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Fitted and Unfitted Finite-Element Methods for Elliptic Equations with Smooth Interfaces

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This paper considers the finite-element approximation of the elliptic interface problem: $-\nabla\cdot(\sigma\nabla u)+cu=f$ in $\Omega\subset\mathbb{R}^n$ ($n=2$ or 3), with $u=0$ on $\partial\Omega$, where σ is discontinuous across a smooth surface Γ in the interior of Ω. First we show that, if the mesh is isoparametrically fitted to Γ using simplicial elements of degree $k-1$, with $k\geqslant 2$, then the standard Galerkin method achieves the optimal rate of convergence in the H^1 and L^2 norms over the approximations Ω_i^h of Ω_i, where $\Omega\equiv\Omega_1\cup\Gamma\cup\Omega_2$. Second, since it may be computationally inconvenient to fit the mesh to Γ, we analyse a fully practical piecewise linear approximation of a related penalized problem, as introduced by Babuska (1970), based on a mesh that is independent of Γ. We show that, by choosing the penalty parameter appropriately, this approximation converges to u at the optimal rate in the H^1 norm over Ω_i^h, and in the L^2 norm over any interior domain Ω_i^* satisfying $\Omega_i^*\Subset\Omega_i^{**}\Subset\Omega_i^h$ for some domain Ω_i^{**}.

1. Introduction

LET Ω be a bounded domain in $\mathbb{R}^n$ ($n=2$ or 3) with a Lipschitz boundary $\partial\Omega$. Let $\Omega_1\Subset\Omega$ be a domain in $\mathbb{R}^n$ with a smooth boundary Γ. Set $\Omega_2=\Omega-\bar{\Omega}_1$. Consider the elliptic interface problem of finding u_1 and u_2 such that

$$A_i u_i=f_i \quad \text{in } \Omega_i \quad (i=1,2), \tag{1.1a}$$

$$u_1=u_2 \quad \text{and} \quad \sigma_1\frac{\partial u_1}{\partial v}=\sigma_2\frac{\partial u_2}{\partial v} \quad \text{on } \Gamma, \tag{1.1b}$$

$$u_2=0 \quad \text{on } \partial\Omega, \tag{1.1c}$$

where

$$A_i w\equiv -\nabla\cdot(\sigma_i\nabla w)+cw, \tag{1.2a}$$

σ_1, σ_2, c, f_1, and f_2 are sufficiently smooth prescribed functions such that

$$\sigma_M\geqslant\sigma_i(x)\geqslant\sigma_L>0 \quad (x\in\Omega_i), \qquad \sigma_i\in C(\bar{\Omega}_i), \tag{1.2b}$$

$$c_M\geqslant c(x)\geqslant c_L\geqslant 0 \quad (x\in\Omega), \qquad c\in C(\bar{\Omega}), \tag{1.2c}$$

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and $\boldsymbol{\nu}$ is the unit normal to Γ pointing into Ω_2 and $\sigma_1 \neq \sigma_2$ on Γ. Throughout this paper, the subscript i is implicitly assumed to mean the integers 1 and 2.

Defining

$$a_i(w, v) = \int_{\Omega_i} (\sigma_i \nabla w \cdot \nabla v + cwv)\,\mathrm{d}x, \qquad l_i(v) = \int_{\Omega_i} fv\,\mathrm{d}x$$

leads to the following variational form of (1.1): find $u \in \mathrm{H}^1_0(\Omega)$ such that

$$\sum_{i=1}^{2} a_i(u_i, v) = \sum_{i=1}^{2} l_i(v) \quad \forall\, v \in \mathrm{H}^1_0(\Omega), \tag{1.3}$$

where $u_i \equiv u \restriction_{\Omega_i}$.

It follows from (1.2) that there exists a unique solution to (1.3). Clearly, (1.1) can be considered to be an elliptic equation with discontinuous coefficients. The purpose of this paper is to consider two finite-element methods for the approximation of (1.1). The first method is the natural fitted-mesh method based on approximating Ω_i by Ω_i^h, using isoparametric elements in the neighbourhood of Γ, so that the approximation Γ^h of Γ is composed of element sides. In Section 2 we show that, if the finite-element space defined over Ω has approximation power h^k in the L^2 norm, and if dist $(\Gamma, \Gamma^h) \leq Ch^k$, then one obtains optimal rates of convergence for the error in the H^1 and L^2 norms over Ω_i^h.

Since it may be computationally inconvenient to fit the mesh to Γ, we wish to consider a finite-element approximation based on a mesh that is independent of Γ. Unfortunately, the standard Galerkin method performs poorly in this case, achieving the rate $O(h^{\frac{1}{2}})$ in the H^1 norm for all $k \geq 2$. An alternative approach is to approximate the penalized problem, as introduced by Babuska (1970), of finding $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ such that

$$A_i u_{i,\varepsilon} = f_i \quad \text{on } \Omega_i \quad (i = 1, 2), \tag{1.4a}$$

$$\sigma_1 \frac{\partial u_{1,\varepsilon}}{\partial \nu} = \sigma_2 \frac{\partial u_{2,\varepsilon}}{\partial \nu} = \varepsilon^{-1}(u_{2,\varepsilon} - u_{1,\varepsilon}) \quad \text{on } \Gamma, \tag{1.4b}$$

$$u_{2,\varepsilon} = 0 \quad \text{on } \partial\Omega, \tag{1.4c}$$

where ε, a small positive constant, is the penalty parameter. In Section 3 we show that $u_{i,\varepsilon} \to u_i$ as $\varepsilon \to 0$. The important problem is how to choose ε with respect to h and k so that the Galerkin finite-element approximation $u_{i,\varepsilon}^h$ of (1.4) converges at the optimal rate as $h \to 0$.

Babuska (1970) analysed the Galerkin approximation of (1.4) in the absence of variational crimes; that is, it was assumed that integrals over Ω_i and Γ could be performed exactly. Setting $\varepsilon = h^\lambda$, with $\lambda > 0$, he showed that

$$\Big(\sum_{i=1}^{2} |u_i - u_{i,\varepsilon}^h|_{1,\Omega_i}^2\Big)^{\frac{1}{2}} \leq Ch^\mu \Big(\sum_{i=1}^{2} \|u_i\|_{k,\Omega_i}^2\Big)^{\frac{1}{2}},$$

where $\mu = \min\{k - 1, k - \frac{1}{2} - \frac{1}{2}\lambda, \frac{1}{2}\lambda\}$. Therefore, for all choices of k $(k \geq 2)$, there is no choice of λ leading to the optimal rate of convergence; the best choice of λ is $k - \frac{1}{2}$, leading to $\mu = \frac{1}{2}(k - \frac{1}{2})$.

King (1975) improved on the rate of convergence by solving a sequence of problems and then extrapolating. He set $\varepsilon = 2^{-j}\gamma h$, successively for $j = 0, 1, 2, \ldots$, and defined $v_{i,2^{-j}h} \equiv u^h_{i,2^{-j}\gamma h}$, where h and γ are fixed. Using Richardson's extrapolation with respect to j, obtaining

$$u^{(1)}_{i,h} = 2v_{i,\frac{1}{2}h} - v_{i,h}, \qquad u^{(j)}_{i,h} = \frac{2^j u^{(j-1)}_{i,\frac{1}{2}h} - u^{(j-1)}_{i,h}}{2^j - 1} \quad (j \geqslant 2),$$

King showed that, for $k \geqslant 3$,

$$\left(\sum_{i=1}^{2} |u_i - u^{(k-2)}_{i,h}|^2_{1,\Omega_i}\right)^{\frac{1}{2}} \leqslant Ch^{k-1}\left(\sum_{i=1}^{2} \|u_i\|^2_{k,\Omega_i}\right)^{\frac{1}{2}}.$$

Once again, this analysis was for the unpractical standard Galerkin method in the absence of variational crimes.

Under the assumption that $u \in \tilde{\mathrm{H}}^{k+2}(\Omega)$, using the techniques of § 2 of Barrett & Elliott (1986), one can improve on the error bounds of Babuska (1970). However, since the penalty method only achieves optimality for piecewise linears, we do not include this analysis here.

In Section 3 we analyse a fully practical piecewise linear Galerkin approximation of (1.4) involving perturbation of the domains (i.e. $\Gamma \to \Gamma^h$, $\Omega_i \to \Omega_i^h$) and numerical integration. We show that the choice $\varepsilon = h^2$ leads to an approximation that converges at the optimal rate in the H^1 norm. Although we are only able to show that the error is $O(h^{\frac{3}{2}})$ in L^2 over Ω_i^h, we show that it is optimal over any interior domain $\Omega_i^* \Subset \Omega_i^{**} \Subset \Omega_i^h$ for h sufficiently small.

