Existence Results for Diffusive Surface Motion Laws

Charles M. Elliott and Harald Garcke

Abstract. Three geometric interface laws for the evolution of curves are considered. They include the motion by surface diffusion and the conserved mean curvature flow. All these laws decrease length and preserve the area of the region enclosed by the curve. We present local existence results and show that a global solution exists if the initial curve is close to a circle. Furthermore it is shown that a global solution converges to a circle.

1 Introduction

In this paper we study geometric evolution laws which describe the motion of interfaces. We assume that the interface at time t is given by a hypersurface Γ_t which is the boundary of a region. All evolution laws we will discuss have the common property that they preserve the volume of the region enclosed by Γ_t and decrease the perimeter. Two laws with this property are the motion by surface diffusion

$$V = -D\Delta_S \kappa \tag{1.1}$$

and the conserved mean curvature flow

$$V = M\left(\kappa - \kappa_{\alpha v}\right) . \tag{1.2}$$

Here V is the normal velocity of the evolving surface, Δ_S is the surface Laplacian, κ is the mean curvature, $\kappa_{\alpha v}$ is the average mean curvature on Γ_t, D is a diffusion coefficient and M is a mobility coefficient. The motion by surface diffusion was first derived by Mullins [26] to describe surface dynamics for phase interfaces when the evolution is purely governed by mass diffusion in the interface. A further discussion and a derivation of motion by surface diffusion in the spirit of Gurtin [21] is given by Davi and Gurtin [13]. The conserved mean curvature flow is a modification of the mean curvature equation $V = \kappa$ which guarantees that the volume of the region enclosed by Γ_t is preserved. Gage [17] showed that convex curves in the plane which evolve according to 1.2 remain convex and converge to a circle as time tends to infinity and Huisken [23] generalized this result to higher dimensions.

Recently Cahn and Taylor [7, 8] showed that the motion by surface diffusion and the conserved mean curvature flow are formally linked by the intermediate laws

$$\begin{aligned} J &= -D\nabla_S w\,, \\ V &= -div_S\, J\,, \\ V &= M(\kappa + w) \end{aligned} \tag{1.3}$$

Here J is the mass flux, w is a potential, ∇_S is the surface gradient and div_S is the surface divergence. These three laws are equivalent to the motion

$$V = \Delta_S \left(\frac{1}{M}\Delta_S - \frac{1}{D}\right)^{-1} \kappa\,. \tag{1.4}$$

One expects for $M \longrightarrow \infty$ that the solutions of the intermediate motions converge to a solution of motion by surface diffusion (1.1), whereas the limit $D \longrightarrow \infty$ should be the motion by the difference of mean curvature and average mean curvature (1.2).

So far we discussed sharp interface models, i.e. two phases are separated by a hypersurface. Another possibility of modeling phase transition phenomena are phase field models which are based on a Ginzburg–Landau functional. In phase field models a continuous real valued order parameter u is introduced, which is assumed to attain prescribed values in the pure phases. These values are the minima of the Ginzburg–Landau functional

$$\mathcal{W}(u) = \int_\Omega |\nabla u|^2 + W(u)\, dx \tag{1.5}$$

where W is usually a smooth double well potential, e.g. $W(u) = (1-u^2)^2$ and Ω is a bounded domain in $\mathbb{R}^n$. In these models the regions where u is approximately ± 1 correspond to the two phases and the region in between these sets is the interfacial region. For a further discussion on phase field models see Langer [25] and Caginalp [5].

All the above sharp interface models are related to phase field models. Well known is the relation between the nonlocal Allen–Cahn equation

$$\varepsilon u_t = \varepsilon \Delta u - \frac{1}{\varepsilon}\left(W'(u) - \int_\Omega W'(u) / \int_\Omega 1\right) \tag{1.6}$$

and the motion by the difference of mean curvature and average mean curvature. Rubinstein and Sternberg [28] used formal asymptotics to show that in the limit $\varepsilon \searrow 0$ solutions of (1.6) converge to solutions of the conserved mean curvature flow (1.2). For a rigorous result in radial symmetry see the work of Bronsard and Stoth [4].

Not as much is known about the relation between the flows (1.1) and (1.4) to phase field models. Cahn, Elliott and Novick–Cohen [6] used formal asymptotics to show that solutions of the Cahn–Hilliard equation with a concentration dependent mobility

$$\begin{aligned} u_t &= \nabla \cdot (B(u)\nabla w)\,, \\ w &= -\varepsilon^2 \Delta u + W'(u)\,. \end{aligned} \tag{1.7}$$

converge to solutions of motion by surface diffusion. Their result is for a mobility $B(u) = max(1-u^2,0)$, the scaling $\tau = \varepsilon^2 t$, a potential

$$W(u) = \frac{\theta}{2}\left((1+u)\ln(1+u) + (1-u)\ln(1-u)\right) + (1-u^2)$$

and in the limit $\theta \searrow 0$ (the so called deep quench limit).

This result is in contrast to the case in which B is a positive constant. Formal asymptotic results by Pego [27] suggest that the solutions of the Cahn–Hilliard equation with constant mobility converge to solutions of the Mullins–Sekerka problem, i.e. the chemical potential w fulfills in the limit

$$\begin{aligned} \Delta w &= 0 && \text{for } x \in \Omega \setminus \Gamma_t\,, \\ w &= \kappa && \text{for } x \in \Gamma_t\,, \\ V &= [\mathbf{n}\cdot\nabla w]_-^+ && \text{for } x \in \Gamma_t \\ \text{and} \quad \nabla w \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \end{aligned}$$

where Γ_t is the interface and $[\,.\,]_-^+$ denotes the jump across Γ_t. Recently Alikakos, Bates and Chen [1] gave a rigorous proof of Pego's result under the assumption that a smooth solution of the Mullins–Sekerka problem exists. Both the Mullins–Sekerka flow and the motion by surface diffusion are volume preserving and perimeter decreasing, but a main difference is that the Mullins–Sekerka flow is nonlocal, i.e. the velocity in each point on Γ_t depends on data away from this point.

A crucial assumption in the formal asymptotics of Cahn, Elliott and Novick–Cohen [6] is the fact that the mobility is zero in the pure phase (i.e. $B(u) = 0$ if $|u| = 1$). Therefore diffusion is only allowed in the interfacial region, which guarantees that in the limit $\varepsilon \searrow 0$ the diffusion is restricted along the sharp interface. But diffusion along a sharp interface results in motion by surface diffusion (see [26]). If the mobility is allowed to be zero the Cahn–Hilliard equation becomes a fourth order degenerate parabolic equation. We refer to [14] for an existence result in this case.

We expect that the order parameter analogue of the intermediate laws is a viscous Cahn–Hilliard type equation of the form

$$\begin{aligned} u_t &= -\nabla\cdot\left(B(u)\nabla w\right)\,, \\ \varepsilon u_t &= M\left(\varepsilon\Delta u - \tfrac{1}{\varepsilon}W'(u) - w\right)\,. \end{aligned}$$

As before the mobility should depend on u and be zero if $|u| = 1$. It seems as if the results of Cahn, Elliott and Novick–Cohen can be generalized to this case. We point out that so far there are no rigorous results concerning the convergence of the (viscous) Cahn–Hilliard equation to the motion by surface diffusion (to the intermediate motion (1.4) respectively).

The motion by mean curvature

$$V = \kappa$$

is the asymptotic limit of the Allen–Cahn equation

$$\varepsilon u_t = \varepsilon\Delta u - \frac{1}{\varepsilon}W'(u)$$

where W is a double well potential or has a double obstacle form. The motion by mean curvature and its relation to the Allen–Cahn equation has been intensively studied (see

[2, 3, 9, 12, 15, 16, 18, 20, 22] and the references therein). In the case of motion of curves it is known that a initial curve will become asymptotically circular and shrinks to a point in finite time. In higher dimensions other singularities are possible and in this case classical methods break down. For example if one starts with a dumbbell in $\mathbb{R}^3$ a so called pinch–off can occur. The motion by mean curvature was studied with several methods (direct mapping, viscosity solutions, varifolds) and the maximum principle was a basic ingredient in crucial arguments. In the case of the velocity laws (1.1)–(1.4) no maximum principle is valid. This is the reason why many of the methods used for studying the mean curvature flow cannot be used for the motions (1.1)–(1.4).

In this paper we study the sharp interface models in the two dimensional case, i.e. the evolution of curves in the plane. The velocity laws (1.1)–(1.4) become

$$\begin{aligned} V &= -D\partial_{ss}\kappa \, , \\ V &= M(\kappa - \kappa_{av}) \, , \\ \text{and} \qquad V &= \partial_{ss}\left(\tfrac{1}{M}\partial_{ss} - \tfrac{1}{D}Id\right)^{-1}\kappa \, . \end{aligned}$$

where ∂_s is the derivative with respect to the arc–length. We will prove a local existence theorem for these geometric motions. Furthermore we show global existence if the initial curve is close to a circle. In the case that a global solution exists, we also prove that the evolving curve converges to a circle. These results show that circles are asymptotically stable. We refer in this context to the work of Coleman, Falk and Moakher [11] who used formal perturbation analysis and numerical simulations to study the stability of cylinders. Their results strongly indicate that cylinders are not asymptotically stable.

