a union of elements. A fitted mesh method would take in (2.7) Ω_h to be a union of elements with the vertices on $\partial_2\Omega_h$ lying also on $\partial_2\Omega$.

The approximation (2.8) was mentioned by Babuška (1971) in early mathematical papers on the finite-element method, but this idea of using an unfitted mesh seems to have been put to one side in the recent literature. The optimal error estimate in the Dirichlet norm for the approximation $U \in S_E^h(\mathscr{D})$ given by (2.8) is easily obtained by the observation that

$$(\nabla u - \nabla U, \nabla V)_\Omega = 0, \quad \forall\ V \in S_0^h(\mathscr{D})$$

so that

$$\begin{aligned} |u-U|_{1,\Omega}^2 &= (\nabla u - \nabla U, \nabla u - \nabla r_h u)_\Omega \\ &\leqslant |u - r_h u|_{1,\Omega}^2, \end{aligned} \tag{2.9}$$

where $r_h u \in S_E^h(\mathscr{D})$ is the interpolate of u. The desired result now follows from the approximation property of $S^h(\mathscr{D})$ and the smoothness of u, that is

$$|u-U|_{1,\Omega} \leqslant Ch|u|_{2,\Omega}. \tag{2.10}$$

It is the aim of this paper to show that the computationally convenient and simple approach (2.7) retains this optimal rate of convergence. To obtain this error estimate we need to study a perturbed mixed boundary-value problem, which forms the basis of the next section. We note that the approximation defined by (2.7) depends on the extended data $\{\hat{f}, \hat{g}_2\}$ as opposed to the given data $\{f, g_2\}$. However, in most problems of interest f and g_2 are smooth functions and thus this extension causes no difficulties. Indeed by employing numerical integration to the right-hand side of (2.7) the dependence of $\hat{f}$ can be removed and for the case of S^h being linears on triangles the dependence on $\hat{g}_2$ can also be removed. This point and further details of the implementation of the technique are described in Section 5. The method is easily applied to the more general equation

$$-\nabla\cdot(a\nabla u) + \nabla\cdot(\mathbf{b}u) + cu = f \text{ in } \Omega.$$

3. A Domain Perturbation of the Boundary-value Problem

Let $\hat{\Omega}(h)$ be a family of bounded open sets in $\mathbb{R}^2$, depending on the parameter $h \in [0, h_0]$, which are obtained from Ω by replacing $\partial_2\Omega$ with a smooth curve $\partial_2\hat{\Omega}(h)$ so that $\hat{\Omega}(0) \equiv \Omega$. The boundary of $\hat{\Omega}(h)$ is then $\partial\hat{\Omega}(h) \equiv \overline{\partial_1\Omega} \cap \overline{\partial_2\hat{\Omega}(h)}$, where $\partial_1\Omega$ and $\partial_2\hat{\Omega}(h)$ are disjoint. We shall assume that $\partial\hat{\Omega}(h)$ is "minimally smooth" in the sense of Stein (1970, p. 189) and that it is so independently of h, i.e. there exists an

$\varepsilon > 0$, an integer N, an $L > 0$ and a sequence of open sets $\mathscr{U}_1, \mathscr{U}_2, \ldots, \mathscr{U}_n, \ldots$, all independent of h, so that:

(a) If $x \in \partial\hat{\Omega}(h)$, then the ball of centre x and radius ε is contained in a $\mathscr{U}_i$ for some i.
(b) No point of $\mathbb{R}^2$ is contained in more than N of the $\mathscr{U}_i$s.
(c) For each i there exists a local co-ordinate system (X^i, Y^i) and a Lipschitz continuous function $\alpha_h^i(X^i)$, whose Lipschitz constant does not exceed L, such that

$$\mathscr{U}_i \cap \hat{\Omega}(h) \equiv \mathscr{U}_i \cap \{(X^i, Y^i) : Y^i > \alpha_h^i(X^i)\}.$$

Under these conditions we have the following result.

LEMMA 3.1 *If $\partial_2\hat{\Omega}(h)$ is minimally smooth, independently of h, there exists a linear operator $\mathscr{E}_h$ mapping functions on $\hat{\Omega}(h)$ to functions on $\mathbb{R}^2$ such that*

$$\text{(a)} \quad \mathscr{E}_h w \equiv w \text{ on } \hat{\Omega}(h), \qquad \text{(b)} \quad \|\mathscr{E}_h w\|_{2,\mathbb{R}^2} \leqslant C_1 \|w\|_{2,\hat{\Omega}(h)}, \tag{3.1}$$

where C_1 is a constant independent of h and w.

Proof. This is proved in Stein (1970, pp. 180–192), where it is shown that the constant C_1 depends upon the Lipschitz constants of the curves $\alpha_h^i(\cdot)$. ■

For each boundary element e of $\mathscr{D}_h$ (i.e. $e \cap \partial_2\Omega_h \neq \{\phi\}$) a local co-ordinate system (X^e, Y^e) may be defined so that $e \cap \partial_2\Omega_h$ is the X^e axis. Then $\partial_2\Omega$ and $\partial_2\hat{\Omega}(h)$ can be parameterized locally by $l_e(\,)$ and $\hat{l}_e(\,)$, respectively, for $0 < X^e < h_e \leqslant h$, where h_e is the length of the boundary edge. Since the boundary $\partial_2\Omega$ is smooth and, by the construction of Ω_h, $l_e(0) = l_e(h_e) = 0$, we have

$$\text{(a)} \quad |l_e(X^e)| \leqslant Ch^2 \quad X^e \in [0, h_e], \qquad \text{(b)} \quad |l_e'(X^e)| \leqslant Ch \quad X^e \in [0, h_e]. \tag{3.2}$$

We shall require that $\hat{\Omega}(h)$ satisfies for $h \in [0, h_e]$

$$\text{(a)} \quad |\hat{l}_e(X^e)| \leqslant Ch^2 \quad X^e \in [0, h_e],$$
$$\text{(b)} \quad |\hat{l}_e'(X^e)| \leqslant Ch \quad X^e \in [0, h_e], \tag{3.3}$$
$$\text{(c)} \quad \Omega_h \subseteq \hat{\Omega}(h) \subseteq \hat{\Omega}(h_0), \quad \Omega \subseteq \hat{\Omega}(h), \quad \overline{\partial_1\Omega} \cap \overline{\partial_2\hat{\Omega}(h)} = \{P_j\}.$$