The following notation will be used. Given $m \in \mathbb{N}$ and a bounded domain G in $\mathbb{R}^n$,

$$\mathrm{W}^{m,p}(G) = \{w \in \mathrm{L}^p(G) : \mathrm{D}^{\eta} w \in \mathrm{L}^p(G) \;\; \forall \, |\eta| \leqslant m\}$$

and $\mathrm{H}^m(G) \equiv \mathrm{W}^{m,2}(G)$ denote, for $1 \leqslant p \leqslant \infty$, the standard Sobolev spaces, where we use the multi-index notation for derivatives. We use the following norms and seminorms on functions w defined by

$$|w|_{0,p,G} = \left(\int_G |w|^p \,\mathrm{d}x\right)^{1/p}, \qquad |w|_{m,G} = |w|_{m,2,G},$$

$$|w|_{m,p,G} = \left(\sum_{|\eta|=m} |\mathrm{D}^{\eta} w|^p_{0,p,G}\right)^{1/p}, \qquad \|w\|_{m,p,G} = \left(\sum_{|\eta|\leqslant m} |\mathrm{D}^{\eta} w|^p_{0,p,G}\right)^{1/p},$$

and $\|w\|_{m,G} = \|w\|_{m,2,G}$ for $m \in \mathbb{N}$ and $1 \leqslant p < \infty$, with the standard modification for $p = \infty$. For ∂G a $\mathrm{C}^{m,1}$ surface, we define $\mathrm{W}^{m,p}(\partial G)$ and $\|w\|_{m,p,\partial G}$ in the standard way; see e.g. Kufner, John, & Fucik (1977: pp. 305, 327). If ∂G is only piecewise $\mathrm{C}^{m,1}$, that is, $\partial G = \bigcup_{i=1}^{M} \partial_i G$, where $\partial_i G$ is $\mathrm{C}^{m,1}$ $(i = 1, \ldots, M)$, then we define

$$\|w\|_{m,p,\partial G} = \left(\sum_{i=1}^{M} \|w\|^p_{m,p,\partial_i G}\right)^{1/p}. \tag{1.5}$$

We define $\mathrm{H}^1_0(G) \equiv \{w \in \mathrm{H}^1(G) : w = 0 \text{ on } \partial G\}$ and adopt the notation

$$\|w\|^2_{\tilde{H}^m(\Omega)} = \sum_{i=1}^{2} \|w_i\|^2_{m,\Omega_i}$$

where $w_i = w\restriction_{\Omega_i}$. Throughout, C denotes a positive constant, independent of h and the penalty parameter, whose value may change in different relations. The measure of a domain G is denoted by $\mathrm{m}(G)$.

We require the trace inequalities: for ∂G of class $C^{0,1}$, we have

$$\|D^\eta w\|_{0,\partial G} \leq C\,\|w\|_{|\eta|+1,G} \quad \forall\, w \in H^{|\eta|+1}(G), \tag{1.6a}$$

and, for ∂G of class $C^{0,1}$ and piecewise $C^{m,1}$, this implies that, for $m \in \mathbb{N}$,

$$\|w\|_{m,\partial G} \leq C\,\|w\|_{m+1,G} \quad \forall\, w \in H^{m+1}(G), \tag{1.6b}$$

$$\left\|\frac{\partial w}{\partial \nu}\right\|_{0,\partial G} \leq C\,\|w\|_{2,G} \quad \forall\, w \in H^2(G). \tag{1.6c}$$

In addition, we require the trace inequality:

$$\|w\|^2_{0,\partial G} \leq \delta\,\|w\|^2_{0,G} + C\delta^{-1}\,\|w\|^2_{1,G} \quad \forall\, w \in H^1(G), \tag{1.6d}$$

where δ is independent of C. In (1.6), ν is the outward pointing unit normal to ∂G, and C is a constant independent of w; see Kufner, John, & Fucik (1977). Finally, we require the Poincaré–Friedrichs' inequality: for all $w \in H^1_0(G)$, the bound

$$\|w\|_{0,G} \leq C\mathrm{m}(G)^{\frac{1}{2}}\,|w|_{1,G} \tag{1.7}$$

holds, where C is a constant independent of G and w; see Ladyzhenskaya & Ural'tseva (1968: p. 46).

2. Isoparametric fitting of Γ

To avoid unnecessary extra complications in the analysis, we shall assume that Ω is convex polyhedral throughout this section. We remark that all the results derived in this section are valid for smooth $\partial\Omega$, provided it is isoparametrically fitted.

Let $\mathcal{T}^h \equiv \mathcal{T}^h_1 \cup \mathcal{T}^h_2$ be a partitioning of Ω into disjoint open elements τ, each of maximum diameter bounded above by h, so that $\bar{\Omega} = \bigcup_{\tau\in\mathcal{T}^h} \bar{\tau}$ and $\bar{\Omega}^h_i = \bigcup_{\tau\in\mathcal{T}^h_i} \bar{\tau}$; for $i = 1$ or 2, the domain Ω^h_i approximates Ω_i and $\Gamma^h \equiv \partial\Omega^h_1$ approximates Γ. Each element has at most one face on Γ^h and only faces on Γ^h are allowed to be curved. Associated with $\mathcal{T}^h$ is a finite-dimensional subspace S^h of $C(\bar{\Omega})$, depending upon an integer $k \geq 2$, such that $\chi\restriction_\tau \in H^2(\tau)\ \forall\, \chi \in S^h$ and $\forall\, \tau \in \mathcal{T}^h$. For convenience, we assume that S^h is generated by a nodal Lagrangian basis on simplicial isoparametric elements. Let Π^h denote the interpolation operator $\Pi^h : C(\bar{\Omega}) \to S^h$. We assume the following approximation property.

(A1) For integers k' and m satisfying $k' \geq m \geq 0$ and $k \geq k' \geq 2$, the bound

$$|w - \Pi^h w|_{m,\tau} \leq Ch^{k'-m}\,\|w\|_{k',\tau} \tag{2.1}$$

holds for each $w \in H^{k'}(\tau)$ and $\tau \in \mathcal{T}^h$.

The boundary Γ^h is constructed with the following properties. Set $\mathcal{B}^h$ to be those elements τ with a curved face on Γ^h. For each element $\tau \in \mathcal{B}^h$, there exists a local co-ordinate system (X_τ, Y_τ) such that $X_\tau \in \Delta_\tau$ and $Y_\tau \in \mathbb{R}$, where Δ_τ is either an interval ($n = 2$) or a triangle ($n = 3$) whose vertices are those vertices of

τ lying on Γ^h. The curved face Γ^h_τ is defined by the surface $\{Y_\tau = \psi^h_\tau(X_\tau), X_\tau \in \Delta_\tau\}$, where ψ^h_τ vanishes at the vertices lying on Γ^h_τ. The surface Γ is locally described by $Y_\tau = \psi_\tau(X_\tau)$, where ψ_τ agrees with ψ^h_τ at all nodes lying on Γ^h_τ. We denote this section of Γ by Γ_τ. We make the following assumption on the boundary approximation:

(A2) $$\|\psi_\tau - \psi^h_\tau\|_{0,\infty,\Delta_\tau} \leq Ch^k. \tag{2.2}$$

The assumptions (A1) and (A2) can be satisfied by a $(k-1)$-regular family of simplicial Lagrangian isoparametric elements as introduced by Ciarlet & Raviart (1972). We make the following regularity assumptions on the data:

(R1) $f_i \in H^k(\Omega_i)$, where $f_i \equiv f\restriction_{\Omega_i}$.

(R2) The function σ_i is the restriction to Ω_i of $\tilde{\sigma}_i$, where $\tilde{\sigma}_i \in C(\tilde{\Omega}_i)$, $\sigma_M \geq \tilde{\sigma}_i(x) \geq \sigma_L > 0$, and $\Omega^h_i \subset \tilde{\Omega}_i$ for all $h < h_0$, with $\partial\tilde{\Omega}_i$ smooth.

(R3) The functions $\tilde{\sigma}_i$ and c and the surface Γ are sufficiently smooth, and

$$\|u\|_{\tilde{H}^{k+2}(\Omega)} \leq C\,\|f\|_{\tilde{H}^k(\Omega)}.$$

In order to define the isoparametric finite element approximation to (1.3), it is convenient to use the notation

$$\tilde{A}_i w \equiv -\nabla\cdot(\tilde{\sigma}_i \nabla w) + cw \quad (i = 1, 2), \tag{2.3}$$

$$\tilde{a}^h_i(w, v) \equiv (\tilde{\sigma}_i \nabla w, \nabla v)_{\Omega^h_i} + (cw, v)_{\Omega^h_i} = (\tilde{A}_i w, v)_{\Omega^h_i} + \left\langle \tilde{\sigma}_i \frac{\partial w}{\partial \nu^h_i}, v\right\rangle_{\partial\Omega^h_i}, \tag{2.4}$$

where ν^h_i is the outward pointing unit normal to $\partial\Omega^h_i$, and

$$\tilde{\ell}^h_i(v) = (\tilde{f}_i, v)_{\Omega^h_i}, \tag{2.5}$$

where $\tilde{f}_i \equiv \tilde{A}_i\tilde{u}_i$ and $\tilde{u}_i = E_i u_i$; here E_i is an extension operator $E_i : H^s(\Omega_i) \to H^s(\tilde{\Omega}_i)$ such that, for all integers $s \geq 1$,

$$\left.\begin{array}{ll} E_i w = w & \text{on } \Omega_i \\ \|E_i w\|_{s,\tilde{\Omega}_i} \leq C\,\|w\|_{s,\Omega_i} & \end{array}\right\} \quad \forall\, w \in H^s(\Omega_i). \tag{2.6a, 2.6b}$$

Clearly,

$$\tilde{f}_i \equiv \tilde{A}_i\tilde{u}_i = Au_i = f \quad \text{on } \Omega_i, \tag{2.7a}$$

$$\|\tilde{f}_i\|_{k,\Omega^h_i} \equiv \|\tilde{A}_i\tilde{u}_i\|_{k,\Omega^h_i} \leq C\,\|\tilde{u}_i\|_{k+2,\Omega^h_i} \leq C\,\|u_i\|_{k+2,\Omega_i}. \tag{2.7b}$$

Remark 2.1. Since $\Omega^h_i \subseteq \tilde{\Omega}_i$ $\forall\, h < h_0$, it follows from the proof of the trace theorem in Necas (1967: p. 15) and Lemma 2.1 in Barrett & Elliott (1985) that the constants C in (1.6) are independent of h when G replaced by Ω^h_i. □

Let $u^h \in S^h_0 \equiv S^h \cap H^1_0(\Omega)$ be the finite-element approximation to u defined by

$$\sum_{i=1}^{2} \tilde{a}^h_i(u^h, \chi) = \sum_{i=1}^{2} \tilde{\ell}^h_i(\chi) \quad \forall\, \chi \in S^h_0. \tag{2.8}$$

We wish to prove the following error bounds.