This paper is organized as follows. In section 2 we introduce a regularized version of (1.2) and (1.4) in order to get approximate solutions for these motions. We show local existence for this problem. The same proof also gives local existence of solutions for the motion by minus the second derivative of the curvature. In section 3 we derive energy identities for the regularized problem. These energy identities enable us to pass to the limit in the regularized problem and we obtain the existence of a local solution to the original problems (see section 4). Parabolic regularity theory is used in section 5 to show that solutions become instantaneously smooth. In section 6 we prove global existence for the motions (1.1)–(1.4) if the initial curve is close to a circle. Under the assumption that a global solution exists we show that the evolving curve converges to a circle as time tends to infinity.

We will give the proof for the intermediate motions (1.4) in detail and because the results for the motions (1.1) and (1.2) are proved with similar methods, we just state the differences.

2 Local existence for the regularized problem

We rewrite the evolution problem for the intermediate motion as follows

$$V = Dw_{ss} \, , \quad Dw_{ss} = M(\kappa + w) \quad \text{in } \Gamma, \tag{2.1}$$

$$\Gamma \cap \{t = 0\} = \Gamma_0$$

where Γ_t is the curve at time t, Γ_0 is a given simple connected closed curve and $\Gamma := \cup_{t\geq 0}\Gamma_t \times \{t\}$.

There are different ways to parametrise evolving curves (see for example Gurtin [21]). In this section we shall express an evolving curve Γ as a graph over a fixed reference curve $\mathcal{M}^0$. But later it will be also convenient to use a parametrisation with the property that the tangential component of the velocity vector vanishes (see section 3).

Before we formulate a regularized version of the intermediate motion let us introduce some notations which we need to parametrise an evolving curve as a graph over a given reference curve. We choose a simple connected curve $\mathcal{M}^0 \in C^4$ close to Γ_0. Let

$$X^0 : I \longrightarrow \mathcal{M}^0, \quad \eta \longmapsto X^0(\eta) \quad (I \subset \mathbb{R} \text{ a compact interval})$$

be an arc–length parametrisation of $\mathcal{M}^0$. Then we obtain

$$\boldsymbol{\tau}^0(\eta) = X^0_\eta(\eta), \quad \boldsymbol{\tau}^0_\eta = \kappa^0(\eta)\mathbf{n}^0(\eta), \quad \mathbf{n}^0_\eta(\eta) = -\kappa^0(\eta)\boldsymbol{\tau}^0(\eta) \tag{2.2}$$

where $\mathbf{n}^0$ is the unit normal such that $(\boldsymbol{\tau}^0, \mathbf{n}^0)$ is positively orientated and κ^0 is the curvature of $\mathcal{M}^0$ with the sign convention for the curvature is chosen such that the Frenet formulas (2.2) hold. We define

$$\mathcal{M}^0_{\delta_0} := \{x \in \mathbb{R}^2 \;\big|\; dist(x, \mathcal{M}^0) < \delta_0\}\,.$$

Then the mapping

$$Y : I \times (-\delta_0, \delta_0) \longrightarrow \mathcal{M}^0_{\delta_0}, \quad Y(\eta, h) := X^0(\eta) + h\mathbf{n}^0(\eta)\ ,$$

is a diffeomorphism if δ_0 is small enough (see [19]). We assume for further use that $\delta_0 \|\kappa_0\|_\infty < \frac{1}{2}$. Every function $d : I \to (-\delta_0, \delta_0)$ defines a simple connected curve X as a graph over $\mathcal{M}^0$ in the following way

$$X(\eta) = X^0(\eta) + d(\eta)\mathbf{n}^0(\eta)\,.$$

For such a curve we obtain

$$X_\eta = (1 - d\kappa^0)\boldsymbol{\tau}^0 + d_\eta \mathbf{n}^0\,.$$

The tangent $\boldsymbol{\tau}$ and the outward normal $\mathbf{n}$ are

$$\begin{aligned} \boldsymbol{\tau} &= \tfrac{1}{J}\left[(1 - d\kappa^0)\boldsymbol{\tau}^0 + d_\eta \mathbf{n}^0\right]\,, \\ \mathbf{n} &= \tfrac{1}{J}\left[-d_\eta \boldsymbol{\tau}^0 + (1 - d\kappa^0)\mathbf{n}^0\right] \end{aligned}$$

where

$$J := |X_\eta| = \sqrt{d_\eta^2 + (1 - d\kappa^0)^2}$$

is the arc–length. The curvature κ becomes

$$\kappa = \frac{1}{J^2} X_{\eta\eta} \cdot \mathbf{n} = \frac{1}{J^3}\left((1 - d\kappa^0)d_{\eta\eta} + 2\kappa^0 d_\eta^2 + d_\eta d\kappa^0_\eta + \kappa^0(1 - d\kappa^0)^2\right)\,. \tag{2.3}$$

An evolving curve we can describe as a function

$$d : [0, T] \times I \longrightarrow (-\delta_0, \delta_0)$$

which is periodic in η. Then we get the outward normal velocity as

$$V = X_t \cdot \mathbf{n} = \frac{1 - d\kappa^0}{J} d_t$$

where $X(t,\eta) := X^0(\eta) + d(t,\eta)\mathbf{n}^0(\eta)$.

To get approximate solutions we introduce a higher order term $-\varepsilon\kappa_{ss}$ $(0 < \varepsilon \leq 1)$ in the first equation of the intermediate motion law (2.1). This leads to

$$V = w_{ss} - \varepsilon\kappa_{ss}, \qquad w_{ss} - w = \kappa\,. \tag{2.4}$$

where we assumed for simplicity $D = 1$ and $M = 1$, but all following proofs are valid in the general case as well. In terms of d the first equation in (2.4) now becomes

$$\frac{1 - d\kappa^0}{J} d_t = w_{ss} - \varepsilon\kappa_{ss} \tag{2.5}$$

with κ as in (2.3). Since $\partial_s = \frac{1}{J}\partial_\eta$ we can rewrite equation (2.5) as

$$d_t + \varepsilon J^{-4} d_{\eta\eta\eta\eta} = \mathcal{F}[d] := \mathcal{F}_1[d] + \varepsilon\mathcal{F}_2[d]\,. \tag{2.6}$$

Here

$$\mathcal{F}_1[d] := \frac{J}{1 - d\kappa^0} w_{ss}\,,$$

where w is the solution of $w_{ss} - w = \kappa$ with periodic boundary conditions. The operator $\mathcal{F}_2$ is defined by

$$\kappa_{ss} = \frac{1 - d\kappa^0}{J^5} d_{\eta\eta\eta\eta} - \frac{1 - d\kappa^0}{J}\mathcal{F}_2[d]\,.$$

For $\delta > 0$ we define

$$\mathcal{M}_\delta := \left\{ d \in C^1(I) \;\middle|\; \|d\|_{C^1(I)} < \delta\,,\ d \text{ is periodic} \right\}\,.$$

To apply a local existence result by Xinfu Chen [10] for a problem similar to (2.6) we have to prove the following lemma.

Lemma 2.1: *There exists a constant C depending on $\mathcal{M}^0$ and δ_0 such that for all $d \in \mathcal{M}_{\delta_0} \cap H^{3,3}(I) \cap H^{2,6}(I)$ the inequalities*

$$\|\mathcal{F}_1[d]\|^2_{L^2(I)} \leq \|\kappa\|^2_{L^2(I)} \leq C\left(\|d_{\eta\eta}\|^2_{L^2(I)} + 1\right) \tag{2.7}$$

and

$$\|\mathcal{F}_2[d]\|^2_{L^2(I)} \leq C\left(1 + \|d_{\eta\eta\eta}\|^3_{L^3(I)} + \|d_{\eta\eta}\|^6_{L^6(I)}\right) \tag{2.8}$$

hold. Furthermore we get for all $\varepsilon \in (0,1]$

$$\|\mathcal{F}[d]\|^2_{L^2(I)} \leq C_0\left(\frac{1}{\varepsilon} + \varepsilon^2\left(\|d_{\eta\eta\eta}\|^3_{L^3(I)} + \|d_{\eta\eta}\|^6_{L^6(I)}\right)\right) \tag{2.9}$$

with a constant C_0 depending on $\mathcal{M}^0$ and δ_0.