We shall assume that $\{f, g_2\}$ are the restrictions to $\bar{\Omega}$ of functions $\{\hat{f}, \hat{g}_2\}$ which are smooth in a neighbourhood $\mathscr{N}^{h_0}$ of $\partial_2\hat{\Omega}(h_0)$ containing $\partial_2\Omega$. There exists a unique solution $\hat{u}(h)$, such that $\hat{u}(0) \equiv u$, of the perturbed boundary-value problem

$$-\nabla^2\hat{u}(h) = \hat{f} \text{ in } \hat{\Omega}(h)$$
$$\hat{u}(h) = g_1 \text{ on } \partial_1\Omega, \quad \frac{\partial\hat{u}(h)}{\partial\hat{n}(h)} = \hat{g}_2 \text{ on } \partial_2\hat{\Omega}(h); \tag{3.4}$$

where $\hat{\mathbf{n}}(h)$ is the outward pointing unit normal on $\partial_2\hat{\Omega}(h)$. We shall assume also that there exist constants C_2 and C_3 dependent only on $\hat{f}, g_1, \hat{g}_2, \Omega$ and h_0 such that

$$\text{(a)} \quad \|\hat{u}(h)\|_{2,\hat{\Omega}(h)} \leqslant C_2, \qquad \text{(b)} \quad \|\hat{u}(h)\|_{C^{1,1}(\mathscr{N}^{h_0})} \leqslant C_3, \tag{3.5}$$

where the space $C^{1,1}(\mathcal{N}^{h_0})$ consists of those functions whose first derivatives are Lipschitz continuous on $\mathcal{N}^{h_0}$ and $\|\cdot\|_{C^{1,1}(\mathcal{N}^{h_0})}$ is its associated norm. It is necessary to justify the strong assumptions (3.3) and (3.5), and we do this in the following remarks.

Remark 3.1. *Construction of* $\hat{\Omega}(h)$. First note that if Ω is locally convex with respect to $\partial_2\Omega$, then $\Omega_h \subseteq \Omega$ and $\hat{\Omega}(h)$ may be taken as Ω, which means that (3.3) is trivially satisfied. Otherwise it is necessary to check that given a domain Ω one is able to construct a family $\hat{\Omega}(h)$ which is minimally smooth, independently of h, and which satisfies (3.3). We give a construction of $\hat{\Omega}(h)$ in the case of $\partial_2\Omega$ being a closed smooth curve with Ω lying on one side of it. In this case for $h \in [0, h_0]$ we set $\partial_2\hat{\Omega}(h)$ to be the envelope of circles centred on $\partial_2\Omega$ with radius $R = C_4h^2$, where C_4 is the constant such that

$$\max_{x \in \partial_2\Omega_h} \operatorname{dist}(x, \partial_2\Omega) \leqslant C_4h^2. \tag{3.6}$$

It is a simple matter to show that this construction of $\hat{\Omega}(h)$ satisfies (3.3). The condition (3.3c) is clearly satisfied. By the construction of Ω_h, $(l_e(0) = l_e(h_e) = 0)$, there exists $X^{e*} \in (0, h_e)$ such that $l'_e(X^{e*}) = 0$. The above construction of $\hat{\Omega}(h)$ is such that $\hat{l}'_e(X^{e*}) = 0$. Expanding $\hat{l}_e(\cdot)$ and $\hat{l}'_e(\cdot)$ in a Taylor series about X^{e*} immediately yields the desired results (3.3a) and (3.3b). Furthermore, $\hat{\Omega}(h)$ is minimally smooth independently of h since the boundary $\partial_2\hat{\Omega}(h)$ has, for small h, essentially the same smoothness properties as $\partial_2\Omega$: that is, the local Lipschitz constants of $\partial_2\hat{\Omega}(h)$ depend only on the local Lipschitz constants of $\partial_2\Omega$ and the constant h_0.

The above construction can be generalized to the case where $\partial_2\Omega$ is a smooth curve whose end-points P_1 and P_2 are also end-points of $\partial_1\Omega$; that is, $\overline{\partial_1\Omega} \cap \overline{\partial_2\Omega} = \{P_1, P_2\}$; and $\partial_2\Omega$ is locally convex at P_1 and P_2. In this case we take $\partial_2\hat{\Omega}(h)$ to be the envelope of circles centred on $\partial_2\Omega$ with radius $R = C_4h^2\gamma(s)$, where s is the arc length of $\partial_2\Omega$ and $\gamma(s)$ is a function which vanishes at the end points P_1 and P_2 and rises smoothly with derivatives bounded above independently of h to take on the constant value 1 on the interior of $\partial_2\Omega$. We omit the details of this construction, since in general for problems where $\overline{\partial_1\Omega} \cap \overline{\partial_2\Omega} \neq \{\phi\}$ we will not be able to ensure the regularity of the resulting boundary-value problem on $\hat{\Omega}(h)$, see Remark 3.2(c).

Remark 3.2. Sufficient conditions for (3.5) to hold

(a) If $\hat{\Omega}(h) \equiv \Omega$, then the conditions (3.5) are those for the original problem.
(b) Suppose $\hat{\partial}_2\Omega(h)$ is a smooth closed curve, $\hat{f} \in L^2(\hat{\Omega}(h))$ and $g_1 \in H^2(\hat{\Omega}(h))$. The estimate (3.5) then follows from the standard estimates for elliptic equations when Ω is convex with respect to $\partial_1\Omega$, $\{\hat{f}, \hat{g}_2\}$ are smooth in $\mathcal{N}^{h_0}$ and the derivatives of the curves defining $\partial_2\hat{\Omega}(h)$ are bounded independently of h by the construction in Remark 3.1. (See Grisvard, 1980, and Agmon *et al.*, 1959).
(c) If $\overline{\partial_1\Omega} \cap \overline{\partial_2\Omega} \neq \{\phi\}$ then the regularity of the solution $\hat{u}(h)$ for any $h \in [0, h_0]$ depends on the behaviour of the data in the neighbourhood of the vertices and on the vertex angles. In general singularities are present, unless compatibility conditions hold, and it is not the purpose of this paper to consider the

problem of singularities. Although we are unable to give conditions to ensure that (3.5) holds for a family of domains in this situation, in Example 3 of Section 5 we give results of a numerical calculation.