THEOREM 2.1 *Let the assumptions* (A1), (A2), (R1), (R2), *and* (R3) *hold. It* *follows that*

$$\sum_{i=1}^{2} (\|\tilde{u}_i - u^h\|_{0,\Omega_i^h}^2 + h^2 \|\tilde{u}_i - u^h\|_{1,\Omega_i^h}^2)^{\frac{1}{2}} \leqslant Ch^k \|\tilde{u}\|_{\tilde{H}^{k+2}(\Omega)}$$

where $\tilde{u}_i \equiv E_i u_i$.

Remark 2.2. Note that u^h depends on the extensions $\tilde{\sigma}_i$ and $\tilde{f}_i$. In practical computations, numerical quadrature rules depending only on evaluation of integrands on $\bar{\Omega}_i$ are used. For simplicity, we do not analyse the effect of numerical integration here, but remark that the error bounds of Theorem 2.1 remain true when standard isoparametric numerical integration is used; see Nedoma (1979) and Barrett & Elliott (1985). In addition, the analysis of Nedoma (1979) can be easily incorporated to show that the result remains true if $\partial\Omega$ is smooth and is isoparametrically approximated. □

The following perturbation lemmas are crucial for the determination of the error bounds.

LEMMA 2.1 *Let* (A2), (R1), (R2), *and* (R3) *hold. It follows that*

$$\left\|\tilde{\sigma}_1 \frac{\partial \tilde{u}_1}{\partial \nu^h} - \tilde{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h}\right\|_{0,\Gamma^h} \leqslant \begin{cases} Ch^{\frac{1}{2}k} \|u\|_{\tilde{H}^2(\Omega)}, & (2.9a) \\ Ch^{k-1} \|u\|_{\tilde{H}^k(\Omega)}, & (2.9b) \end{cases}$$

where $\tilde{u}_i \equiv E_i u_i$ *and* ν^h *is the unit normal to* Γ^h *pointing into* Ω_2^h.

Proof. (i) First note that

$$\nu = \begin{bmatrix} -\nabla\psi \\ 1 \end{bmatrix} J \quad \text{and} \quad \nu^h = \begin{bmatrix} -\nabla\psi^h \\ 1 \end{bmatrix} J^h,$$

where $J = (1 + |\nabla\psi|^2)^{-\frac{1}{2}}$, $J^h = (1 + |\nabla\psi^h|^2)^{-\frac{1}{2}}$, and $dX = J\, ds = J^h\, ds^h$.

For notational convenience, the subscript τ has been suppressed in the above. It follows from the construction of Γ^h that (see Barrett & Elliott, 1985)

$$\|J - J^h\|_{0,\infty,\Delta_\tau} \leqslant Ch^k, \qquad \|\nu - \nu^h\|_{0,\infty,\Delta_\tau} \leqslant Ch^{k-1} \qquad (2.10a, b)$$

where the constants C depend on derivatives of ψ, which are independent of h. Adopting the notation

$$\tilde{u}_i(\Gamma^h) \equiv \tilde{u}_i(X, \psi^h(X)), \qquad u_i(\Gamma) \equiv u_i(X, \psi(X)),$$

etc., we have, for each element face Γ_τ^h ($\tau \in \mathcal{B}^h$), that

$$\left\|\tilde{\sigma}_1 \frac{\partial \tilde{u}_1}{\partial \nu^h} - \tilde{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h}\right\|_{0,\Gamma_\tau^h}^2$$

$$= \int_{\Delta_\tau} [\tilde{\sigma}_1(\Gamma^h)\nabla\tilde{u}_1(\Gamma^h) \cdot \nu^h - \tilde{\sigma}_2(\Gamma^h)\nabla\tilde{u}_2(\Gamma^h) \cdot \nu^h]^2 (J^h)^{-1}\, dX$$

$$\leqslant \sum_{i=1}^{2} \int_{\Delta_\tau} [\tilde{\sigma}_i(\Gamma^h)\nabla\tilde{u}_i(\Gamma^h) \cdot \nu^h - \sigma_i(\Gamma)\nabla u_i(\Gamma) \cdot \nu]^2 (J^h)^{-1}\, dX$$

where the interface condition $\sigma_1(\Gamma)\nabla u_1 \cdot \nu - \sigma_2(\Gamma)\nabla u_2 \cdot \nu = 0$ has been used. It follows that

$$\left\|\bar{\sigma}_1 \frac{\partial \bar{u}_1}{\partial v^h} - \bar{\sigma}_2 \frac{\partial \bar{u}_2}{\partial v^h}\right\|^2_{0,\Gamma^h_\tau} \leqslant$$

$$2\sum_{i=1}^{2}\int_{\Delta_\tau} \{[\bar{\sigma}_i(\Gamma^h)\nabla\bar{u}_i(\Gamma^h) - \sigma_i(\Gamma)\nabla u_i(\Gamma)]\cdot v^h + \sigma_i(\Gamma)\nabla u_i(\Gamma)\cdot(v^h - v)\}^2 \frac{dX}{J^h}$$

$$\leqslant 4\sum_{i=1}^{2}\int_{\Delta_\tau}\left[\left(\int_{\psi(X)}^{\psi^h(X)} \frac{\partial}{\partial Y}(\bar{\sigma}_i\nabla\bar{u}_i)\cdot v^h\, dY\right)^2 + [\sigma_i(\Gamma)\nabla u_i(\Gamma)\cdot(v^h - v)]^2\right]\frac{dX}{J^h} \qquad (2.11)$$

$$\leqslant 4\sum_{i=1}^{2}\left[\|(J^h)^{-1}(\psi - \psi^h)\|_{0,\infty,\Delta_\tau}\int_{\Delta_\tau}\int_{\psi(X)}^{\psi^h(X)}\left(\frac{\partial}{\partial Y}(\bar{\sigma}_i\nabla\bar{u}_i)\cdot v^h\right)^2 dX\, dY\right.$$

$$\left. + \|J(J^h)^{-1}\|_{0,\infty,\Delta_\tau}\|v - v^h\|^2_{0,\infty,\Delta_\tau}\int_{\Delta_\tau}|\sigma_i(\Gamma)\nabla u_i(\Gamma)|^2 J^{-1}\, dX\right].$$

Applying (A2) and the bound (2.10b), summing over all $\tau \in \mathscr{B}^h$, and applying the trace inequality (1.6a), we obtain

$$\left\|\bar{\sigma}_1 \frac{\partial \bar{u}_1}{\partial v^h} - \bar{\sigma}_2 \frac{\partial \bar{u}_2}{\partial v^h}\right\|^2_{0,\Gamma^h} \leqslant Ch^k\sum_{i=1}^{2}\|\bar{u}_i\|^2_{2,\bar{\Omega}_i} + Ch^{2(k-1)}\sum_{i=1}^{2}\|u_i\|^2_{2,\Omega_i}.$$

The desired result (2.9a) then follows from (2.6b).

Returning to (2.11), we have that this can be bounded by

$$C\sum_{i=1}^{2} \mathrm{m}(\Delta_\tau)(\|\psi - \psi^h\|^2_{0,\infty,\Delta_\tau}\|\bar{u}_i\|^2_{2,\infty,\bar{\Omega}_i} + \|v - v^h\|^2_{0,\infty,\Delta_\tau}\|u_i\|^2_{1,\infty,\Omega_i}).$$

Applying (A2) and the bound (2.10b), summing over all $\tau \in \mathscr{B}^h$, and noting that $\sum_{\tau\in\mathscr{B}^h} \mathrm{m}(\Delta_\tau) = O(1)$, the desired result (2.9b) then follows from Sobolev's embedding theorem and (2.6b). □

LEMMA 2.2 *Let* (A2) *hold.*

(i) *It follows, for all* $w \in \mathrm{H}^1(\bar{\Omega}_i)$, *that*

$$\|w\|_{0,\Omega_i^h\setminus\Omega_i} \leqslant C(h^k|w|_{1,\Omega_i^h\setminus\Omega_i} + h^{\frac{1}{2}k}\|w\|_{0,\Gamma^h}), \qquad (2.12a)$$

$$\|w\|_{0,\Omega_i\setminus\Omega_i^h} \leqslant C(h^k|w|_{1,\Omega_i\setminus\Omega_i^h} + h^{\frac{1}{2}k}\|w\|_{0,\Gamma}). \qquad (2.12b)$$

(ii) *If* $w \in \mathrm{H}^2(\bar{\Omega}_i)$ *and* $w = 0$ *on* Γ, *then*

$$\|w\|_{0,\Gamma^h} \leqslant Ch^k\|w\|_{2,\bar{\Omega}_i}. \qquad (2.13)$$

Proof. (i) The proof of (2.12a) is given in Lemma 3.2 of Barrett and Elliott (1987). The proof of (2.12b) follows in a similar manner.