Proof: From (2.3) and $d \in \mathcal{M}_{\delta_0}$ it follows (where we have in mind that $\delta_0\|\kappa_0\|_\infty < \frac{1}{2}$)

$$\|\kappa\|^2_{L^2(I)} \leq C\left(\|d_{\eta\eta}\|^2_{L^2(I)} + 1\right)$$

Furthermore the solution w of the equation $w_{ss} - w = \kappa$ with periodic boundary conditions fulfills the energy estimate

$$\|w_{ss}\|^2_{L^2(I)} + \|w_s\|^2_{L^2(I)} \le C\|\kappa\|^2_{L^2(I)}$$

which proves the first assertion. The second inequality follows directly from the definition of $\mathcal{F}_2$ by an application of Hölder's inequality. Now we apply Young's inequality to get

$$\|d_{\eta\eta}\|^2_{L^2(I)} \le C\left(\frac{1}{\varepsilon} + \varepsilon^2\|d_{\eta\eta}\|^6_{L^6(I)}\right)$$

which together with (2.7) and (2.8) gives (2.9).

□

Now we state the local existence result for the regularized motion (2.4). We assume that Γ_0 is given as $\Gamma_0 = \left\{Y(\eta, h) \;\Big|\; h = d_0(\eta), \eta \in I\right\}$.

Theorem 2.2: *Assume that $d_0 \in C^4(I) \cap \mathcal{M}_{\delta_0/2}$ and $\varepsilon \in (0,1]$. Then*
a) there exists a time $T > 0$ (depending on ε) such that the equation (2.6) has a solution

$$d \in L^\infty(0,T;H^2(I)) \cap L^2(0,T;H^4(I)) \cap H^{1,2}(0,T;L^2(I)) \cap C^0([0,T];\mathcal{M}_{\delta_0})$$

with $d(0) = d_0$.
b) there exists a $\delta_1 > 0$ depending on $\mathcal{M}^0$ (but independent of ε) such that a solution of (2.6) with regularity as in a) can be extended as long as

$$\sup_{0 \le t \le T} \|d(t) - d_0\|_{C^1(I)} < \delta_1 .$$

The proof of this theorem follows the same line as the proof of Theorem 2.5 in [10]. One linearizes the equation, shows apriori estimates and uses Schauder's fixed–point theorem. The basic estimates are proved in Lemma 2.1. For details we refer to the paper of Xinfu Chen [10].

Remark 2.3: 1) Under the same assumptions on d_0 we can prove existence for the motion

$$V = \kappa - \kappa_{\alpha v} - \varepsilon \kappa_{ss}$$

with $\kappa_{\alpha v} := \int_{\Gamma_t} \kappa / \int_{\Gamma_t} 1$. In this case $\mathcal{F}_1$ is defined as

$$\mathcal{F}_1[d] := \kappa - \kappa_{\alpha v} .$$

We get

$$\|\mathcal{F}_1[d]\|^2_{L^2(I)} \le C\left(1 + \|d_{\eta\eta}\|^2_{L^2(I)}\right) .$$

and the proof remains the same.
2) If we choose $\mathcal{F}_1[d] \equiv 0$ and $\varepsilon = 1$ the same method gives a local existence result for the law

$$V = -\kappa_{ss} .$$

Furthermore one can allow an anisotropic law of the form

$$V = -\partial_{ss}\kappa_\gamma$$

where

$$\kappa_\gamma := (\gamma(\theta) + \gamma''(\theta))\,\kappa\,. \tag{2.10}$$

Here γ is a smooth function defined for the normal angle θ which describes anisotropy in the interfacial energy. Under the assumption that $\gamma(\theta) + \gamma''(\theta) > 0$, we can prove the same result as above for the anisotropic law (2.10). Therefore one has to observe that an estimate analogous to (2.7) holds for the anisotropic motion. Then one gets local existence as outlined in the proof of Theorem 2.2. For further details on anisotropic laws see Gurtin [21] and Cahn and Taylor [7, 8].

□

3 Energy Identities for the Regularized Problem

To prove local existence for the limit problem (i.e. $\varepsilon = 0$ in (2.4)), we show in a first step that the regularized problems have a common existence interval. On this interval we show energy estimates uniformly in ε which enable us to pass to the limit. The key for both of these arguments are the energy identities derived in this section.

Let Γ be the evolving curve which solves (2.4) with initial condition d_0. The existence of such a solution is guaranteed by Theorem 2.2. In this section we assume that d_0 is smooth, which gives via a bootstrap argument that Γ is smooth as well. In the following we choose a parametrisation

$$\begin{array}{rcl} X^* : [0,1]\times[0,T] & \longrightarrow & \mathbb{R}^2\,, \\ (p,t) & \longmapsto & X^*(p,t) \end{array}$$

of Γ such that

$$X_t^*(p,t)\cdot X_p^*(p,t) = 0$$

for all $(p,t) \in [0,1]\times[0,T]$. If we express the curvature κ, the arc–length $J^* := |X_p^*|$ and the normal velocity V in the variable (p,t) we get

$$\begin{aligned} J_t^* &= -J^*\kappa V\,, && (3.1)\\ \kappa_t &= V_{ss} + \kappa^2 V\,, && (3.2)\\ \frac{d}{dt}\int_{\Gamma_t} f &= \int_{\Gamma_t} f_t - \int_{\Gamma_t} f\kappa V\,, && (3.3)\\ \text{and}\qquad f_{ts} &= f_{st} - f_s\kappa V && (3.4) \end{aligned}$$

where we have used

$$f_s := \frac{1}{J^*}f_p \text{ and } \int_{\Gamma_t} f := \int_0^1 f(p,t)J^*(p,t)dp$$

for a real valued function f defined on Γ. Using the parametrisation X^* we can interpret f as a function of p and t. The identities (3.1)–(3.4) are proved in Gurtin [21].

The following theorem states energy identities for the regularized problem. By $L(t)$ we mean the length of the evolving curve at time t.

Lemma 3.1: *A smooth solution of*

$$V = w_{ss} - \varepsilon\kappa_{ss}\,, \tag{3.5}$$

$$w_{ss} - w = \kappa \tag{3.6}$$

fulfills the energy identities

a) $$\frac{d}{dt}L(t) = -\int_{\Gamma_t}\kappa(w_{ss} - \varepsilon\kappa_{ss}) = -\int_{\Gamma_t}(w_{ss}^2 + w_s^2 + \varepsilon\kappa_s^2)\,,$$

b) $$\frac{d}{dt}\int_{\Gamma_t}\kappa^2 + 2\varepsilon\int_{\Gamma_t}\kappa_{ss}^2 + 2\int_{\Gamma_t}\kappa_s w_{sss} = \int_{\Gamma_t}\kappa^3 V\,,$$

c) $$\frac{d}{dt}\int_{\Gamma_t}(w_s^2 + w_{ss}^2 + \varepsilon\kappa_s^2) + 2\int_{\Gamma_t}V_s^2 = 2\int_{\Gamma_t}\kappa^2V^2 + \int_{\Gamma_t}\kappa V(\varepsilon\kappa_s^2 + w_s^2 - w_{ss}^2)\,.$$

Proof: If we choose $f \equiv 1$ in (3.3) and use identities (3.5) and (3.6) we get

$$\frac{d}{dt}L(t) = -\int_{\Gamma_t}\kappa V = -\int_{\Gamma_t}\kappa(w_{ss} - \varepsilon\kappa_{ss}) = -\int_{\Gamma_t}(w_{ss}^2 + w_s^2 + \varepsilon\kappa_s^2).$$

In order to prove assertion b) we differentiate $\int_{\Gamma_t}\kappa^2$ with respect to t to get

$$\frac{d}{dt}\int_{\Gamma_t}\kappa^2 = 2\int_{\Gamma_t}\kappa\kappa_t - \int_{\Gamma_t}\kappa^3 V = 2\int_{\Gamma_t}\kappa(V_{ss} + \kappa^2 V) - \int_{\Gamma_t}\kappa^3 V = 2\int_{\Gamma_t}\kappa V_{ss} + \int_{\Gamma_t}\kappa^3 V\,.$$

The identity (3.5) and integration by parts gives

$$\int_{\Gamma_t}\kappa V_{ss} = \int_{\Gamma_t}\kappa(-\varepsilon\kappa_{ss} + w_{ss})_{ss} = -\int_{\Gamma_t}\varepsilon\kappa_{ss}^2 - \int_{\Gamma_t}\kappa_s w_{sss}\,.$$