We now present a lemma relating the solution u of the mixed boundary-value problem (2.2) to the solution $\hat{u}(h)$ of the perturbed problem (3.4). To simplify the notation in the remainder of this paper we omit the dependence on h of the perturbed problem; that is, we refer to $\hat{u}(h)$ as $\hat{u}$, etc.

LEMMA 3.2

(i) *If* $\Omega \subseteq \hat{\Omega}$, *then*

$$|u-\hat{u}|_{1,\Omega} \leqslant C_5 \left|\frac{\partial \hat{u}}{\partial n} - g_2\right|_{0,\partial_2\Omega}. \tag{3.7a}$$

(ii) *If* (3.3) *and* (3.5) *hold, then*

$$\left|\frac{\partial \hat{u}}{\partial n} - g_2\right|_{0,\partial_2\Omega} \leqslant C_6 h \tag{3.7b}$$

$$\left|\frac{\partial \hat{u}}{\partial n_h} - \hat{g}_2\right|_{0,\partial_2\Omega_h} \leqslant C_7 h, \tag{3.7c}$$

where $\mathbf{n}_h$ *is the outward-pointing unit normal to* $\partial_2\Omega_h$ *and the constants* C_5, C_6 *and* C_7 *are independent of* h.

Proof.

(i) Setting $w = u - \hat{u}$, then w satisfies $-\nabla^2 w = 0$ in $\Omega \subseteq \hat{\Omega}$, $w = 0$ on $\partial_1\Omega$ and $\partial w/\partial n = g_2 - \partial\hat{u}/\partial n$ on $\partial_2\Omega$ and Green's theorem implies

$$|w|^2_{1,\Omega} = \left\langle w, \frac{\partial w}{\partial n}\right\rangle_{\partial_2\Omega} \leqslant |w|_{0,\partial_2\Omega}\left|g_2 - \frac{\partial \hat{u}}{\partial n}\right|_{0,\partial_2\Omega}. \tag{3.8}$$

For the domain Ω one has the standard trace inequality

$$|w|_{0,\partial_2\Omega} \leqslant C_5 |w|_{1,\Omega} \quad \forall\, w \in H^1_{E_0}(\Omega).$$

Applying this inequality to the right-hand side of (3.8) yields the desired result (3.7a).

(ii) For $e \in \mathscr{D}_h$ such that $e \cap \partial_2\Omega_h \neq \{\phi\}$ we may write

$$\begin{aligned}\left|\frac{\partial \hat{u}}{\partial n} - g_2\right|^2_{0,\partial_2\Omega} &= \sum_e \left|\frac{\partial \hat{u}}{\partial n} - g_2\right|^2_{0,e\cap\partial_2\Omega} \\ &= \sum_e \int_0^{h_e} \left[\frac{\partial \hat{u}}{\partial n}(X^e, l_e(X^e)) - g_2(X^e, l_e(X^e))\right]^2 \times \\ &\quad [1 + l_e'(X^e)^2]^{\frac{1}{2}}\, dX^e.\end{aligned} \tag{3.9}$$

Noting that $\partial\hat{u}/\partial\hat{n} = \hat{g}_2$ on $\partial_2\hat{\Omega}$ we obtain

$$\begin{aligned}&\left|\frac{\partial \hat{u}}{\partial n}(X^e, l_e(X^e)) - g_2(X^e, l_e(X^e))\right| \\ &\qquad \leqslant \left|\frac{\partial \hat{u}}{\partial n}(X^e, l_e(X^e)) - \frac{\partial \hat{u}}{\partial \hat{n}}(X^e, \hat{l}_e(X^e))\right| + \\ &\qquad\quad |g_2(X^e, l_e(X^e)) - \hat{g}_2(X^e, \hat{l}_e(X^e))|.\end{aligned} \tag{3.10}$$

Expanding the first term on the right-hand side of (3.10) yields

$$\left|\frac{\partial \hat{u}}{\partial n}(X^e, l_e(X^e)) - \frac{\partial \hat{u}}{\partial \hat{n}}(X^e, \hat{l}_e(X^e))\right|$$

$$= \left|[1+l_e'(X^e)^2]^{-\frac{1}{2}}\left[\frac{\partial \hat{u}}{\partial Y^e}(X^e, l_e(X^e)) - l_e'(X^e)\frac{\partial \hat{u}}{\partial X^e}(X^e, l_e(X^e))\right] - \right.$$

$$\left.[1+\hat{l}_e'(X^e)^2]^{-\frac{1}{2}}\left[\frac{\partial \hat{u}}{\partial Y^e}(X^e, \hat{l}_e(X^e)) - \hat{l}_e'(X^e)\frac{\partial \hat{u}}{\partial X^e}(X^e, \hat{l}_e(X^e))\right]\right|$$

$$\leqslant [1+\hat{l}_e'(X^e)^2]^{-\frac{1}{2}}\left|\frac{\partial \hat{u}}{\partial Y^e}(X^e, l_e(X^e)) - \frac{\partial \hat{u}}{\partial Y^e}(X^e, \hat{l}(X^e))\right| +$$

$$\left|\frac{\partial \hat{u}}{\partial Y^e}(X^e, l_e(X^e))\right| |[1+l_e'(X^e)^2]^{-\frac{1}{2}} - [1+\hat{l}_e'(X^e)^2]^{-\frac{1}{2}}| +$$

$$|l_e'(X^e)|[1+l_e'(X^e)^2]^{-\frac{1}{2}}\left|\frac{\partial \hat{u}}{\partial X^e}(X^e, l_e(X^e))\right| +$$

$$|\hat{l}_e'(X^e)|[1+\hat{l}_e'(X^e)^2]^{-\frac{1}{2}}\left|\frac{\partial \hat{u}}{\partial X^e}(X^e, \hat{l}_e(X^e))\right|. \tag{3.11}$$