(ii) The proof of (2.13) is given in Lemma 3.4 of Barrett and Elliott (1986). □

H^1 error bound

Since

$$\sum_{i=1}^{2}\|\bar{u}_i - u^h\|^2_{1,\Omega_i^h} \leqslant 2\sum_{i=1}^{2}(\|\bar{u}_i - \Pi^h\bar{u}_i\|^2_{1,\Omega_i^h} + \|\Pi^h\bar{u}_i - u^h\|^2_{1,\Omega_i^h})$$

and the assumptions (A1) and (R3) hold, it is sufficient to bound the H^1 norm of $\chi \equiv \Pi^h \tilde{u} - u^h \in S_0^h$, where

$$(\Pi^h \tilde{u})\restriction_{\Omega_i^h} \equiv \Pi^h \tilde{u}_i \quad (i = 1, 2).$$

Clearly,

$$\sum_{i=1}^{2} \tilde{a}_i^h(\chi, \chi) = \sum_{i=1}^{2} [\tilde{a}_i^h(\tilde{u}_i - \Pi^h \tilde{u}_i, \chi) + \tilde{\ell}_i^h(\chi) - \tilde{a}_i^h(\tilde{u}_i, \chi)],$$

and, using the Poincaré–Friedrichs inequality (1.7), we obtain

$$\|\chi\|_{1,\Omega}^2 \leqslant C\,|\chi|_{1,\Omega}^2 = C \sum_{i=1}^{2} |\chi|_{1,\Omega_i^h}^2 \leqslant C \sum_{i=1}^{2} \tilde{a}_i^h(\chi, \chi)$$
$$\leqslant C \sum_{i=1}^{2} \|\tilde{u}_i - \Pi^h \tilde{u}_i\|_{1,\Omega_i^h} \|\chi\|_{1,\Omega_i^h} + \left|\sum_{i=1}^{2} [\tilde{\ell}_i^h(\chi) - \tilde{a}_i^h(\tilde{u}_i, \chi)]\right|. \quad (2.14)$$

It follows from (2.4) and (2.7a) that

$$\sum_{i=1}^{2} [\tilde{\ell}_i^h(\chi) - \tilde{a}_i^h(\tilde{u}_i, \chi)] = \int_{\Gamma^h} \left(\bar{\sigma}_1 \frac{\partial \tilde{u}_1}{\partial \nu^h} - \bar{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h}\right) \chi \, \mathrm{d}s^h. \quad (2.15)$$

The H^1 error bound is now a direct consequence of (2.14), (2.15), (A1), (2.9b), and the trace inequality (1.6b).

L^2 error bound

Let

$$\tilde{e} = \begin{cases} \tilde{u}_1 - u^h & \text{in } \Omega_1^h, \\ \tilde{u}_2 - u^h & \text{in } \Omega_2^h, \end{cases}$$

and note that

$$\|\tilde{e}\|_{0,\Omega} = \sup_{\eta \in \mathrm{L}^2(\Omega)} \frac{(\tilde{e}, \eta)_\Omega}{\|\eta\|_{0,\Omega}}. \quad (2.16a)$$

For any $\eta \in \mathrm{L}^2(\Omega)$, we take $z = \{z_i\}_{i=1}^2$ to be the unique solution of (1.1) with $f \equiv \eta$, so that

$$\|z_i\|_{2,\Omega_i} \leqslant C\,\|\eta\|_{0,\Omega}, \quad (2.16b)$$

and we define $\tilde{z}_i = E_i z_i$. It follows that

$$(\tilde{e}, \eta)_\Omega = \sum_{i=1}^{2} (\tilde{e}, \tilde{A}_i \tilde{z}_i)_{\Omega_i^h} + \sum_{i=1}^{2} (\tilde{e}, \eta - \tilde{A}_i \tilde{z}_i)_{\Omega_i^h \setminus \Omega_i}$$
$$= \sum_{i=1}^{2} \tilde{a}_i^h(\tilde{e}, \tilde{z}_i) + \int_{\Gamma^h} \left(\bar{\sigma}_2 \frac{\partial \tilde{z}_2}{\partial \nu^h} (\tilde{u}_2 - u^h) - \bar{\sigma}_1 \frac{\partial \tilde{z}_1}{\partial \nu^h} (\tilde{u}_1 - u^h)\right) \mathrm{d}s^h$$
$$+ \sum_{i=1}^{2} (\tilde{e}, \eta - \tilde{A}_i \tilde{z}_i)_{\Omega_i^h \setminus \Omega_i}$$
$$= \sum_{i=1}^{2} \tilde{a}_i^h(\tilde{e}, \tilde{z}_i - \Pi^h \tilde{z}_i) + \sum_{i=1}^{2} [\tilde{a}_i^h(\tilde{u}_i, \Pi^h \tilde{z}_i) - \tilde{\ell}_i^h(\Pi^h \tilde{z}_i)]$$
$$+ \int_{\Gamma^h} \left(\bar{\sigma}_2 \frac{\partial \tilde{z}_2}{\partial \nu^h} - \bar{\sigma}_1 \frac{\partial \tilde{z}_1}{\partial \nu^h}\right)(\tilde{u}_1 - u^h) \, \mathrm{d}s^h + \int_{\Gamma^h} \bar{\sigma}_2 \frac{\partial \tilde{z}_2}{\partial \nu^h} (\tilde{u}_2 - \tilde{u}_1) \, \mathrm{d}s^h$$
$$+ \sum_{i=1}^{2} (\tilde{e}, \eta - \tilde{A}_i \tilde{z}_i)_{\Omega_i^h \setminus \Omega_i} \equiv \sum_{j=1}^{5} I_j. \quad (2.17)$$

We now proceed to bound each of the I_j in turn and show that, for each j,

$$|I_j| \leq Ch^k \|\eta\|_{0,\Omega} \|u\|_{\tilde{H}^{k+2}(\Omega)}. \tag{2.18}$$

Using the interpolation bound (2.1), the known regularity of z_i, (2.16b), and the properties of E_i, we have that

$$|I_1| \leq C \sum_{i=1}^{2} \|\tilde{u}_i - u^h\|_{1,\Omega_i^h} \|\tilde{z}_i - \Pi^h \tilde{z}_i\|_{1,\Omega_i^h}$$
$$\leq Ch\Big(\sum_{i=1}^{2} \|\tilde{u}_i - u^h\|_{1,\Omega_i^h}\Big) \|\eta\|_{0,\Omega},$$

which, together with the H^1 error bound, yields (2.18). Using (1.4), we find that

$$|I_2| \leq \sum_{i=1}^{2} |\tilde{a}_i^h(\tilde{u}_i, \Pi^h\tilde{z}_i - \tilde{z}_i) - \tilde{\ell}_i^h(\Pi^h\tilde{z}_i - \tilde{z}_i)|$$
$$+ \sum_{i=1}^{2} |\tilde{a}_i^h(\tilde{u}_i, \tilde{z}_i) - a_i(u_i, z_i)| + \sum_{i=1}^{2} |\ell_i(z_i) - \tilde{\ell}_i^h(\tilde{z}_i)|$$
$$\leq \left|\int_{\Gamma^h} \Big(\tilde{\sigma}_1 \frac{\partial \tilde{u}_1}{\partial \nu^h} - \tilde{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h}\Big)(\tilde{z}_1 - \Pi^h \tilde{z}_1)\, \mathrm{d}s^h\right|$$
$$+ \left|\int_{\Gamma^h} \tilde{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h} (\tilde{z}_1 - \Pi^h\tilde{z}_1 + \Pi^h\tilde{z}_2 - \tilde{z}_2)\, \mathrm{d}s^h\right|$$
$$+ \sum_{i=1}^{2} \left|\int_{\Omega_i^h \setminus \Omega_i} (\tilde{\sigma}_i \nabla \tilde{u}_i \cdot \nabla \tilde{z}_i + c\tilde{u}_i \tilde{z}_i)\, \mathrm{d}x - \int_{\Omega_i \setminus \Omega_i^h} (\sigma_i \nabla u_i \cdot \nabla z_i + c u_i z_i)\, \mathrm{d}x\right|$$
$$+ \sum_{i=1}^{2} \left|\int_{\Omega_i \setminus \Omega_i^h} f z_i\, \mathrm{d}x - \int_{\Omega_i^h \setminus \Omega_i} \tilde{f} \tilde{z}_i\, \mathrm{d}x\right| = \sum_{j=1}^{4} S_j.$$

From (2.9b), the trace inequality (see (1.6b)), the interpolation error bound, (2.6b), and (2.16b), it follows that

$$|S_1| \leq Ch^k \|\eta\|_{0,\Omega} \|u\|_{\tilde{H}^k(\Omega)}.$$

Noting that $\Pi^h\tilde{z}_1 = \Pi^h\tilde{z}_2$ on Γ^h and $\tilde{z}_1 - \tilde{z}_2 = 0$ on Γ, we have from (2.13), (2.16b), (2.6b), and (1.6c) that

$$|S_2| \leq Ch^k \|\eta\|_{0,\Omega} \|u\|_{\tilde{H}^2(\Omega)}.$$

Setting w in (2.12a, b) to be, in turn, u_i, $\tilde{u}_i$, z_i, $\tilde{z}_i$, and their first derivatives, and then applying the trace inequality (1.6a), (2.6b), and (2.16b), yields

$$|S_3| \leq C \sum_{i=1}^{2} \{\|\tilde{u}_i\|_{1,\Omega_i^h\setminus\Omega_i} \|\tilde{z}_i\|_{1,\Omega_i^h\setminus\Omega_i} + \|u_i\|_{1,\Omega_i\setminus\Omega_i^h} \|z_i\|_{1,\Omega_i\setminus\Omega_i^h}\}$$
$$\leq Ch^k \|\eta\|_{0,\Omega} \|u\|_{\tilde{H}^2(\Omega)}.$$