To prove c) we want to multiply $V = w_{ss} - \varepsilon\kappa_{ss}$ by κ_t and integrate over Γ_t. The terms we get are

$$\begin{aligned}
\int_{\Gamma_t} V\kappa_t &= \int_{\Gamma_t} V(V_{ss} + \kappa^2 V) = -\int_{\Gamma_t} V_s^2 + \int_{\Gamma_t}\kappa^2 V^2\,,\\
\int_{\Gamma_t} -\kappa_{ss}\kappa_t &= \int_{\Gamma_t}\kappa_s\kappa_{ts} = \int_{\Gamma_t}\kappa_s(\kappa_{st} - \kappa V\kappa_s) = \frac{1}{2}\frac{d}{dt}\int_{\Gamma_t}\kappa_s^2 - \frac{1}{2}\int_{\Gamma_t}\kappa V\kappa_s^2\,,\\
\int_{\Gamma_t} w_{ss}\kappa_t &= \int_{\Gamma_t} w_{ss}(w_{ss} - w)_t = \int_{\Gamma_t}\frac{1}{2}\frac{d}{dt}w_{ss}^2 - \int_{\Gamma_t} w_{ss}w_t =\\
&= \int_{\Gamma_t}\frac{1}{2}\frac{d}{dt}w_{ss}^2 + \int_{\Gamma_t} w_s w_{ts} = \int_{\Gamma_t}\frac{1}{2}\frac{d}{dt}w_{ss}^2 + \int_{\Gamma_t}\frac{1}{2}\frac{d}{dt}w_s^2 - \int_{\Gamma_t}\kappa V w_s^2 =\\
&= \frac{1}{2}\frac{d}{dt}\int_{\Gamma_t}\left(w_{ss}^2 + w_s^2\right) + \frac{1}{2}\int_{\Gamma_t}\kappa V(w_{ss}^2 + w_s^2) - \int_{\Gamma_t}\kappa V w_s^2\,.
\end{aligned}$$

Using the above identities we derive

$$\frac{d}{dt}\int_{\Gamma_t}\left(w_{ss}^2+w_s^2+\varepsilon\kappa_s^2\right)+2\int_{\Gamma_t}V_s^2=2\int_{\Gamma_t}\kappa^2V^2+\int_{\Gamma_t}\kappa V(\varepsilon\kappa_s^2+w_s^2-w_{ss}^2)\,.$$

This completes the proof of Lemma 3.1.

□

Remark 3.2: For the motion $V=\kappa-\kappa_{\alpha v}-\varepsilon\kappa_{ss}$ we get with a similar calculation

$$\begin{aligned}\frac{d}{dt}L(t)+\int_{\Gamma_t}(\kappa-\kappa_{\alpha v})^2+\varepsilon\int_{\Gamma_t}\kappa_s^2 &= 0\,,\\ \frac{d}{dt}\int_{\Gamma_t}\kappa^2+2\varepsilon\int_{\Gamma_t}\kappa_{ss}^2+2\int_{\Gamma_t}\kappa_s^2 &= \int_{\Gamma_t}\kappa^3V\end{aligned}$$

and

$$\frac{d}{dt}\int_{\Gamma_t}\left((\kappa-\kappa_{\alpha v})^2+\varepsilon\kappa_s^2\right)+2\int_{\Gamma_t}V_s^2=-\int_{\Gamma_t}(\kappa-\kappa_{\alpha v})^2\kappa V+\int_{\Gamma_t}\varepsilon\kappa V\kappa_s^2+2\int_{\Gamma_t}\kappa^2V^2\,.$$

□

4 Local existence for the limit problem

In this section we are going to prove the following theorem which states a local existence result for the intermediate motions.

Theorem 4.1: *Assume $d_0\in\mathcal{M}_{\delta_1/2}$ and $\int_{\Gamma_0}\kappa_0^2$ is bounded. Then there exists a time $T>0$ such that the evolution problem*

$$\left.\begin{aligned}V &= w_{ss}\\ w_{ss}-w &= \kappa\\ \Gamma\cap\{t=0\} &= \Gamma_0\end{aligned}\right\}\tag{4.1}$$

has a solution $d\in C([0,T];\mathcal{M}_{\delta_1})$ with $d_t\in L^2(I\times(0,T))$ and $d_{\eta\eta}\in L^\infty(0,T;L^2(I))$.

We assumed as before that Γ_0 is defined as a graph over $\mathcal{M}_0$ with the distance function d_0. We want to approximate solutions of (4.1) by solutions of the motion which is regularized by $-\varepsilon\kappa_{ss}$. Therefore we approximate the distance function d_0 with C^∞ functions $d_0^\varepsilon\in\mathcal{M}_{\delta_1/2}$ such that

$$\begin{aligned}d_0^\varepsilon &\longrightarrow d_0 \quad\text{in}\quad H^{2,2}(I)\\ \text{and}\quad \varepsilon\int_{\Gamma_0}(\kappa_0^\varepsilon)_s^2 &\le 1\end{aligned}\tag{4.2}$$

This is possible since the facts that $\int_{\Gamma_0}\kappa_0^2$ is bounded and $d_0\in\mathcal{M}_{\delta_0}$ imply $d_0\in H^{2,2}(I)$. The inequality (4.2) is just a matter of scaling. Now we can apply Theorem 2.2 to get the existence of a solution Γ^ε to the evolution problem

$$\left.\begin{aligned}V^\varepsilon &= w_{ss}^\varepsilon-\varepsilon\kappa_{ss}^\varepsilon\\ w_{ss}^\varepsilon-w^\varepsilon &= \kappa^\varepsilon\\ \Gamma^\varepsilon\cap\{t=0\} &= \Gamma_0^\varepsilon\end{aligned}\right\}\tag{4.3}$$

where Γ_0^ε is defined through d_0^ε.

Let $[0, T_\varepsilon)$ be the maximal existence interval of a solution of (4.3). In the next step we show that there exists a time $T^* > 0$ and an $\varepsilon_0 > 0$ such that $T^* \le T_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$.

This is done by deriving estimates for $\|d - d_0\|_{C^1}$ and using assertion b) of Theorem 2.2. Furthermore we prove on $[0, T^*]$ energy estimates uniformly in ε which enable us to use a compactness argument to show that a subsequence of solutions of the regularized problems converge to a solution of the limit motion (4.1).

Let Γ^ε be a solution of (4.3). Then we define the quantities

$$\begin{aligned} A_\varepsilon(t) &= L^\varepsilon(t) + \int_{\Gamma_t^\varepsilon} (\kappa^\varepsilon)^2 + \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon w_{ss}^\varepsilon + \varepsilon \int_{\Gamma_t^\varepsilon} (\kappa_s^\varepsilon)^2 , \\ B_\varepsilon(t) &= \int_{\Gamma_t^\varepsilon} \left(\kappa^\varepsilon w_{ss}^\varepsilon + \varepsilon(\kappa_s^\varepsilon)^2 + 2\varepsilon(\kappa_{ss}^\varepsilon)^2 + 2\kappa_s^\varepsilon w_{sss}^\varepsilon + 2(V_s^\varepsilon)^2 \right) \end{aligned}$$

where $L^\varepsilon(t)$ is the length of Γ_t^ε. From Lemma 3.1 we get

$$\frac{d}{dt} A_\varepsilon(t) + B_\varepsilon(t) = \int_{\Gamma_t^\varepsilon} (\kappa^\varepsilon)^3 V^\varepsilon + 2 \int_{\Gamma_t^\varepsilon} (\kappa^\varepsilon V^\varepsilon)^2 + \int_{\Gamma_t^\varepsilon} \kappa^\varepsilon V^\varepsilon \left(\varepsilon(\kappa_s^\varepsilon)^2 + (w_s^\varepsilon)^2 - (w_{ss}^\varepsilon)^2 \right) . \quad (4.4)$$

In the following two Lemmas we are going to derive estimates for A_ε and B_ε.

Lemma 4.2: *There exists a positive constant C_0 such that for all $\mu > 0$*

$$\frac{d}{dt} A_\varepsilon(t) + B_\varepsilon(t) \le C_0(\sqrt{\varepsilon} + \mu) A_\varepsilon B_\varepsilon + C_\mu A_\varepsilon^2$$

holds with a positive constant C_μ depending on μ.