Now (3.2), (3.3), (3.5) and the smoothness of $\hat{g}_2$ imply that

$$\left|\frac{\partial \hat{u}}{\partial n}(X^e, l_e(X^e)) - g_2(X^e, l_e(X^e))\right| \leqslant Ch \tag{3.12}$$

through combining (3.10) and (3.11). The inequality (3.7b) follows from (3.9), (3.12) and the fact that the number of boundary elements in $O(h^{-1})$. In a similar fashion we obtain (3.7c) since

$$\left|\frac{\partial \hat{u}}{\partial n_h} - \hat{g}_2\right|^2_{0,\partial_2\Omega_h} = \sum_e \int_0^{h_e} \left[\frac{\partial \hat{u}}{\partial n_h}(X^e, 0) - \hat{g}_2(X^e, 0)\right]^2 dX^e$$

and noting $\partial\hat{u}/\partial\hat{n} = \hat{g}_2$ on $\partial_2\hat{\Omega}$ we have

$$\left|\frac{\partial \hat{u}}{\partial n_h}(X^e, 0) - \hat{g}_2(X^e, 0)\right| \leqslant \left|\frac{\partial \hat{u}}{\partial Y^e}(X^e, 0) - [1+\hat{l}_e'(X^e)^2]^{-\frac{1}{2}} \times \right.$$

$$\left.\left[\frac{\partial \hat{u}}{\partial Y^e}(X^e, \hat{l}_e(X^e)) - \hat{l}_e'(X^e)\frac{\partial \hat{u}}{\partial X^e}(X^e, \hat{l}(X^e))\right]\right| +$$

$$|\hat{g}_2(X^e, \hat{l}_e(X^e)) - \hat{g}_2(X^e, 0)|.$$

The smoothness of $\hat{g}_2$, (3.3) and (3.5) imply the desired result. ∎

4. The Error Estimate

To estimate the error in the approximation (2.7) it is convenient to consider the perturbed problem (3.4) studied in the last section. Let U^* be the best fit to $\hat{u}$ in $S_E^h(\mathcal{D}_h)$ with respect to the Dirichlet norm over Ω_h, then we have the following approximation result.

11

LEMMA 4.1 *Let $U^* \in S_E^h(\mathscr{D}_h)$ be the unique solution of the projection*

$$(\nabla U^* - \nabla \hat{u}, \nabla V)_{\Omega_h} = 0 \;\; \forall \; V \in S_0^h(\mathscr{D}_h), \tag{4.1}$$

then

$$|\hat{u} - U^*|_{1,\Omega_h} \leqslant C_8 h \|\hat{u}\|_{2,\hat{\Omega}}. \tag{4.2}$$

Proof. The projection (4.1) implies that

$$|\hat{u} - U^*|_{1,\Omega_h} \leqslant |\hat{u} - V|_{1,\Omega_h} \leqslant |\mathscr{E}_h \hat{u} - V|_{1,\mathscr{D}_h} \quad \forall \; V \in S_E^h(\mathscr{D}_h)$$

and the interpolation estimate (2.6) together with the extension result (3.1) yield the desired result (4.2). ■

LEMMA 4.2 *There exist constants C_9 and C_{10}, independent of h and w, such that*

$$|w|_{0,\Omega_h} \leqslant C_9 |w|_{1,\Omega_h} \quad \forall \; w \in H^1_{E_0}(\Omega_h), \tag{4.3a}$$

$$|w|_{0,\partial_2\Omega_h} \leqslant C_{10} |w|_{1,\Omega_h} \quad \forall \; w \in H^1_{E_0}(\Omega_h). \tag{4.3b}$$

Proof. For a domain $G \subseteq \mathbb{R}^2$ there exists a constant C independent of w and G such that $|w|_{0,G} \leqslant C[\mathrm{m}(G)]^{\frac{1}{2}} |w|_{1,G}$, see Ladyzhenskaya & Ural'tseva (1968, p. 46). Thus (4.3a) holds. The proof of (4.3b) follows from the proof of the trace theorem in Nečas (1967, p. 15), (4.3a) and the inclusion $\Omega_h \subseteq \hat{\Omega}(h_0)$. ■

The error between u, the solution of (2.2), and U, the solution of (2.7), on $\Omega \cap \Omega_h$ satisfies

$$|u - U|_{1,\Omega\cap\Omega_h} \leqslant |u - \hat{u}|_{1,\Omega} + |\hat{u} - U^*|_{1,\Omega_h} + |U^* - U|_{1,\Omega_h}. \tag{4.4}$$

The first term on the right-hand side of (4.4) is the perturbation in the exact solutions due to the change of domain Ω to $\hat{\Omega}$, the second term is the approximation error due to the choice of trial space and the third term is the difference in the discrete solutions due to the change of domain. Lemma 3.2 relates the difference between the exact solutions; its analogue for the discrete solutions is given below.

LEMMA 4.3 *If* (3.3c) *holds then the solutions of* (2.7) *and* (4.1) *satisfy*

$$|U^* - U|_{1,\Omega_h} \leqslant C_{10} \left| \frac{\partial \hat{u}}{\partial n_h} - \hat{g}_2 \right|_{0,\partial_2\Omega_h}. \tag{4.5}$$

Proof. Subtracting (2.7) from (4.1), we obtain

$$(\nabla U^* - \nabla U, \nabla V)_{\Omega_h} = (\nabla \hat{u}, \nabla V)_{\Omega_h} - (\hat{f}, V)_{\Omega_h} - \langle \hat{g}_2, V \rangle_{\partial_2\Omega_h} = \left\langle \frac{\partial \hat{u}}{\partial n_h} - \hat{g}_2, V \right\rangle_{\partial_2\Omega_h}, \quad \forall \; V \in S_0^h(\mathscr{D}_h), \tag{4.6}$$

by recalling (3.4) and that $\Omega_h \subseteq \hat{\Omega}$. Then the result (4.5) follows by taking $V = U^* - U$ in (4.6) and applying the trace inequality (4.3b). ■

Combining Lemmas 3.2, 4.1 and 4.3 with Equation (4.4) results in the following theorem.

THEOREM 4.1 *Assuming that the domain $\hat{\Omega}$ can be constructed such that* (3.3) *and* (3.5) *hold then the solution u of* (2.2) *and the approximation U defined by* (2.7) *satisfy the* *following error estimate:*

$$|u-U|_{1,\Omega\cap\Omega_h} \leqslant C_{11}h, \tag{4.7}$$

where

$$C_{11} = C_5C_6 + C_8\|\hat{u}\|_{2,\hat{\Omega}} + C_{10}C_7.$$

Remark 4.1. The preceding analysis holds for fitted meshes.