Setting $w = z_i$ and $w = \tilde{z}_i$ in (2.12a, b), and using (1.6b), (2.6b), (2.16b), (2.2), (2.7b), and Sobolev's embedding theorem, yields

$$|S_4| \leq \sum_{i=1}^{2} [\mathrm{m}(\Omega_i \setminus \Omega_i^h)^{\frac{1}{2}} \|z_i\|_{0,\Omega_i\setminus\Omega_i^h} + \mathrm{m}(\Omega_i^h \setminus \Omega_i)^{\frac{1}{2}} \|\tilde{z}_i\|_{0,\Omega_i^h\setminus\Omega_i}] \|\tilde{f}_i\|_{0,\infty,\hat{\Omega}_i}$$
$$\leq Ch^k \|\eta\|_{0,\Omega} \sum_{i=1}^{2} \|\tilde{u}_i\|_{2,\infty,\hat{\Omega}_i} \leq Ch^k \|\eta\|_{0,\Omega} \|u\|_{\tilde{H}^k(\Omega)}.$$

Collecting these bounds for the S_j, we obtain (2.18) for I_2.

Since $\sigma_1\,\partial z_1/\partial\nu=\sigma_2\,\partial z_2/\partial\nu$ on Γ, we have, from (2.9a), that

$$|I_3|\leqslant Ch^{\frac{1}{2}k}\,\|z\|_{\tilde{H}^2(\Omega)}\,\|\tilde{u}_1-u^h\|_{0,\Gamma^h}.$$

Applying (2.16b), (1.6a), and the H^1 error bound yields (2.18).

Since $\tilde{u}_1-\tilde{u}_2=0$ on Γ, it follows from (2.13) and (1.6c) that

$$|I_4|\leqslant Ch^k\,\|\tilde{z}_2\|_{2,\tilde{\Omega}_2}\,\|\tilde{u}_2\|_{2,\tilde{\Omega}_2},$$

and again (2.18) holds.

Finally, we have from (2.12a) and (1.6b) that

$$\begin{aligned}|I_5|&\leqslant\sum_{i=1}^{2}\|\tilde{e}\|_{0,\Omega_i^h\setminus\Omega_i}(\|\eta\|_{0,\Omega_i^h\setminus\Omega_i}+C\,\|\tilde{z}_i\|_{2,\tilde{\Omega}_i})\\&\leqslant Ch^{\frac{1}{2}k}\Big(\sum_{i=1}^{2}\|\tilde{u}_i-u^h\|_{1,\Omega_i^h}\Big)\,\|\eta\|_{0,\Omega},\end{aligned}$$

and the H^1 error bound yields (2.18).

The L^2 error bound is an immediate consequence of (2.16a), (2.17), and (2.18).

3. A finite-element penalty method

In order to avoid unnecessary extra complications in the analysis, we shall assume, throughout this section, that Ω is convex and has a smooth boundary. The smoothness of $\partial\Omega$ is required in the derivation of negative norm estimates. The analysis given is for a practical piecewise linear finite-element approximation of (1.4) based on a mesh fitted to $\partial\Omega$ but totally independent of Γ. We remark that the results derived are valid with the mesh either isoparametrically fitted to $\partial\Omega$ or independent of $\partial\Omega$; in the latter case, the Dirichlet boundary condition is imposed weakly using the boundary penalty method (see Barrett & Elliott, 1986).

The variational form of (1.4) is: find $u_{1,\varepsilon}\in H^1(\Omega_1)$ and $u_{2,\varepsilon}\in H^1_E(\Omega_2)\equiv\{w\in H^1(\Omega_2):w=0\text{ on }\partial\Omega\}$ such that

$$\sum_{i=1}^{2}a_i(u_{i,\varepsilon},v_i)+\frac{1}{\varepsilon}\langle u_{1,\varepsilon}-u_{2,\varepsilon},v_1-v_2\rangle_\Gamma=\sum_{i=1}^{2}l_i(v_i)\tag{3.1}$$

for all $v_1\in H^1(\Omega_1)$ and $v_2\in H^1_E(\Omega_2)$. We now prove that $u_{i,\varepsilon}\to u_i$ as $\varepsilon\to 0$.

THEOREM 3.1 *The solutions of* (1.3) *and* (3.1) *satisfy*

$$\sum_{i=1}^{2}\|u_i-u_{i,\varepsilon}\|_{0,\Omega_i}\leqslant C\varepsilon\,\|u\|_{\tilde{H}^2(\Omega)}.\tag{3.2}$$

Proof. Observe that

$$\sum_{i=1}^{2}a_i(u_i,v_i)-\Big\langle\sigma_1\frac{\partial u_1}{\partial\nu},v_1\Big\rangle_\Gamma+\Big\langle\sigma_2\frac{\partial u_2}{\partial\nu},v_2\Big\rangle_\Gamma=\sum_{i=1}^{2}\ell_i(v_i)\tag{3.3}$$

for all $v_1 \in H^1(\Omega_1)$ and $v_2 \in H^1_E(\Omega_2)$. Taking $v_i = u_i - u_{i,\varepsilon}$ in (3.1) and (3.3), we find that, upon subtraction,

$$\sum_{i=1}^{2} a_i(u_i - u_{i,\varepsilon}, u_i - u_{i,\varepsilon}) + \varepsilon^{-1} \|(u_1 - u_{1,\varepsilon}) - (u_2 - u_{2,\varepsilon})\|^2_{0,\Gamma}$$

$$= \left\langle \sigma_1 \frac{\partial u_1}{\partial \nu}, (u_1 - u_{1,\varepsilon}) - (u_2 - u_{2,\varepsilon}) \right\rangle_\Gamma, \quad (3.4)$$

where we have used (1.1b). It follows from (3.4) and the trace inequality (1.6c) that

$$\|(u_1 - u_{1,\varepsilon}) - (u_2 - u_{2,\varepsilon})\|_{0,\Gamma} \leq C\varepsilon \|u\|_{\tilde{H}^2(\Omega)}. \quad (3.5)$$

Set

$$e^\varepsilon = e^\varepsilon_i = u_i - u_{i,\varepsilon} \quad \text{in } \Omega_i \quad (i = 1, 2). \quad (3.6)$$

We wish to bound

$$\|e^\varepsilon\|_{0,\Omega} = \sup_{\eta \in L^2(\Omega)} \frac{(e^\varepsilon, \eta)_\Omega}{\|\eta\|_{0,\Omega}}. \quad (3.7)$$

For each η in $L^2(\Omega)$, let z be the unique solution of (1.1) with f replaced by η. It follows from (R3) and elliptic regularity that

$$\|z\|_{\tilde{H}^2(\Omega)} \leq C \|\eta\|_{0,\Omega}. \quad (3.8)$$

Observe from (3.1) and (3.3) that, if $v_1 = v_2$ on Γ, then $\sum_{i=1}^2 a_i(e^\varepsilon_i, v_i) = 0$ and, in particular,

$$0 = \sum_{i=1}^{2} a_i(e^\varepsilon_i, z_i) = \sum_{i=1}^{2} (A_i z_i, e^\varepsilon_i)_{\Omega_i} + \left\langle \sigma_1 \frac{\partial z_1}{\partial \nu}, e^\varepsilon_1 \right\rangle_\Gamma - \left\langle \sigma_2 \frac{\partial z_2}{\partial \nu}, e^\varepsilon_2 \right\rangle_\Gamma$$

$$= (\eta, e^\varepsilon)_\Omega + \left\langle \sigma_1 \frac{\partial z_1}{\partial \nu}, e^\varepsilon_1 - e^\varepsilon_2 \right\rangle_\Gamma.$$

It follows from (1.6c) that

$$(\eta, e^\varepsilon) \leq C \|z\|_{\tilde{H}^2(\Omega)} \|(u_1 - u_{1,\varepsilon}) - (u_2 - u_{2,\varepsilon})\|_{0,\Gamma}$$

and so (3.2) is obtained by noting (3.8) and (3.5). □

Consider the finite-element approximation of (3.1). The domain Ω is approximated by a convex polyhedron $\Omega^h \subset \Omega$. Let $\mathcal{T}^h$ be a partitioning of Ω^h into disjoint open simplices τ, each of maximum diameter bounded above by h, so that $\bar{\Omega}^h = \bigcup_{\tau \in \mathcal{T}^h} \bar{\tau}$. This partitioning is independent of Γ. Each element has at most one face on $\partial\Omega^h$ and the vertices of this face lie on $\partial\Omega$. For $\partial\Omega$ sufficiently smooth it follows that $\operatorname{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$. We approximate Γ by choosing Γ^h to be the piecewise linear interpolation of Γ that is linear in each element $\tau \in \mathcal{B} \equiv \{\tau \in \mathcal{T}^h : \operatorname{meas}(\bar{\tau} \cap \Gamma) \neq 0 \text{ in } \mathbb{R}^{n-1}\}$, the interpolation points being where Γ crosses either the element sides ($n = 2$) or the element edges ($n = 3$). It follows that $\bar{\Omega}^h = \bar{\Omega}^h_1 \cup \bar{\Omega}^h_2$ where $\partial\Omega^h_1 = \Gamma^h$ and $\partial\Omega^h_2 = \Gamma^h \cup \partial\Omega^h$. Further, there exist local coordinate systems (X_τ, Y_τ) such that $X_\tau \in \Delta_\tau$, $Y_\tau \in \mathbb{R}$, Δ_τ is either an interval ($n = 2$) or a triangle ($n = 3$), and Γ^h and Γ are as described by $Y_\tau = \psi^h_\tau(X_\tau)$ and $Y_\tau = \psi_\tau(X_\tau)$. It follows, by the construction of Γ^h and the smoothness of ψ_τ, that

$$\|\psi_\tau - \psi_\tau^h\|_{0,\infty,\Delta_\tau} \leq Ch^2. \tag{3.9}$$

Defining

$$\mathcal{T}_i^h \equiv \{\tau \in \mathcal{T}^h : \tau \cap \Omega_i^h \neq \emptyset\}, \tag{3.10a}$$

we have that

$$\bar{\Omega}_i^h \subseteq \bar{D}_i^h = \bigcup_{\tau \in \mathcal{T}_i^h} \bar{\tau}. \tag{3.10b}$$