Proof: We have to estimate the right hand side of (4.4) in terms of A_ε and B_ε. First of all let us derive an estimate for $\|V^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)}$. Because $\int_{\Gamma_t^\varepsilon} V^\varepsilon = 0$ we get

$$\|V^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)}^2 \le C \|V^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} \|V_s^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} \le C \|V^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} B_\varepsilon^{\frac{1}{2}} .$$

The evolution equation (4.3) yields

$$\|V^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} \le \varepsilon \|\kappa_{ss}^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} + \|w_{ss}^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} \le C\, (\varepsilon)^{\frac{1}{2}}\, B_\varepsilon^{\frac{1}{2}} + A_\varepsilon^{\frac{1}{2}} .$$

From these two estimates we derive

$$\|V^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)} \le C \left(\varepsilon^{\frac{1}{4}} B_\varepsilon^{\frac{1}{2}} + A_\varepsilon^{\frac{1}{4}} B_\varepsilon^{\frac{1}{4}} \right) .$$

Now we can estimate the first term on the right hand side of (4.4)

$$\begin{aligned} \left| \int_{\Gamma_t^\varepsilon} (\kappa^\varepsilon)^3 V^\varepsilon \right| &\le \|\kappa^\varepsilon\|_{L^2(\Gamma_t)}^2 \|\kappa^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)} \|V^\varepsilon\|_{L^\infty(\Gamma_t^\varepsilon)} \\ &\le C A_\varepsilon \left(\|\kappa^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} + \|\kappa_s^\varepsilon\|_{L^2(\Gamma_t^\varepsilon)} \right) \left(\varepsilon^{\frac{1}{4}} B_\varepsilon^{\frac{1}{2}} + A_\varepsilon^{\frac{1}{4}} B_\varepsilon^{\frac{1}{4}} \right) \\ &\le C A_\varepsilon (A_\varepsilon^{\frac{1}{2}} + B_\varepsilon^{\frac{1}{2}}) (\varepsilon^{\frac{1}{4}} B_\varepsilon^{\frac{1}{2}} + A_\varepsilon^{\frac{1}{4}} B_\varepsilon^{\frac{1}{4}}) \\ &\le C \left((\varepsilon^{\frac{1}{4}} + \mu) A_\varepsilon B_\varepsilon + C_\mu A_\varepsilon^2 \right) \end{aligned}$$

where we used Young's inequality to get the last estimate. In a next step we estimate

$$
\begin{aligned}
\left| \int_{\Gamma_t^\varepsilon} (\kappa^\varepsilon V^\varepsilon)^2 \right| &\leq \|V^\varepsilon\|^2_{L^\infty(\Gamma_t^\varepsilon)} \|\kappa^\varepsilon\|^2_{L^2(\Gamma_t^\varepsilon)} \\
&\leq C\left(\varepsilon^{\frac{1}{2}} B_\varepsilon + A_\varepsilon^{\frac{1}{2}} B_\varepsilon^{\frac{1}{2}}\right) A_\varepsilon \\
&\leq C\left((\varepsilon^{\frac{1}{2}} + \mu) A_\varepsilon B_\varepsilon + C_\nu A_\varepsilon^2\right) .
\end{aligned}
$$

One can estimate the remaining terms in a similar manner, which completes the proof of the lemma.

□

Now we state a lemma which gives uniform estimates for A_ε and B_ε on a small interval.

Lemma 4.3: *If we choose ε, μ such that*

$$C_0(\sqrt{\varepsilon} + \mu) \leq \frac{1}{2(A_\varepsilon(0) + 1)}$$

then for all t with

$$0 \leq t \leq min\left(T_\varepsilon, [C_\mu A_\varepsilon(0)(A_\varepsilon(0) + 1)]^{-1}\right)$$

the inequalities

$$A_\varepsilon(t) \leq A_\varepsilon(0) + 1$$

and

$$\int_0^t B_\varepsilon(t) \leq 2C_\mu t(A_\varepsilon(0) + 1)^2$$

hold.

Proof: We omit the index ε in this proof. From Lemma 4.2 we know

$$\frac{d}{dt}A(t) + B(t) \leq C_0(\sqrt{\varepsilon} + \mu)AB + C_\mu A^2 .$$

Hence

$$\frac{d}{dt}A(t) + \left(1 - \frac{A(t)}{2(A(0) + 1)}\right) B(t) \leq C_\mu A(t)^2 .$$

If we define

$$\tilde{T} := \sup\{t \in [0, T_\varepsilon) \mid A(\tau) \leq A(0) + 1 \quad \text{for all} \quad \tau \in [0, t]\}$$

we get

$$\frac{A(t)}{2(A(0) + 1)} \leq \frac{1}{2} \text{ for all } t \in [0, \tilde{T}] .$$

This implies

$$\frac{d}{dt}A(t) \leq C_\mu A^2 \text{ for all } t \in [0, \tilde{T}] .$$

From this differential inequality we can conclude that

$$A(t) \leq \frac{A(0)}{1 - C_\mu A(0)t} \text{ for all } t \in [0, \tilde{T}] .$$

Now we get

$$A(t) \leq A(0) + 1$$

as long as

$$t < \min\left(T_\varepsilon, [C_\mu A(0)(A(0)+1)]^{-1}\right) .$$

The second inequality follows because the inequality

$$\int_0^t B(t) \leq 2C_\mu \int_0^t A(t)^2$$

holds as long as $A(t) \leq A(0) + 1$.

□

To establish a lower bound on the T_ε we want to apply the assertion b) of Theorem 2.2, i.e. we have to show

$$\|d^\varepsilon(t) - d_0\|_{C^1(I)} \leq \delta_1$$

on a time interval $[0, T^*]$. To do so we need the following lemma proved by Chen [10]. We use the notation $I_T := I \times (0, T)$.

Lemma 4.4: *Assume that $f \in L^\infty(0,T;H^2(I)) \cap H^1(0,T;L^2(I))$ and $f(t)$ is periodic for all t. Then*
a) for all $0 \leq \tau < t \leq T$, f satisfies

$$\|f(t) - f(\tau)\|_{C^1(I)} \leq (t-\tau)^{1/8} \|f_t\|_{L^2(I_T)}^{1/4} \left(T^{1/4}\|f_t\|_{L^2(I_T)}^{1/2} + 2\|(f(t) - f(\tau))_{\eta\eta}\|_{L^2(I)}^{1/2}\right)^{3/2} \ ;$$

b) there exists a constant C depending only on T such that

$$\|f\|_{C^{1,\frac{3}{8}}(I_T)} + \|f_\eta\|_{C^{\frac{1}{2},\frac{1}{8}}(I_T)} \leq C\left(\|f(0)\|_{L^2(I)} + \sup_{0\leq t\leq T} \|f_{\eta\eta}(t)\|_{L^2(I)} + \|f_t\|_{L^2(I_T)}\right) .$$

The next lemma states that a common existence interval for the regularized motions exists.

Lemma 4.5: *There exists a $T^* > 0$ and a $\varepsilon_0 > 0$ (depending on $\mathcal{M}_0$ and $A(0)$) such that the evolution problem (4.3) has a solution in $[0, T^*]$ for all $\varepsilon \in (0, \varepsilon_0]$.*

Proof: Lemma 4.3 gives the existence of an $\varepsilon_0 > 0$ and a $\mu > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and t with

$$0 \leq t < \min\left(T_\varepsilon, [C_\mu A_\varepsilon(0)(A_\varepsilon(0)+1)]^{-1}\right) =: T_\varepsilon^*$$

the inequalities

$$A_\varepsilon(t) \leq A_\varepsilon(0) + 1$$

and

$$\int_0^t B_\varepsilon(t) \leq 2C_\mu t(A_\varepsilon(0)+1)^2$$

hold. The estimates on A_ε and B_ε together with $\int_{\Gamma_t^\varepsilon} V^\varepsilon = 0$ give

$$\int_0^{T_\varepsilon^*} \|V^\varepsilon\|^2_{L^2(\Gamma_t)} + \sup_{0\le t\le T_\varepsilon^*} \|\kappa^\varepsilon\|^2_{L^2(\Gamma_t)} \le C\,.$$

From

$$V^\varepsilon = \frac{1-d^\varepsilon\kappa^0}{J^\varepsilon} d_t^\varepsilon$$

and $d^\varepsilon \in \mathcal{M}_{\delta_0}$ it follows that d_t^ε is uniformly bounded in $L^2(I_{T_\varepsilon^*})$ (for $\varepsilon \in (0,\varepsilon_0]$). Similarly we derive from the estimate on κ^ε that

$$\sup_{0\le t< T_\varepsilon^*} \|d_{xx}^\varepsilon(t)\|_{L^2(I)} \le C\,.$$

Now Lemma 4.4 yields

$$\|d^\varepsilon(t) - d_0\|_{C^1(I)} \le t^{1/8} C\,.$$

Since the right hand side does not depend on ε we can apply assertion b) of Theorem 2.2. to conclude the existence of a $T^* > 0$ such that a solution of (4.3) exists on $[0,T^*]$ for all $\varepsilon \in (0,\varepsilon_0)$, which finishes the proof.

□

Now we are in a position to prove Theorem 4.1. It remains to pass to the limit in the regularized problem. To do so we want to exploit the energy estimates established in Lemmas 4.2 and 4.3.