Remark 4.2. If Ω is concave with respect to the surface $\partial_2\Omega$, then $\Omega \subset \Omega_h$ and the error analysis given in Strang & Fix (1973) for fitted meshes may be generalized to apply to the approximation (2.7). Let $\hat{u}$ be the extension $\mathscr{E}_h u$ of u from Ω to the plane as defined in (3.1). Then we have

$$|u-U|_{1,\Omega} \leqslant |\hat{u}-U^*|_{1,\Omega_h} + |U^*-U|_{1,\Omega_h}, \tag{4.8}$$

where U^* is defined by (4.1) and satisfies (4.2) with $\hat{\Omega} \equiv \Omega$. Hence it only remains to bound $|U^*-U|_{1,\Omega_h}$. Subtracting (2.7) from (4.1) yields

$$\begin{aligned}(\nabla U^*-\nabla U, \nabla V)_{\Omega_h} &= (\nabla\hat{u}, \nabla V)_{\Omega_h} - (\hat{f}, V)_{\Omega_h} - \langle \hat{g}_2, V\rangle_{\partial_2\Omega_h}\\ &= [(\nabla\hat{u}, \nabla V)_{\Omega_h} - (\nabla u, \nabla V)_{\Omega}] + [(f, V)_{\Omega} - (\hat{f}, V)_{\Omega_h}] +\\ &\quad [\langle g_2, V\rangle_{\partial_2\Omega} - \langle \hat{g}_2, V\rangle_{\partial_2\Omega_h}], \quad \forall\ V \in S_0^h(\mathscr{D}_h).\end{aligned} \tag{4.9}$$

The "skin" $\mathscr{R}$ is then $\Omega_h\backslash\Omega$ and $\mathrm{m}(\mathscr{R}) \leqslant Ch^2$ by the construction of Ω_h. Taking $V = e^* \equiv U^*-U$ in (4.9) and noting that $\hat{u} \equiv u$ on Ω implies

$$|e^*|^2_{1,\Omega_h} = (\nabla\hat{u}, \nabla e^*)_{\mathscr{R}} - (\hat{f}, e^*)_{\mathscr{R}} + \langle g_2, e^*\rangle_{\partial_2\Omega} - \langle \hat{g}_2, e^*\rangle_{\partial_2\Omega_h}.$$

Consider the case $g_2 = 0$, and setting $\hat{g}_2 = 0$, that is homogeneous Neumann boundary condition on $\partial_2\Omega$. Then assuming that $\nabla\hat{u}$ and $\hat{f}$ are bounded on $\mathscr{R}$ we have

$$\begin{aligned}|e^*|^2_{1,\Omega_h} &\leqslant [\mathrm{m}(\mathscr{R})]^{\frac{1}{2}}\|\nabla\hat{u}\|_{0,\infty,\mathscr{R}}|e^*|_{1,\mathscr{R}} + [\mathrm{m}(\mathscr{R})]^{\frac{1}{2}}\|\hat{f}\|_{0,\infty,\mathscr{R}}|e^*|_{0,\mathscr{R}}\\ &\leqslant Ch[|e^*|_{1,\Omega_h} + |e^*|_{0,\Omega_h}],\end{aligned} \tag{4.10}$$

since $\mathscr{R} \subset \Omega_h$. As $e^* = 0$ on $\partial_1\Omega$ we can apply the Poincaré–Friedrichs inequality (4.3a) to obtain

$$|e^*|_{1,\Omega_h} \leqslant Ch. \tag{4.11}$$

Thus substituting (4.11) and the approximation error (4.2) into (4.8) we have shown that

$$|u-U|_{1,\Omega} \leqslant Ch. \tag{4.12}$$

This result applies to an important class of problems—for example exterior flow past a convex body. One can view (4.12) as a generalization of a result buried in Oganesyan (1966). Oganesyan shows that for any domain Ω if one constructs an approximation Ω_h such that $\Omega \subset \Omega_h$ and which satisfies dist $(\partial\Omega, \partial\Omega_h) \leqslant Ch^2$, then a piecewise linear approximation U based on a triangulation of Ω_h, that is $\mathscr{D}_h \equiv \Omega_h$ a fitted mesh, satisfies (4.12). Oganesyan gives a construction for Ω_h which is extremely technical and certainly not easy to implement.

5. Implementation and Numerical Examples

For a given Ω and $\mathscr{D}^*$ the construction of Ω_h is straightforward. Then to obtain the approximation $U \in S_E^h(\mathscr{D}_h)$ defined by (2.7) integrals over this polygonal domain

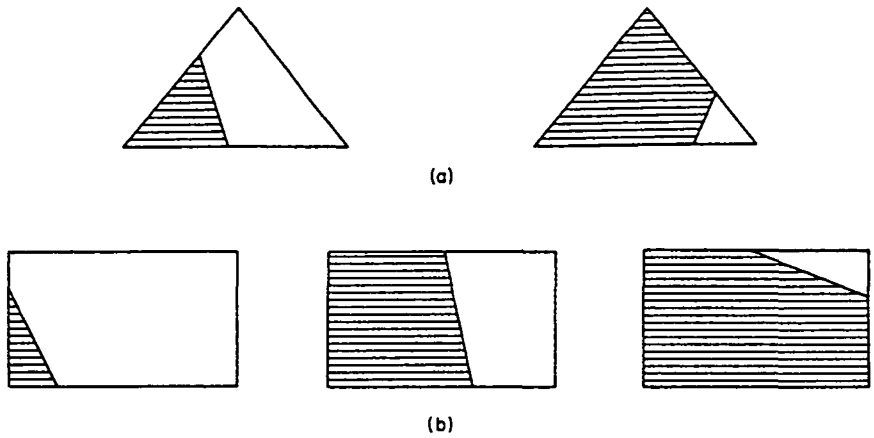

FIG. 3. (a) Examples of typical subregions $e \cap \Omega_h$ when using triangular elements. (b) Examples of typical subregions $e \cap \Omega_h$ when using rectangular elements.