It is convenient to introduce smooth domains $\tilde{\Omega}_i$ such that

$$\Omega_i \subseteq \tilde{\Omega}_i \quad \text{and} \quad D_i^h \subseteq \tilde{\Omega}_i \quad \forall\, h < h_0.$$

Once again, the constants C are independent of h for the trace inequalities (1.6) with $G \equiv \Omega_i^h$.

Associated with each $\mathcal{T}_i^h$ is the space $S_i^h \subset C(\bar{D}_i^h)$ whose functions are linear on each element $\tau \in \mathcal{T}_i^h$. Denoting by $\Pi_i^h : C(\bar{D}_i^h) \to S_i^h$ the interpolation operator that interpolates values at the vertices of elements we have that, for $m = 0$ and $m = 1$, and for each $\tau \in \mathcal{T}_i^h$ and $p \in [2, \infty]$,

$$|w - \Pi_i^h w|_{m,p,\tau} \leq Ch^{2-m}\, |w|_{2,p,\tau} \quad \forall\, w \in W^{2,p}(\tau). \tag{3.11}$$

Our finite-element penalty method for approximating the solution of (1.1) is: find $u_{1,\varepsilon}^h \in S_1^h$ and $u_{2,\varepsilon}^h \in S_2^h(0) = \{\chi \in S_2^h : \chi = 0 \text{ on } \partial\Omega^h\}$ such that

$$\sum_{i=1}^{2} a_i^h(u_{i,\varepsilon}^h, \chi_i) + \frac{1}{\varepsilon}\langle u_{1,\varepsilon}^h - u_{2,\varepsilon}^h, \chi_1 - \chi_2\rangle_{\Gamma^h} = \sum_{i=1}^{2} \ell_i^h(\chi_i) \tag{3.12}$$

for all $\chi_1 \in S_1^h$ and $\chi_2 \in S_2^h(0)$. Here,

$$a_i^h(\bullet, \bullet) \equiv (\bar{\sigma}_i \nabla \bullet, \nabla \bullet)_i^h + (c\bullet, \bullet)_i^h, \qquad \ell_i^h(\bullet) \equiv (\bar{f}_i, \bullet)_i^h,$$

where $(\bullet, \bullet)_i^h$ approximates $(\bullet, \bullet)_{\Omega_i^h}$. In particular, we use a quadrature rule for linear finite elements on unfitted meshes, which is defined in Barrett & Elliott (1985). This rule only uses evaluations of the data $\{\sigma_i, f_i, c\}$ on $\tilde{\Omega}_i$, and, in the analysis that follows, it is convenient to take $\bar{f}_i \equiv \tilde{A}_i \tilde{u}_i$ as in (2.3) and (2.7). Further, for all $\chi \in S_i^h$, we have

$$|\tilde{a}_i^h(w^h, \chi) - a_i^h(w^h, \chi)|$$
$$\leq Ch(h\, \|w\|_{2,\Omega_i^h} + \|w - w^h\|_{1,\Omega_i^h})\, \|\chi\|_{1,\Omega_i^h} \quad \forall\, w^h \in S_i^h \quad \forall\, w \in H^2(\tilde{\Omega}_i), \tag{3.13a}$$

$$a_i^h(\chi, \chi) \geq C\, |\chi|_{1,\Omega_i^h}^2, \qquad |\ell_i^h(\chi) - \tilde{\ell}_i^h(\chi)| \leq Ch^2\, \|\bar{f}_i\|_{2,\Omega_i^h}\, \|\chi\|_{1,\Omega_i^h}. \tag{3.13b, c}$$

Lemma 3.1 *For all $w_1 \in H^1(\Omega_1^h)$ and $w_2 \in H_E^1(\Omega_2^h) \equiv \{w \in H^1(\Omega_2^h) : w = 0$ on $\partial\Omega^h\}$ we have the generalized Friedrichs' inequality*:

$$\sum_{i=1}^{2} \|w_i\|_{0,\Omega_i^h}^2 \leq C\Big(\sum_{i=1}^{2} |w_i|_{1,\Omega_i^h}^2 + \|w_1 - w_2\|_{0,\Gamma^h}^2\Big), \tag{3.14}$$

where C is independent of w_i and h.

Proof. From the proof of Friedrichs' inequality given in Rektorys (1977: Chs 18 & 30), we have that there exists a constant C, independent of w_1 and h, such that

$$\|w_1\|^2_{0,\Omega_1^h} \leq C(|w_1|^2_{1,\Omega_1^h} + \|w_1\|^2_{0,\Gamma^*}) \quad \forall\, w_1 \in \mathrm{H}^1(\Omega_1^h).$$

From the trace theorem (1.6b), it follows that

$$\|w_1\|_{0,\Gamma^*} \leq \|w_1 - w_2\|_{0,\Gamma^*} + \|w_2\|_{0,\Gamma^*} \leq \|w_1 - w_2\|_{0,\Gamma^*} + C\,\|w_2\|_{1,\Omega_2^h}.$$

Therefore, the result (3.14) follows if

$$\|w_2\|_{0,\Omega_2^h} \leq C\,|w_2|_{1,\Omega_2^h} \quad \forall\, w_2 \in \mathrm{H}^1_{\mathrm{E}}(\Omega_2^h),$$

with C independent of h and w_2. This result follows by introducing $\hat{\Omega}_2^h = \Omega\backslash\bar{\Omega}_1^h$, and by extending w_2 from Ω_2^h to $\hat{\Omega}_2^h$ by setting w_2 to be zero on $\hat{\Omega}_2^h\backslash\Omega_2^h$. Then, from Rektorys (1977), it follows that

$$\|w_2\|_{0,\hat{\Omega}_2^h} \leq C\,|w_2|_{1,\hat{\Omega}_2^h},$$

where C depends only on $\partial\Omega$ and the area of $\hat{\Omega}_2^h$, and is hence independent of h. □

LEMMA 3.2 *The following error bound holds*:

$$\varepsilon^{-1}\,\|(\bar{u}_1 - u^h_{1,\varepsilon}) - (\bar{u}_2 - u^h_{2,\varepsilon})\|^2_{0,\Gamma^*} + \sum_{i=1}^{2} \|\bar{u}_i - u^h_{i,\varepsilon}\|^2_{1,\Omega_i^h} \leq C(h^2 + \varepsilon^{-1}h^4 + \varepsilon)\,\|u\|^2_{\tilde{H}^2(\Omega)}, \quad (3.15)$$

where $\bar{u}_i = E_i u_i$.

Proof. Clearly the inequality

$$\begin{aligned}\tfrac{1}{2}\varepsilon^{-1}\,\|(\bar{u}_1 - u^h_{1,\varepsilon}) - (\bar{u}_2 - u^h_{2,\varepsilon})\|^2_{0,\Gamma^*} + \tfrac{1}{2}\sum_{i=1}^{2} \|\bar{u}_i - u^h_{i,\varepsilon}\|^2_{1,\Omega_i^h} \\ \leq \varepsilon^{-1}\,\|(\bar{u}_1 - \Pi_1^h\bar{u}_1) - (\bar{u}_2 - \Pi_2^h\bar{u}_2)\|^2_{0,\Gamma^*} + \sum_{i=1}^{2} \|\bar{u}_i - \Pi_i^h\bar{u}_i\|^2_{1,\Omega_i^h} \\ + \varepsilon^{-1}\,\|\chi_1 - \chi_2\|^2_{0,\Gamma^*} + \sum_{i=1}^{2} \|\chi_i\|^2_{1,\Omega_i^h} \quad (3.16)\end{aligned}$$

holds, where $\chi_i = u^h_{i,\varepsilon} - \Pi_i^h\bar{u}_i$. Since, for all $w_i \in \mathrm{H}^1(\Omega_i^h)$, we have

$$|w_i|^2_{1,\Omega_i^h} \leq C\bar{a}_i^h(w_i, w_i) \quad (3.17)$$

and the Friedrichs' inequality (3.14), it follows that

$$\varepsilon^{-1}\,\|\chi_1 - \chi_2\|^2_{0,\Gamma^*} + \sum_{i=1}^{2} \|\chi_i\|^2_{l,\Omega_i^h} \leq C\Big(\varepsilon^{-1}\,\|\chi_1 - \chi_2\|^2_{0,\Gamma^*} + \sum_{i=1}^{2} \bar{a}_i^h(\chi_i, \chi_i)\Big). \quad (3.18)$$