For all $\varepsilon \in (0,\varepsilon_0]$ we have

$$\sup_{0\le t\le T^*}\left(L^\varepsilon(t) + \int_{\Gamma_t^\varepsilon}(\kappa^\varepsilon)^2 + \int_{\Gamma_t^\varepsilon}\kappa^\varepsilon w_{ss}^\varepsilon\right) + \int_0^{T^*}\int_{\Gamma_t^\varepsilon}\left(\kappa_s^\varepsilon w_{sss}^\varepsilon + \varepsilon(\kappa_{ss}^\varepsilon)^2 + (V_s^\varepsilon)^2\right) \le C\,.$$

Since $\int_{\Gamma_t^\varepsilon} V^\varepsilon = 0$ and $d^\varepsilon \in C([0,T^*],\mathcal{M}_{\delta_1})$ the identity

$$V^\varepsilon = \frac{1-d^\varepsilon\kappa^0}{J^\varepsilon} d_t^\varepsilon$$

implies that d_t^ε is uniformly bounded in $L^2(I_{T^*})$. Similarly we can use the estimate on κ^ε to conclude that $d_{\eta\eta}^\varepsilon$ is uniformly bounded in $L^\infty(0,T^*;L^2(I))$. This implies the existence of a subsequence (which we still denote by d^ε) such that

$$\begin{aligned} d^\varepsilon &\longrightarrow d \quad \text{weak}-* \text{ in } L^\infty(0,T^*;H^2(I)) \\ \text{and}\quad d_t^\varepsilon &\longrightarrow d_t \quad \text{weakly in } L^2(I_{T^*})\,. \end{aligned}$$

An application of Lemma 4.3 yields

$$\|d^\varepsilon\|_{C^{1,\frac{3}{8}}(I_{T^*})} + \|d_\eta^\varepsilon\|_{C^{\frac12,\frac18}(I_{T^*})} \le C$$

and therefore we get for a subsequence

$$\begin{aligned} d^\varepsilon &\longrightarrow d \quad \text{in } C^{1-\nu,\frac38-\nu}(I_{T^*}) \\ \text{and}\quad d_\eta^\varepsilon &\longrightarrow d_\eta \quad \text{in } C^{\frac12-\nu,\frac18-\nu}(I_{T^*}) \end{aligned}$$

for all $\nu \in (0, \frac{1}{8})$. This implies $J^\varepsilon \longrightarrow J$ uniformly.

It remains to show convergence in w^ε. The estimate on $\int_{\Gamma_t^\varepsilon} w_{ss}^\varepsilon \kappa^\varepsilon$ implies that w_{ss}^ε and w_s^ε are uniformly bounded in $L^\infty(0,T^*;L^2(I))$. The identity $w^\varepsilon = \kappa^\varepsilon - w_{ss}^\varepsilon$ yields that the same is true for w_ε.

For a subsequence we get

$$\begin{aligned} w^\varepsilon &\longrightarrow w \quad \text{weak-}* \text{ in } L^\infty(0,T^*;L^2(I))\,, \\ w_s^\varepsilon &\longrightarrow w_1 \quad \text{weak-}* \text{ in } L^\infty(0,T^*;L^2(I))\,, \\ w_{ss}^\varepsilon &\longrightarrow w_2 \quad \text{weak-}* \text{ in } L^\infty(0,T^*;L^2(I))\,. \end{aligned}$$

Since $J^\varepsilon \longrightarrow J$ uniformly we can conclude $w_1 = w_s$ and $w_2 = w_{ss}$. Finally we want to pass to the limit in

$$\begin{aligned} V^\varepsilon &= \frac{1 - d^\varepsilon \kappa_0}{J^\varepsilon} d_t^\varepsilon = w_{ss}^\varepsilon - \varepsilon \kappa_{ss}^\varepsilon \\ \text{and} \qquad w_{ss}^\varepsilon - w^\varepsilon &= \kappa^\varepsilon \end{aligned} \tag{4.5}$$

The estimate

$$\int_0^{T^*} \int_{\Gamma_t^\varepsilon} \varepsilon (\kappa_{ss}^\varepsilon)^2 \leq C$$

gives that

$$\varepsilon \kappa_{ss}^\varepsilon \longrightarrow 0 \quad \text{strongly in } L^2(I_{T^*})\,.$$

Besides this we use the convergence properties we have proved for $d^\varepsilon, J^\varepsilon$ and w^ε to conclude

$$V = w_{ss} \quad \text{a.e. in } (0,T^*) \times I\,.$$

Now we remember that we can express the curvature κ^ε in terms of d^ε and J^ε as follows

$$\kappa^\varepsilon = \frac{1}{(J^\varepsilon)^3} \Big((1 - d^\varepsilon \kappa^0) d_{\eta\eta}^\varepsilon + 2\kappa^0 (d_\eta^\varepsilon)^2 + d_\eta^\varepsilon d^\varepsilon \kappa_\eta^0 + \kappa^0 (1 - d^\varepsilon \kappa^0)^2 \Big)\,.$$

The convergence properties of d^ε and J^ε imply

$$\kappa^\varepsilon \longrightarrow \kappa \quad \text{weakly in } L^2(I_{T^*})\,.$$

Therefore we can pass to the limit in (4.5) to get

$$w_{ss} - w = \kappa \quad \text{a.e. in } (0,T^*) \times I\,.$$

This completes the proof of Theorem 4.1.

□

Remark 4.6: The uniform estimate for $\int_0^{T^*} \int_{\Gamma_t^\varepsilon} \kappa_s^\varepsilon w_{sss}^\varepsilon$ imply that $\int_0^{T^*} \int_{\Gamma_t^\varepsilon} (\kappa_s^\varepsilon)^2$. is uniformly bounded. Therefore we get for a subsequence $\kappa_s^\varepsilon \longrightarrow \kappa_s$ weakly in $L^2(I_{T^*})$ and hence $\int_0^{T^*} \int_{\Gamma_t} (\kappa_s)^2$ is bounded. Similarly we can conclude that $\int_0^{T^*} \int_{\Gamma_t} (V_s)^2$ is bounded.

□

Remark 4.7: Under the same assumptions as in Theorem 4.1 we can establish an existence result for the conserved curvature flow

$$V = \kappa - \kappa_{av}\,.$$

We use the regularization $V = \kappa - \kappa_{av} - \varepsilon\kappa_{ss}$ to get approximate solutions. Then we can exploit the identities stated in Remark 3.1 to establish estimates similar to Lemmas 4.2 and 4.3. These estimates can be used to conclude that the regularized problems have a common existence interval. Passing to the limit in the regularized equation is then similar as in the proof of Theorem 4.1.

□

5 Regularity Results

From now on we assume $\mathcal{M}_0 \in C^\infty$. This is not a very restrictive assumption. For example for every C^2–initial curve Γ_0 we can find a C^∞–curve $\mathcal{M}_0$ in a neighborhood of Γ_0 such that Γ_0 can be parametrised as a graph over $\mathcal{M}_0$.

In the following we show that curves evolving to the intermediate law become instantaneously smooth.

Theorem 5.1: *Assume $d \in L^\infty(0,T;H^2(I)) \cap H^1(0,T;L^2(I)) \cap C([0,T];\mathcal{M}_{\delta_0})$ defines a solution $\Gamma = \{X(t,\eta)\,|\,X(t,\eta) := X^0(\eta) + d(t,\eta)\mathbf{n}^0(\eta)\}$ of the motion $V = w_{ss}$, $w_{ss} - w = \kappa$. Then*

$$d \in C^\infty([T_1,T]\times I)$$

for all $0 < T_1 < T$.

Proof: We write the motion as follows

$$V = \kappa + w \quad \text{and} \quad w_{ss} - w = \kappa\,.$$

In terms of d the first equation becomes

$$d_t = \frac{1}{J^2}d_{\eta\eta} + \frac{1}{J^2(1-d\kappa^0)}\left[2\kappa^0 d_\eta^2 + d_\eta d\kappa_\eta^0 + \kappa^0(1-d\kappa^0)^2\right] + \frac{J}{1-d\kappa^0}w\,. \tag{5.1}$$

Our goal is to apply parabolic regularity theory. The function $v = d_\eta$ is a generalized solution of

$$v_t = \left(\frac{1}{J^2}v_\eta\right)_\eta + (a_1 + a_2 w)v_\eta + a_3 w + a_4 w_\eta + a_5$$

where $a_1,..,a_5$ are appropriate functions which depend smoothly on (η, d, d_η). Since d, d_η are Hölder continuous and $w, w_\eta \in L^\infty$ parabolic regularity theory (see Ladyzhenskaya et al. [24], Chapter III, §12) gives

$$v_\eta \in C^{\alpha,\frac{\alpha}{2}}([T_1,T]\times I)$$

for an $\alpha > 0$ appropriate and all $T_1 \in (0,T)$. This implies that κ is Hölder continuous on $[T_1,T]\times I$. Using $w_{ss} - w = \kappa$ we can conclude that $w_{\eta\eta}$ is Hölder continuous. A bootstrap argument now gives the claimed assertion.

Figure 1:

□

Remark 5.2: For the motion $V = \kappa - \kappa_{\alpha v}$ we can establish the same regularity result, i.e. $d \in C^\infty([T_1, T] \times I)$ for all $T_1 > 0$. The proof is similar to the one for the intermediate motion. We just have to use the facts that $\kappa \in L^\infty(0, T; L^2(I))$ implies $\kappa_{\alpha v} \in L^\infty(0, T)$ and that $\kappa_{\alpha v}$ does not depend on the space variable.