Ω_h and the piecewise linear curve $\partial_2\Omega_h$ have to be calculated. These integrals are performed in each element e individually as in the normal manner, but now, as we are dealing with an unfitted mesh, in some elements the integration is calculated only over the subregion $e \cap \Omega_h$; examples of which are given in Fig. 3. Since Ω_h is polygonal these integrals are easy to evaluate.

In evaluating the left-hand side of (2.7) for each basis function of $S_0^h(\mathcal{D}_h)$ a constant function has to be integrated over $e \cap \Omega_h$ when using a piecewise linear trial space on triangles or a quadratic function when using a piecewise bilinear trial space on rectangles. In either case, by inspecting Fig. 3, the region $e \cap \Omega_h$ can be split into one, two or three subtriangles t on which an appropriate integration rule can be used to evaluate the integral exactly: that is sampling the integrand at the centroid for a linear trial space and averaging the value of the integrand at the midpoint of the sides for a bilinear trial space. Therefore it is a simple matter to evaluate the left-hand side of (2.7) exactly.

The use of numerical integration plays a more important role in evaluating the terms on the right-hand side of (2.7). The numerical integration rules chosen should be of sufficiently high order so as to retain the optimal rate of convergence given by (4.7), but at the same time it would be desirable that their sampling points were contained in Ω for evaluating the integral over Ω_h and on $\partial_2\Omega$ for evaluating the integral on $\partial_2\Omega_h$, respectively; for then the numerical approximation U would be independent of the extensions of f and g_2. In employing numerical integration our approximation scheme (2.7) becomes: find $U^h \in S_E^h(\mathcal{D}_h)$ such that

$$(\nabla U^h, \nabla V)_{\Omega_h} = (\hat{f}, V)^h_{\Omega_h} + \langle \hat{g}_2, V\rangle^h_{\partial_2\Omega_h}, \quad \forall\ V \in S_0^h(\mathcal{D}_h), \tag{5.1}$$

where $(w_1, w_2)^h_{\Omega_2}$ and $\langle w_1, w_2\rangle^h_{\partial_2\Omega_h}$ are approximations to the integrals $(w_1, w_2)_{\Omega_h}$ and $\langle w_1, w_2\rangle_{\partial_2\Omega_h}$, respectively. We now have the following result:

LEMMA 5.1 *If the numerical integration rules are such that*

$$|(\hat{f}, V)_{\Omega_h} - (\hat{f}, V)^h_{\Omega_h}| \leqslant C(\hat{f})h|V|_{1,\Omega_h}, \quad \forall\ V \in S_0^h(\mathcal{D}_h), \tag{5.2a}$$

and

$$|\langle \hat{g}_2, V\rangle_{\partial_2\Omega_h} - \langle \hat{g}_2, V\rangle^h_{\partial_2\Omega_h}| \leqslant C(\hat{g}_2)h|V|_{1,\Omega_h}, \quad \forall\ V \in S^h_0(\mathscr{D}_h); \tag{5.2b}$$

then the solutions U of (2.7) *and U^h of* (5.1) *satisfy*

$$|U - U^h|_{1,\Omega_h} \leqslant C_{12}h, \tag{5.3}$$

where C_{12} is a constant independent of h.

Proof. Subtracting (5.1) from (2.7) with $V = U - U^h$ and using the bounds (5.2) yields the desired result. ■

Thus if the assumptions of Lemma 5.1 hold we have that the approximation U^h has the optimal rate of convergence in the Dirichlet norm through combining (5.3) with (4.7).

A numerical integration rule $(\hat{f}, V)^h_{\Omega_h}$ which depends only on f, and not on $\hat{f}$, is to average the value of the integrand at the vertices of each subtriangle t, since each vertex lies in $\bar{\Omega}$ through the construction of Ω_h. It is a simple matter to show that this rule satisfies the condition (5.2a). For it is equivalent to integrating exactly the piecewise linear interpolate $s^1_h(\hat{f}V)$ of $\hat{f}V$, which is linear on each subtriangle t and interpolates $\hat{f}V$ at the vertices, and so we have

$$\begin{aligned}|(\hat{f}, V)_{\Omega_h} - (\hat{f}, V)^h_{\Omega_h}| &= \left|\int_{\Omega_h} [\hat{f}V - s^1_h(\hat{f}V)]\,dx\,dy\right| \\ &\leqslant C|\hat{f}V - s^1_h(\hat{f}V)|_{0,\Omega_h}.\end{aligned} \tag{5.4}$$

From standard interpolation theory (see Ciarlet, 1978, p. 123) we have for sufficiently smooth $\hat{f}$

$$|\hat{f}V - s^1_h(\hat{f}V)|_{0,t} \leqslant Ch_t^2|\hat{f}V|_{2,t}, \tag{5.5}$$

where h_t is the diameter of the triangle t. Note that for (5.5) to hold we do *not* require that the smallest angle of each subtriangle t to be bounded below independently of h_t, which we could not guarantee with the given construction of Ω_h and t. Combining (5.4) with (5.5) yields

$$\begin{aligned}|(\hat{f}, V)_{\Omega_h} - (f, V)^h_{\Omega_h}| &\leqslant C(\sum_t |\hat{f}V - s^1_h(\hat{f}V)|^2_{0,t})^{\frac{1}{2}} \\ &\leqslant C(\sum_t h_t^4|\hat{f}V|^2_{2,t})^{\frac{1}{2}}.\end{aligned} \tag{5.6}$$

However, we have that

$$|\hat{f}V|_{2,t} \leqslant C[|\hat{f}|_{2,\infty,t}|V|_{0,t} + |\hat{f}|_{1,\infty,t}|V|_{1,t} + |\hat{f}|_{0,\infty,t}|V|_{2,t}].$$

For the case $S^h(\mathscr{D}_h)$ linears on triangles we have $|V|_{2,t} \equiv 0$. For bilinears on rectangles we have the inverse inequality $|V|_{2,t} \leqslant Ch_t^{-1}|V|_{1,t}$. Thus in both cases the right-hand side of (5.6) can be bounded as follows

$$\begin{aligned}(\sum_t h_t^4|\hat{f}V|^2_{2,t})^{\frac{1}{2}} &\leqslant (\sum_t h_t^2\|\hat{f}\|^2_{2,\infty,t}\|V\|^2_{1,t})^{\frac{1}{2}} \\ &\leqslant h\|\hat{f}\|_{2,\infty,\Omega_h}\|V\|_{1,\Omega_h}.\end{aligned} \tag{5.7}$$

The desired result (5.2a) follows by applying the Poincaré–Friedrichs inequality (4.3a) to the right-hand side of (5.7).