We note that

$$\|\bar{u}_i - \Pi_i^h \bar{u}_i\|_{0,\Gamma^h} \leq C \|\bar{u}_i - \Pi_i^h \bar{u}_i\|_{0,\infty,\Gamma^h} \leq C \|\bar{u}_i - \Pi_i^h \bar{u}_i\|_{0,\infty,\Omega_i^h}$$
$$\leq Ch^2 |\bar{u}_i|_{2,\infty,D_i^h} \leq Ch^2 \|\bar{u}_i\|_{4,D_i^h} \leq Ch^2 \|u_i\|_{4,\Omega_i}, \quad (3.19)$$

where we have used (3.11), (3.10b), and (2.6). Observe that, using (3.12),

$$\varepsilon^{-1} \|\chi_1 - \chi_2\|_{0,\Gamma^h}^2 + \sum_{i=1}^{2} \bar{a}_i^h(\chi_i, \chi_i) = \varepsilon^{-1} \langle \Pi_2^h \bar{u}_2 - \Pi_1^h \bar{u}_1, \chi_1 - \chi_2 \rangle_{\Gamma^h}$$
$$+ \sum_{i=1}^{2} \bar{a}_i^h(\bar{u}_i - \Pi_i^h \bar{u}_i, \chi_i) + \sum_{i=1}^{2} [\bar{\ell}_i^h(\chi_i) - \bar{a}_i^h(\bar{u}_i, \chi_i)]$$
$$+ \sum_{i=1}^{2} [\bar{a}_i^h(\Pi_i^h \bar{u}_i, \chi_i) - a_i^h(\Pi_i^h \bar{u}_i, \chi_i)] + \sum_{i=1}^{2} [\ell_i^h(\chi_i) - \bar{\ell}_i^h(\chi_i)]. \quad (3.20)$$

We now proceed to bound the right-hand side of (3.20).

First note that

$$\|\Pi_1^h \bar{u}_1 - \Pi_2^h \bar{u}_2\|_{0,\Gamma^h} \leq \|\bar{u}_1 - \bar{u}_2\|_{0,\Gamma^h} + \sum_{i=1}^{2} \|\bar{u}_i - \Pi_i^h \bar{u}_i\|_{0,\Gamma^h},$$

and using (3.18), the construction of Γ^h, (3.9), and (2.13) we obtain

$$\|\Pi_1^h \bar{u}_1 - \Pi_2^h \bar{u}_2\|_{0,\Gamma^h} \leq Ch^2 \|u\|_{\tilde{H}^4(\Omega)}. \quad (3.21)$$

Second, note that, using the definition of $\bar{f}_i$, (2.7), and (2.4),

$$\left| \sum_{i=1}^{2} [\bar{\ell}_i^h(\chi_i) - \bar{a}_i^h(\bar{u}_i, \chi_i)] \right| = \left| \sum_{i=1}^{2} \left\langle \bar{\sigma}_i \frac{\partial \bar{u}_i}{\partial v_i^h}, \chi_i \right\rangle_{\partial\Omega_i^h} \right|$$
$$\leq \left| \left\langle \bar{\sigma}_1 \frac{\partial \bar{u}_1}{\partial v^h} - \bar{\sigma}_2 \frac{\partial \bar{u}_2}{\partial v^h}, \chi_1 \right\rangle_{\Gamma^h} \right| + \left| \left\langle \bar{\sigma}_2 \frac{\partial \bar{u}_2}{\partial v^h}, \chi_1 - \chi_2 \right\rangle_{\Gamma^h} \right|$$
$$\leq C \left\| \bar{\sigma}_1 \frac{\partial \bar{u}_1}{\partial v^h} - \bar{\sigma}_2 \frac{\partial \bar{u}_2}{\partial v^h} \right\|_{0,\Gamma^h} \|\chi_1\|_{0,\Gamma^h} + \varepsilon^{\frac{1}{2}} \left\| \bar{\sigma}_2 \frac{\partial \bar{u}_2}{\partial v^h} \right\|_{0,\Gamma^h} \cdot \varepsilon^{-\frac{1}{2}} \|\chi_1 - \chi_2\|_{0,\Gamma^h}.$$

It follows from the trace inequalities (1.6) and (2.9) that

$$\left| \sum_{i=1}^{2} [\bar{\ell}_i^h(\chi_i) - \bar{a}_i^h(\bar{u}_i, \chi_i)] \right|$$
$$\leq Ch \|u\|_{\tilde{H}^4(\Omega)} \|\chi_1\|_{1,\Omega_1^h} + C\varepsilon^{\frac{1}{2}} \|u\|_{\tilde{H}^2(\Omega)} \cdot \varepsilon^{-\frac{1}{2}} \|\chi_1 - \chi_2\|_{0,\Gamma^h}. \quad (3.22)$$

Finally, note that the numerical integration error bounds (3.13a, c), together with the interpolation error bound and (2.7b), imply that

$$\sum_{i=1}^{2} [|\bar{a}_i^h(\Pi_i^h \bar{u}_i, \chi_i) - a_i^h(\Pi_i^h \bar{u}_i, \chi_i)| + |\ell_i^h(\chi_i) - \bar{\ell}_i^h(\chi_i)|]$$
$$\leq Ch^2 \|u\|_{\tilde{H}^4(\Omega)} \sum_{i=1}^{2} \|\chi_i\|_{1,\Omega_i^h}. \quad (3.23)$$

From (3.13b), (3.14), (3.18), (3.20), (3.21), (3.22), and (3.23), it follows that

$$\varepsilon^{-1} \|\chi_1 - \chi_2\|_{0,\Gamma^h}^2 + \sum_{i=1}^{2} \|\chi_i\|_{1,\Omega_i^h}^2 \le C(h^2 + \varepsilon^{-1}h^4 + \varepsilon) \|u\|_{\tilde{H}^4(\Omega)}^2. \tag{3.24}$$

The error bound (3.15) is now a consequence of (3.16), (3.11), (3.19), and (3.24). □

We see from (3.15) that the optimal rate of convergence in the H[1] norm is achieved for $\varepsilon = O(h^2)$.

THEOREM 3.2 *Let the assumptions* (R1), (R2), *and* (R3) *hold, and let* $\varepsilon = O(h^2)$. *Then the solutions* u_i *and* $u_{i,\varepsilon}^h$ *of* (1.1) *and* (3.12) *satisfy*

$$\|(\tilde{u}_1 - \tilde{u}_2) - (u_{1,\varepsilon}^h - u_{2,\varepsilon}^h)\|_{0,\Gamma^h} \le Ch^2 \|u\|_{\tilde{H}^4(\Omega)}, \tag{3.25a}$$

$$\sum_{i=1}^{2} \|\tilde{u}_i - u_{i,\varepsilon}^h\|_{1,\Omega_i^h} \le Ch \|u\|_{\tilde{H}^4(\Omega)}, \tag{3.25b}$$

$$\sum_{i=1}^{2} \|\tilde{u}_i - u_{i,\varepsilon}^h\|_{0,\Omega_i^h} \le Ch^{\frac{3}{2}} \|u\|_{\tilde{H}^4(\Omega)}, \tag{3.25c}$$

$$\sum_{i=1}^{2} \|\tilde{u}_i - u_{i,\varepsilon}^h\|_{-2,\Omega_i^h} \le Ch^2 \|u\|_{\tilde{H}^4(\Omega)}, \tag{3.25d}$$

where $\tilde{u}_i \equiv E_i u_i$.

Proof. The results (3.25a) and (3.25b) are a direct consequence of (3.15). To prove (3.25c, d), we introduce the standard auxiliary problem.

Set

$$e = \begin{cases} \tilde{u}_i - u_{i,\varepsilon}^h & \text{on } \Omega_i^h \quad (i = 1, 2), \\ 0 & \text{on } \Omega \backslash \Omega^h, \end{cases} \tag{3.26a}$$

and note that, for $r \ge 0$, we have

$$\|e\|_{-r,\Omega} = \sup_{\eta \in H^r(\Omega)} \frac{(e, \eta)_\Omega}{\|\eta\|_{r,\Omega}}. \tag{3.26b}$$

Let z be the solution to (1.1) with f replaced by η. It follows from (R3) and elliptic regularity, since $\partial\Omega$ is smooth, that

$$\|z\|_{\tilde{H}^{r+2}(\Omega)} \le C \|\eta\|_{r,\Omega}. \tag{3.27}$$

From the definition of z and using (2.4) we obtain, with $\tilde{z}_i = E_i z_i$, that

$$\begin{aligned}(e, \eta)_\Omega &= \sum_{i=1}^{2} (e, \bar{A}_i \tilde{z}_i)_{\Omega_i^h} + \sum_{i=1}^{2} (e, \eta - \bar{A}_i \tilde{z}_i)_{\Omega_i^h \cap \Omega_i} \\ &= \sum_{i=1}^{2} \bar{a}_i^h(e, \tilde{z}_i) + \left\langle \bar{\sigma}_2 \frac{\partial \tilde{z}_2}{\partial \nu^h}, \tilde{u}_2 - u_{2,\varepsilon}^h \right\rangle_{\Gamma^h} - \left\langle \bar{\sigma}_1 \frac{\partial \tilde{z}_1}{\partial \nu^h}, \tilde{u}_1 - u_{1,\varepsilon}^h \right\rangle_{\Gamma^h} \\ &\quad - \left\langle \bar{\sigma}_2 \frac{\partial \tilde{z}_2}{\partial \nu^h}, \tilde{u}_2 - u_{2,\varepsilon}^h \right\rangle_{\partial\Omega^h} + \sum_{i=1}^{2} (e, \eta - \bar{A}_i \tilde{z}_i)_{\Omega_i^h \cap \Omega_i}.\end{aligned}$$