□

6 Global existence for perturbations of a circle

All three motions (1.1)–(1.3) have the common property that circles are the only simple connected curves which are stationary. Furthermore all motions decrease the length of Γ_t and preserve the area of the region enclosed by Γ_t. Therefore one would expect that a curve which evolves to one of the these motions will converge to a circle as time tends to infinity. This is in fact true for the motion $V = \kappa - \kappa_{\alpha v}$ if the initial curve is convex (see Gage [17]). But an example by Gage [17] indicates that, in general, simple curves which evolve by the law $V = \kappa - \kappa_{\alpha v}$ may not remain simple.

There is also some evidence that the same is true for the two other laws. Let us consider the flow $V = -\kappa_{ss}$ with an initial curve where points come close to each other as in Figure 1. Such an initial curve can be chosen such that the velocity vector near the points which lie close to each other is pointing inwards. Since κ_{ss} produces a fourth order term, we can choose initial curves which yield to a normal velocity which is initially arbitrary large, with just a minor change in the local shape of the curve. Furthermore we can make the distance between such points arbitrarily small. Although we cannot give a rigorous counter–example the above example suggests that in general self intersections can occur.

In this section we prove that initial curves which are close to a circle have a global solution. This means especially that no self intersections occur. In the following section we shall prove that global solutions in fact converge to a circle.

Let us state the global existence result for the intermediate motion. We assume that the initial curve is given as a graph over a circle and is close to this circle.

Theorem 6.1: *For every radius $R_0 > 0$ there exists a $\delta(R_0) > 0$ such that for all initial curves Γ_0 which satisfy*
a) Γ_0 is given as $\Gamma_0 = \{d_0(\theta)(cos\theta, sin\theta) | \theta \in [0, 2\pi)$ and d_0 is periodic $\}$,

b) $\int_{\Gamma_0} \kappa_0^2$ *is bounded,*
c) $\|d_0 - R_0\|_{C^1([0,2\pi))} \le \delta(R_0)$ *and*
d) $\int_{\Gamma_0} \kappa_0 (w_0)_{ss} \le \delta(R_0)$ *where* w_0 *is the periodic solution of* $(w_0)_{ss} - w_0 = \kappa_0$
a global solution of the evolution problem

$$V = w_{ss}, \quad w_{ss} - w = \kappa\,,$$
$$\Gamma \cap \{t = 0\} = \Gamma_0$$

exists.

To prove this result we want to apply part b) of Theorem 2.2. Therefore we show that the evolving curve remains a graph over a circle and that $\|d(t,.) - R_0\|_{C^1([0,2\pi))}$ fulfills the estimate in part b) of Theorem 2.1. Essential for the proof is a geometric lemma proved by Xinfu Chen [10]. It states that one can parametrise a curve γ as a graph over a circle provided $\int_\gamma |\kappa - \kappa_{\alpha v}|$ is small enough. Furthermore this lemma estimates the distance between γ and a circle in terms of $\int_\gamma |\kappa - \kappa_{\alpha v}|$.

Lemma 6.2: *Assume* γ *is a simple connected curve with length* L *and that the curvature of* γ *lies in* $L^2(\gamma)$ *and satisfies*

$$\int_\gamma |\kappa - \kappa_{\alpha v}| \le \frac{1}{5}\,.$$

Then there exists a point $(x_0, y_0) \in \mathbb{R}^2$ *and a periodic function* $R \in C^{1,\frac{1}{2}}([0, 2\pi])$ *such that*

$$\gamma = \{(x_0, y_0) + R(\theta)(cos\theta, sin\theta)\,|\,\theta \in [0, 2\pi)\}\,;$$

in addition, the function R *satisfies*

$$\|R(.) - \bar{R}\|_{C^0} \le \frac{8}{5}\bar{R}\int_\gamma |\kappa - \kappa_{\alpha v}| \quad \text{and} \quad \|R_\theta\|_{C^0} \le \frac{15}{8}\bar{R}^2\int_\gamma |\kappa - \kappa_{\alpha v}|$$

where $\bar{R} := |\kappa_{\alpha v}|^{-1} = \frac{L}{2\pi}$.

To apply this lemma we need to establish an estimate for $\int_{\Gamma_t} |\kappa - \kappa_{\alpha v}|$. The following lemma shows that we can use estimates for $\int_{\Gamma_t} \kappa w_{ss}$ to do so.

Lemma 6.3: *Let* γ *be a simple connected* C^1*–curve. Then the inequality*

$$\int_\gamma (f - f_{\alpha v})^2 \le \left(1 + \left(\frac{L}{2\pi}\right)^2\right)\int_\gamma f g_{ss} \tag{6.1}$$

holds for all $f \in L^2(\gamma)$*. Here* $g \in H^{2,2}(\gamma)$ *is the solution of* $g_{ss} - g = f$ *and* L *is the length of* γ.

Proof: We write f in a Fourier series

$$f(s) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(n\frac{2\pi}{L}s\right) + b_n \sin\left(n\frac{2\pi}{L}s\right)\right)$$

where s is the arc–length parameter. Solving $g_{ss} - g = f$ with periodic boundary conditions we get

$$g(s) = -a_0 - \sum_{n=1}^{\infty} \left(\left(n\frac{2\pi}{L} \right)^2 + 1 \right)^{-1} \left(a_n \cos\left(n\frac{2\pi}{L} s \right) + b_n \sin\left(n\frac{2\pi}{L} s \right) \right)$$

and

$$g_{ss}(s) = \sum_{n=1}^{\infty} \left(n\frac{2\pi}{L} \right)^2 \left(\left(n\frac{2\pi}{L} \right)^2 + 1 \right)^{-1} \left(a_n \cos\left(n\frac{2\pi}{L} s \right) + b_n \sin\left(n\frac{2\pi}{L} s \right) \right) .$$

Now we calculate

$$\|f - f_{\alpha v}\|^2_{L^2(\gamma)} = \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right) \tag{6.2}$$

and

$$\int_\gamma f g_{ss} = \sum_{n=1}^{\infty} \left(n\frac{2\pi}{L} \right)^2 \left(\left(n\frac{2\pi}{L} \right)^2 + 1 \right)^{-1} \left(a_n^2 + b_n^2 \right) . \tag{6.3}$$

Comparing the sums in (6.2) and (6.3) gives (6.1).

□

Proof of Theorem 6.1: To make use of Lemmas 5.2 and 5.3 we have to establish an estimate on $\int_{\Gamma_t} \kappa w_{ss}$. We shall do this by using the identities

$$L(t) - L(0) + \int_0^t \int_{\Gamma_\tau} \kappa w_{ss} \, d\tau \; = 0 \tag{6.4}$$

and

$$\int_{\Gamma_t} \kappa w_{ss} - \int_{\Gamma_0} \kappa(0) w_{ss}(0) + 2 \int_0^t \int_{\Gamma_\tau} V_s^2 \, d\tau \; = \; \int_0^t \int_{\Gamma_\tau} \kappa V (2\kappa V + w_s^2 - w_{ss}^2) d\tau \tag{6.5}$$

which hold for solutions of $V = w_{ss}$, $w_{ss} - w = \kappa$ and can be proved in the same way as Lemma 3.1.

Our next goal is to estimate the right hand side in (6.5). We have

$$\int_{\Gamma_\tau} \kappa^2 V^2 = \int_{\Gamma_\tau} (\kappa - \kappa_{\alpha v})^2 V^2 + 2 \int_{\Gamma_\tau} (\kappa - \kappa_{\alpha v}) \kappa_{\alpha v} V^2 + \kappa_{\alpha v}^2 \int_{\Gamma_\tau} V^2 .$$

The first term on the right hand side can be estimated as follows

$$\begin{aligned} \int_{\Gamma_\tau} (\kappa - \kappa_{\alpha v})^2 V^2 \;\; &\leq \;\; \|\kappa - \kappa_{\alpha v}\|^2_{L^2(\Gamma_\tau)} \|V\|^2_{L^\infty(\Gamma_\tau)} \\ &\leq \;\; C \|\kappa - \kappa_{\alpha v}\|^2_{L^2(\Gamma_\tau)} \|V_s\|^2_{L^2(\Gamma_\tau)} \\ &\leq \;\; C \int_{\Gamma_\tau} \kappa w_{ss} \, \|V_s\|^2_{L^2(\Gamma_\tau)} \end{aligned}$$

Here we used the fact that $(4\pi A(0))^{\frac{1}{2}} \leq L(t) \leq L(0)$ where $A(0)$ is the area of the region enclosed by Γ_0.