If the trapezium rule is used for each section $e \cap \partial_2\Omega_h$ of the boundary integral, then once again this rule depends only on g_2, and not on the extension $\hat{g}_2$. However, we shall see that the bound (5.2b) only holds when $S^h(\mathscr{D}_h)$ is the space of piecewise linear functions on triangles. In the case of bilinears on rectangles a higher order numerical integration rule, such as Simpson's rule, has to be used to satisfy the bound (5.2b) with the disadvantage that it is dependent on the extension $\hat{g}_2$. The trapezium rule on the boundary edge $[0, h_e] \equiv e \cap \partial_2\Omega_h$ is equivalent to integrating exactly the linear interpolate $q_h^1(\hat{g}_2 V)$ of $\hat{g}_2 V$, which is linear on $e \cap \partial_2\Omega_h$ and interpolates $\hat{g}_2 V$ at the end points, and so we have

$$\begin{aligned} |\langle \hat{g}_2, V\rangle_{\partial_2\Omega_h} - \langle \hat{g}_2, V\rangle^h_{\partial_2\Omega_h}| &\leqslant C|\hat{g}_2 V - q_h^1(\hat{g}_2 V)|_{0,\partial_2\Omega_h} \\ &\leqslant C[\sum_e h_e^4 |\hat{g}_2 V|^2_{2,e\cap\partial_2\Omega_h}]^{\frac{1}{2}}, \end{aligned} \tag{5.8}$$

for sufficiently smooth $\hat{g}_2$ from standard interpolation theory. Expanding $|\hat{g}_2 V|_{2,e\cap\partial_2\Omega_h}$, we obtain

$$\begin{aligned} |\hat{g}_2 V|_{2,e\cap\partial_2\Omega_h} \leqslant C\{&|\hat{g}_2|_{2,\infty,e\cap\partial_2\Omega_h}|V|_{0,e\cap\partial_2\Omega_h} + |\hat{g}_2|_{1,\infty,e\cap\partial_2\Omega_h}|V|_{1,e\cap\partial_2\Omega_h} + \\ &|\hat{g}_2|_{0,\infty,e\cap\partial_2\Omega_h}|V|_{2,e\cap\partial_2\Omega_h}\}. \end{aligned}$$

For the case $S^h(\mathscr{D}_h)$ linears on triangles we have $|V|_{2,e\cap\partial_2\Omega_h} \equiv 0$ and applying the inverse inequality

$$|V|_{1,e\cap\partial_2\Omega_h} \leqslant Ch_e^{-1}|V|_{0,e\cap\partial_2\Omega_h}, \tag{5.9}$$

we obtain

$$|\hat{g}_2 V|_{2,e\cap\partial_2\Omega_h} \leqslant C(\hat{g}_2)h_e^{-1}|V|_{0,e\cap\partial_2\Omega_h}. \tag{5.10}$$

Substituting (5.10) into (5.8) yields

$$|\langle \hat{g}_2, V\rangle_{\partial_2\Omega_h} - \langle \hat{g}_2, V\rangle^h_{\partial_2\Omega_h}| \leqslant C(\hat{g}_2)h|V|_{0,\partial_2\Omega_h},$$

and applying the trace inequality (4.3b) we obtain the desired result (5.2b). For the case $S^h(\mathscr{D}_h)$ bilinears on rectangles we have the inverse inequalities:

$$|V|_{2,e\cap\partial_2\Omega_h} \leqslant Ch_e^{-1}|V|_{1,e\cap\partial_2\Omega_h} \leqslant Ch_e^{-2}|V|_{0,e\cap\partial_2\Omega_h}, \tag{5.11}$$

which imply that

$$|\hat{g}_2 V|_{2,e\cap\partial_2\Omega_h} \leqslant C(\hat{g}_2)h_e^{-2}|V|_{0,e\cap\partial_2\Omega_h} \tag{5.12}$$

instead of (5.10). Thus the desired bound (5.2b) is not obtained. Indeed, in practice $O(h)$ convergence of the error $u - U^h$ in the Dirichlet norm was not observed in this case.

If Simpson's rule is used for the case $S^h(\mathscr{D}_h)$ bilinears on quadrilaterals we do recover the bound (5.2b). For Simpson's rule is equivalent to integrating exactly the quadratic interpolate $q_h^2(\hat{g}_2 V)$ of $\hat{g}_2 V$ on each segment $[0, h_e] \equiv e \cap \partial_2\Omega_h$ and so we have

$$\begin{aligned} |\langle \hat{g}_2, V\rangle_{\partial_2\Omega_h} - \langle \hat{g}_2, V\rangle^h_{\partial_2\Omega_h}| &\leqslant C|\hat{g}_2 V - q_h^2(\hat{g}_2 V)|_{0,\partial_2\Omega_h} \\ &\leqslant C\left[\sum_e h_e^6 |\hat{g}_2 V|^2_{3,e\cap\partial_2\Omega_h}\right]^{\frac{1}{2}}, \end{aligned} \tag{5.13}$$

for sufficiently smooth $\hat{g}_2$. Expanding $|\hat{g}_2 V|_{3,e\cap\partial_2\Omega_h}$ and noting that $|V|_{3,e\cap\partial_2\Omega_h} \equiv 0$

TABLE 1
Results for Example 5.1

h	$\lvert u-U^h\rvert_{1,\Omega\cap\Omega_h}$	$\lvert u-U^h\rvert_{0,\Omega\cap\Omega_h}$	$\max_{\substack{\text{nodes}\\ x_j\in\Omega}} \lvert u(x_j)-U^h(x_j)\rvert$
2/4	1·13017	0·24544	0·14771
2/5	0·87594	0·15422	0·07636
2/6	0·74415	0·10979	0·07993
2/8	0·55648	0·06120	0·03953
2/10	0·44503	0·03975	0·03147
2/12	0·37033	0·02711	0·01819

and using the inverse inequalities (5.11) we obtain

$$|\hat{g}_2 V|_{3,e\cap\partial_2\Omega_h} \leqslant C(\hat{g}_2)h_e^{-2}|V|_{0,e\cap\partial_2\Omega_h}. \tag{5.14}$$

Substituting (5.14) into (5.13) and applying the trace inequality yields the desired result (5.2b).