Rewriting this, using (3.12):

$$(e, \eta)_\Omega = \sum_{i=1}^{2} \tilde{a}_i^h(e, \tilde{z}_i - \Pi_i^h \tilde{z}_i) + \sum_{i=1}^{2} [\tilde{a}_i^h(\tilde{u}_i, \Pi_i^h \tilde{z}_i) - \tilde{\ell}_i^h(\Pi_i^h \tilde{z}_i)]$$
$$+ \varepsilon^{-1}\langle u_{1,\varepsilon}^h - u_{2,\varepsilon}^h, \Pi_1^h \tilde{z}_1 - \Pi_2^h \tilde{z}_2\rangle_{\Gamma^h} - \left\langle \tilde{\sigma}_2 \frac{\partial \tilde{z}_2}{\partial \nu^h}, \tilde{u}_2 - u_{2,\varepsilon}^h \right\rangle_{\partial\Omega^h}$$
$$+ \sum_{i=1}^{2} (e, \eta - \tilde{A}_i \tilde{z}_i)_{\Omega_i^h \setminus \Omega_i} + \sum_{i=1}^{2} [a_i^h(u_{i,\varepsilon}^h, \Pi_i^h \tilde{z}_i) - \tilde{a}_i^h(u_{i,\varepsilon}^h, \Pi_i^h \tilde{z}_i)]$$
$$+ \sum_{i=1}^{2} [\tilde{\ell}_i^h(\Pi_i^h \tilde{z}_i) - \ell_i^h(\Pi_i^h \tilde{z}_i)] + \left\langle \tilde{\sigma}_2 \frac{\partial \tilde{z}_2}{\partial \nu^h} - \tilde{\sigma}_1 \frac{\partial \tilde{z}_1}{\partial \nu^h}, \tilde{u}_1 - u_{1,\varepsilon}^h \right\rangle_{\Gamma^h}$$
$$+ \left\langle \tilde{\sigma}_2 \frac{\partial \tilde{z}_2}{\partial \nu^h}, \tilde{u}_2 - u_{2,\varepsilon}^h - \tilde{u}_1 + u_{1,\varepsilon}^h \right\rangle_{\Gamma^h} \equiv \sum_{j=1}^{9} E_j. \quad (3.28)$$

We proceed to estimate the E_j in turn. It is an immediate consequence of (3.25b) and (3.11) that

$$|E_1| \leq Ch^2 \|u\|_{\tilde{H}^4(\Omega)} \|z\|_{\tilde{H}^2(\Omega)}. \quad (3.29)$$

The term E_2 arises in the analysis of L^2 error in Section 2 as I_2 of (2.17), and we can bound it in a similar way. We obtain

$$|E_2| \leq \sum_{i=1}^{2} |\tilde{a}_i^h(\tilde{u}_i, \Pi_i^h \tilde{z}_i - \tilde{z}_i) - \tilde{\ell}_i^h(\Pi_i^h \tilde{z}_i - \tilde{z}_i)|$$
$$+ \sum_{i=1}^{2} |\tilde{a}_i^h(\tilde{u}_i, \tilde{z}_i) - a_i(u_i, z_i)| + \sum_{i=1}^{2} |\ell_i(z_i) - \tilde{\ell}_i^h(\tilde{z}_i)|. \quad (3.30)$$

The second and third terms can be bounded by $Ch^k \|z\|_{\tilde{H}^2(\Omega)} \|u\|_{\tilde{H}^4(\Omega)}$ using the same analysis as that for I_2. The first term is bounded by

$$\left|\int_{\partial\Omega^h} \tilde{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h} (\tilde{z}_2 - \Pi_2^h \tilde{z}_2)\, ds^h\right| + \left|\int_{\Gamma^h} \left(\tilde{\sigma}_2 \frac{\partial \tilde{u}_1}{\partial \nu^h} - \tilde{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h}\right)(\tilde{z}_1 - \Pi_1^h \tilde{z}_1)\, ds^h\right|$$
$$+ \left|\int_{\Gamma^h} \tilde{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h} (\tilde{z}_1 - \Pi_1^h \tilde{z}_1)\, ds^h\right| + \left|\int_{\Gamma^h} \tilde{\sigma}_2 \frac{\partial \tilde{u}_2}{\partial \nu^h} (\tilde{z}_2 - \Pi_2^h \tilde{z}_2)\, ds^h\right|. \quad (3.31)$$

The trace inequality (1.6d) with $\delta = h$ and (3.11) imply that

$$\|\tilde{z}_i - \Pi_i^h \tilde{z}_i\|_{0,\partial\Omega_i^h} \leq Ch^{\frac{3}{2}} \|\tilde{z}_i\|_{2,D_i^h} \leq Ch^{\frac{3}{2}} \|\tilde{z}_i\|_{2,\Omega_i}; \quad (3.32a)$$

whereas the analysis of (3.19) implies that

$$\|\tilde{z}_i - \Pi_i^h \tilde{z}_i\|_{0,\partial\Omega_i^h} \leq Ch^2 \|\tilde{z}_i\|_{4,\Omega_i}. \quad (3.32b)$$

Noting (1.6c) and (2.9b), we obtain, through combining (3.30), (3.31), and (3.32),

$$|E_2| \leq \begin{cases} Ch^{\frac{3}{2}} \|u\|_{\tilde{H}^4(\Omega)} \|z\|_{\tilde{H}^2(\Omega)}, & (3.33a) \\ Ch^2 \|u\|_{\tilde{H}^4(\Omega)} \|z\|_{\tilde{H}^4(\Omega)}. & (3.33b) \end{cases}$$

Turning to E_3, we have that

$$|E_3| \leq \varepsilon^{-1}[\|(\tilde{u}_1 - u^h_{1,\varepsilon}) - (\tilde{u}_2 - u^h_{2,\varepsilon})\|_{0,\Gamma^*} + \|\tilde{u}_1 - \tilde{u}_2\|_{0,\Gamma^*}] \times \left(\|\tilde{z}_1 - \tilde{z}_2\|_{0,\Gamma^*} + \sum_{i=1}^{2} \|\tilde{z}_i - \Pi^h_i \tilde{z}_i\|_{0,\Gamma^*}\right).$$

Therefore (2.13), (3.25a), and (3.33) yield

$$|E_3| \leq \begin{cases} Ch^{\frac{3}{2}} \|u\|_{\tilde{H}^4(\Omega)} \|z\|_{\tilde{H}^2(\Omega)}, & (3.34a) \\ Ch^2 \|u\|_{\tilde{H}^4(\Omega)} \|z\|_{\tilde{H}^4(\Omega)}. & (3.34b) \end{cases}$$

Since $u^h_{2,\varepsilon} = 0$ on $\partial\Omega^h$ and $\operatorname{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$, the bounds (2.13) and (1.6c) imply that

$$|E_4| \leq Ch^2 \|u\|_{\tilde{H}^4(\Omega)} \|z\|_{\tilde{H}^2(\Omega)}. \tag{3.35}$$

The term E_5 is bounded as I_5 in (2.17). The terms E_6 and E_7 are due to numerical integration, and are bounded using (3.13a, c), We bound E_8 by using (2.9a), (1.6b), and (3.25b); and E_9 is bounded using (1.6c) and (3.25a).

Therefore, we obtain

$$(e, \eta)_{\Omega^h} \leq \begin{cases} Ch^{\frac{3}{2}} \|u\|_{\tilde{H}^4(\Omega)} \|z\|_{\tilde{H}^2(\Omega)} \quad (r = 0), & (3.36a) \\ Ch^2 \|u\|_{\tilde{H}^4(\Omega)} \|z\|_{\tilde{H}^4(\Omega)} \quad (r = 2). & (3.36b) \end{cases}$$

The results (3.25c) and (3.25d) are an immediate consequence of (3.26), (3.27), and (3.36). □

Although we have only been able to prove that the approximation (3.12) with $\varepsilon = h^2$ converges at the rate $O(h^{\frac{3}{2}})$ globally in L^2, we can show that it converges at the optimal rate in L^2 over any interior domain Ω^*_i satisfying $\Omega^*_i \Subset \Omega^{**}_i \Subset \Omega^h_i$, for some domain Ω^{**}_i, using the techniques of Nitsche & Schatz (1974) for obtaining interior error estimates.

THEOREM 3.3 *Let the assumptions* (R1), (R2), *and* (R3) *hold, and let* $\varepsilon = O(h^2)$. *Then, for h sufficiently small, the solutions* u_i *and* $u^h_{i,\varepsilon}$ *of* (1.1) *and* (3.12) *satisfy*:

$$\|u_i - u^h_{i,\varepsilon}\|_{0,\Omega^*_i} \leq Ch^2 \|u\|_{\tilde{H}^4(\Omega)} \quad (i = 1, 2), \tag{3.37}$$

where $\Omega^*_i \Subset \Omega^{**}_i \Subset \Omega^h_i$ *and* $\Omega^*_i \subset \Omega_i$.

Proof. The proof is completely analogous to the proofs of Lemma 3.5 and Theorem 3.3 in Barrett & Elliott (1986). □

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