We now report on some numerical examples, each solving a Poisson equation in a square with a section removed. For our trial space we choose piecewise bilinears on squares of size h, resulting from a uniform partition of the complete square. Numerical integration of the type described previously in this section is employed.

Example 5.1. The domain Ω consists of the square $[-2,2]\times[-2,2]$ with the unit disc removed. Dirichlet conditions are specified on the sides of the square and a Neumann condition on $x^2+y^2=1$ so that the solution u of Poisson's equation with $f=-(x^2+y^2)$ is

$$u(r,\theta)=(r^2+r^{-2})\cos 2\theta+\frac{r^4}{12}[\sin^4\theta+\cos^4\theta], \tag{5.15}$$

where

$$r^2=x^2+y^2 \quad\text{and}\quad \tan\theta=y/x.$$

From (5.15) we see that $u_n=(x^4+y^4)/3$ on $x^2+y^2=1$ and thus for the extension $\hat{g}_2$ we choose the obvious candidate $\hat{g}_2=(x^4+y^4)/3$. Owing to symmetry one can solve this problem in the quadrant $[0,2]\times[0,2]$ with $x^2+y^2\geqslant 1$. For our error estimate

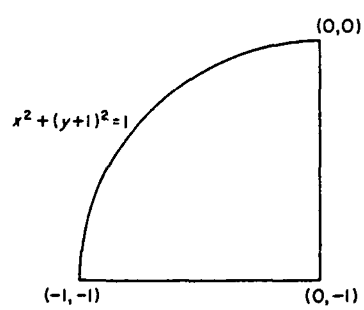

FIG. 4. The domain Ω for Example 5.2.

TABLE 2
Results for Example 5.2

h	$\lvert u-U^h\rvert_{1,\Omega\cap\Omega_h}$	$\lvert u-U^h\rvert_{0,\Omega\cap\Omega_h}$	$\max_{\text{nodes } x_j\in\Omega} \lvert u(x_j)-U^h(x_j)\rvert$
1/4	0·14994	0·01014	0·01359
1/5	0·12462	0·00661	0·00791
1/6	0·10544	0·00484	0·00804
1/8	0·07997	0·00274	0·00509
1/10	0·06429	0·00177	0·00360
1/12	0·05372	0·00126	0·00282

to hold we have to construct a domain $\hat{\Omega}(h)$ such that (3.3) and (3.5) hold. Clearly this is easily achieved by taking $\partial_2\hat{\Omega}(h)$ to be the circle $x^2+y^2 = 1-C_4h^2$, where C_4 is a sufficiently large positive constant such that $\Omega \subseteq \hat{\Omega}_h \subseteq \hat{\Omega}(h)$. The error between the true solution and the finite-element solution is shown in Table 1 for various values of h. We see that the predicted $O(h)$ convergence in the Dirichlet norm is obtained. Also the L^2 error appears to be $O(h^2)$ and the maximum error at a node lying inside Ω appears to be approximately $O(h^2)$, this maximum error usually being attained at a node lying near the boundary $\partial_2\Omega$.

Example 5.2. The domain Ω is depicted in Fig. 4. Dirichlet conditions are specified on the sides $x = 0$ and $y = -1$ and a Neumann condition on the curved boundary $x^2+(y+1)^2 = 1$ so that the solution u of Laplace's equation is

$$u(r) = \ln r, \tag{5.16}$$

where

$$r^2 = (x-0{\cdot}25)^2+y^2.$$

From (5.16) we see that $u_n = [x(x-0{\cdot}25)+y(y+1)]r^{-2}$ on $x^2+(y+1)^2 = 1$ and this is what we set the extension $\hat{g}_2$ to be. From the results given in Table 2 we see that the error in the Dirichlet norm is $O(h)$, the error in L_2 norm is $O(h^2)$ and the maximum nodal error is approximately $O(h^2)$. Since Ω is convex, we have $\Omega_h \subseteq \Omega$ and so we can take $\hat{\Omega} \equiv \Omega$ to apply our error estimate.

TABLE 3
Results for Example 5.3

h	$\lvert u-U^h\rvert_{1,\Omega\cap\Omega_h}$	$\lvert u-U^h\rvert_{0,\Omega\cap\Omega_h}$	$\max_{\text{nodes } x_j\in\Omega} \lvert u(x_j)-U^h(x_j)\rvert$
1/4	0·07378	0·00838	0·03063
1/5	0·05880	0·00609	0·01386
1/6	0·04943	0·00475	0·01624
1/8	0·03494	0·00201	0·00715
1/10	0·02804	0·00142	0·00505
1/12	0·02350	0·00090	0·00326

Example 5.3. This example is similar to Example 5.2, the only change being that we take the curved boundary to be $(y+\frac{1}{2}) = 4(x+\frac{1}{2})^3$, so now Ω is not convex and thus $\Omega_h \not\subseteq \Omega$. Dirichlet and Neumann boundary conditions are specified as before so that the solution u is given by (5.16). From (5.16) we see that

$$u_n = [y-12(x+\tfrac{1}{2})^2(x-0{\cdot}25)]r^{-2}/[144(x+\tfrac{1}{2})^4+1]^{\frac{1}{2}}$$

on $(y+\frac{1}{2}) = 4(x+\frac{1}{2})^3$ and this is what we set the extension $\hat{g}_2$ to be. From the results given in Table 3 we see once again that the error in the Dirichlet norm is $O(h)$, the error in the L_2 norm is $O(h^2)$ and the maximum nodal error is approximately $O(h^2)$. However, in this case it is not obvious how to construct the domain $\hat{\Omega}(h)$ such that the conditions (3.3) and (3.5) hold in order for our error estimate to apply. The main problem is that $\partial_2\Omega$ is locally concave near the origin. Although u is smooth at the origin, if we construct a curve $\partial_2\hat{\Omega}(h)$ satisfying (3.3) we cannot guarantee the regularity of $\hat{u}$ at the origin.

To conclude, we see that the technique presented is easy to implement and produces an approximation with the same convergence properties as one would obtain with a fitted mesh. Thus the method retains accuracy with a considerable saving in effort and computer time.

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