Furthermore we can estimate

$$\begin{aligned} \left| \int_{\Gamma_\tau} (\kappa - \kappa_{\alpha v}) \kappa_{\alpha v} V^2 \right| \;\; &\leq \;\; |\kappa_{\alpha v}| \, \|\kappa - \kappa_{\alpha v}\|_{L^2(\Gamma_\tau)} \|V\|_{L^2(\Gamma_\tau)} \|V\|_{L^\infty(\Gamma_\tau)} \\ &\leq \;\; C \left(\int_{\Gamma_\tau} \kappa w_{ss} \right)^{\frac{1}{2}} \|V_s\|^2_{L^2(\Gamma_\tau)} \end{aligned}$$

For the integral $\int_{\Gamma_\tau} \kappa V w_{ss}^2$ we get

$$\begin{aligned}
|\int_{\Gamma_\tau} \kappa V w_{ss}^2| &\leq |\int_{\Gamma_\tau} (\kappa - \kappa_{av}) V w_{ss}^2| + |\kappa_{av}| \int_{\Gamma_\tau} |V w_{ss}^2| \\
&\leq \|\kappa - \kappa_{av}\|_{L^2(\Gamma_\tau)} \|V\|_{L^\infty(\Gamma_\tau)} \|w_{ss}\|_{L^\infty(\Gamma_\tau)} \|w_{ss}\|_{L^2(\Gamma_\tau)} + C\|w_{ss}\|^2_{L^2(\Gamma_\tau)} \|V\|_{L^\infty(\Gamma_\tau)} \\
&\leq C\left(\int_{\Gamma_\tau} \kappa w_{ss}\right) \|V_s\|^2_{L^2(\Gamma_\tau)} + C\left(\int_{\Gamma_\tau} \kappa w_{ss}\right)^{\frac{1}{2}} \|V_s\|^2_{L^2(\Gamma_\tau)} \\
&\leq C\left(\left(\int_{\Gamma_\tau} \kappa w_{ss}\right)^{\frac{1}{2}} + \int_{\Gamma_\tau} \kappa w_{ss}\right) \|V_s\|^2_{L^2(\Gamma_\tau)}.
\end{aligned}$$

The same estimate can be established for $| \int_{\Gamma_\tau} \kappa V w_s^2 |$. These estimates together with (6.5) give

$$\begin{aligned}
\int_{\Gamma_t} \kappa w_{ss} + \int_0^t \left(2 - C_0\left(\left(\int_{\Gamma_\tau} \kappa w_{ss}\right)^{\frac{1}{2}} + \int_{\Gamma_\tau} \kappa w_{ss}\right)\right) \|V_s\|^2_{L^2(\Gamma_\tau)} d\tau & \\
\leq \int_{\Gamma_0} \kappa(0) w_{ss}(0) + 2\int_0^t \kappa_{av}^2(\tau) \int_{\Gamma_\tau} V^2 d\tau & \qquad (6.6) \\
\leq \int_{\Gamma_0} \kappa(0) w_{ss}(0) + C\int_0^t \int_{\Gamma_\tau} w_{ss}^2 d\tau & \\
\leq \int_{\Gamma_0} \kappa(0) w_{ss}(0) + C_1 (L(0) - L(t)) &
\end{aligned}$$

where we used (6.4) to get the last inequality. Now we assume that the initial data fulfill

$$C_0 \left(\rho^{\frac{1}{2}} + \rho\right) \leq 1$$

where

$$\rho := \int_{\Gamma_0} \kappa(0) w_{ss}(0) + C_1 (L(0) - L_\infty) .$$

and L_∞ is the length of a sphere which encloses a circle with area $A(0)$.

(*) We claim: $\int_{\Gamma_t} \kappa w_{ss} \leq \rho$ as long as the solution exists.

1) If $L(0) = L(\infty) := \lim_{t\to\infty} L(t)$, then the initial curve is a circle and the motion is stationary.

2) If $L(0) > L_\infty$ we get $\int_{\Gamma_0} \kappa(0) w_{ss}(0) < \rho$. Assume now there exists a time t such that $\int_{\Gamma_t} \kappa w_{ss} = \rho$. Let t^* the first time at which this happens. From inequality (6.6) we conclude

$$\int_{\Gamma_{t^*}} \kappa w_{ss} + \int_0^{t^*} \|V_s\|^2_{L^2} \leq \rho .$$

This contradicts $V_s \not\equiv 0$ and therefore we proved (*).

Now we use the facts that

$$\int_{\Gamma_t} |\kappa - \kappa_{av}| \leq L(t)^{\frac{1}{2}} \|\kappa - \kappa_{av}\|_{L^2(\Gamma_t)} \leq L(t)^{\frac{1}{2}} \left(1 + \left(\frac{L(t)}{2\pi}\right)^2\right)^{\frac{1}{2}} \left(\int_{\Gamma_t} \kappa w_{ss}\right)^{\frac{1}{2}}$$

and that we can make $\int_{\Gamma_t} \kappa w_{ss}$ as small as we want if the initial data are chosen such that ρ is small enough to fulfill the conditions of Lemma 6.2. Therefore we get the following representation for Γ_t

$$\Gamma_t = \{(x_0(t), y_0(t)) + R(\theta,t)(\cos\theta, \sin\theta) \,|\, \theta \in [0, 2\pi) \text{ and } R(.,t) \text{ is periodic}\}$$

with a function R such that

$$\|R(.,t) - \bar{R}(t)\|_{L^\infty} \leq \frac{8}{5}\bar{R}(t) \int_{\Gamma_t} |\kappa - \kappa_{\alpha v}|$$

and

$$\|R_\theta(.,t)\|_{L^\infty} \leq \frac{15}{8}\bar{R}(t)^2 \int_{\Gamma_t} |\kappa - \kappa_{\alpha v}| \,.$$

If we now choose ρ small enough we can fulfill the conditions of Theorem 2.1 part b) to conclude that a global solution exists.

□

The following theorem states that a global solution converges to a circle.

Theorem 6.4: *Assume a global simply connected solution Γ to the motion*

$$V = w_{ss} , \; w_{ss} - w = \kappa \,,$$

$$\Gamma \cap \{t = 0\} = \Gamma_0$$

exists. Then there exists a time T_0 such that Γ_t is given by a periodic function $R(.,t) : [0, 2\pi) \longrightarrow \mathbb{R}^+$ in the following way

$$\Gamma_t = \{(x_0(t), y_0(t)) + R(\theta,t)(\cos\theta, \sin\theta) \,|\, \theta \in [0, 2\pi)\}$$

for all $t > T_0$. Moreover

1.) $L(t) \searrow L_\infty$

and 2.) $\|R(t,.) - R_\infty\|_{L^\infty} \longrightarrow 0$

where L_∞ (R_∞) is the length (radius) of a sphere which encloses a ball with the area $A(0)$.

Proof: Let us define $L(\infty) := \lim_{t\to\infty} L(t)$. We know that for all $0 \leq t_1 < t_2$

$$\int_{t_1}^{t_2} \int_{\Gamma_\tau} \kappa w_{ss} = L(t_1) - L(t_2) \leq L(0)$$

holds. Therefore $\int_0^\infty \int_{\Gamma_\tau} \kappa w_{ss}$ is bounded. This means that we find for all $\varepsilon > 0$ and $T > 0$ a time $t_1 > T$ such that $\int_{\Gamma_{t_1}} \kappa w_{ss} < \varepsilon$. Using Lemmas 6.2 and 6.3 we get that for ε small enough Γ_{t_1} fulfills the conditions on an initial curve in Theorem 6.1. As in the proof of Theorem 6.1 we get

$$\int_{\Gamma_{t_2}} \kappa w_{ss} \leq \int_{\Gamma_{t_1}} \kappa w_{ss} + C_1 \left(L(t_1) - L(t_2)\right) \tag{6.7}$$

for all $t_1 \leq t_2$. The right hand side in (6.7) is arbitrarily small if we choose t_1 appropriate. Therefore

$$\int_{\Gamma_t} \kappa w_{ss} \longrightarrow 0 \quad \text{when} \quad t \longrightarrow \infty \,.$$

The Lemmas 6.2 and 6.3 now give

$$\|R(.,t) - \bar{R}\|_{C^1} \le C \int_{\Gamma_t} |\kappa - \kappa_{av}| \le C \left(\int_{\Gamma_t} \kappa w_{ss} \right)^{\frac{1}{2}} \longrightarrow 0 \,.$$

Since area is preserved we get

$$\lim_{t \to \infty} \bar{R}(t) = \frac{L_\infty}{2\pi} \quad \text{and} \quad \lim_{t \to \infty} L(t) = L_\infty \,.$$

□

Remark 6.5: a) A result similar to Theorem 6.1 can also be proved for the motion $V = -\kappa_{ss}$. In this case we have to replace assumption d) by a smallness condition for $\int_{\Gamma_0} \kappa_s(0)^2$. The prove is essentially the same as the one for Theorem 6.1. In the proof of Theorem 6.1 we used the energy identities (6.4) and (6.5) to show that $\int_{\Gamma_t} \kappa w_{ss}$ remains small. In the case $V = -\kappa_{ss}$ we apply the identities

$$L(t) - L(0) + \int_0^t \int_{\Gamma_\tau} \kappa_s^2 d\tau = 0 \tag{6.8}$$

and

$$\int_{\Gamma_t} \kappa_s^2 - \int_{\Gamma_0} \kappa_s(0)^2 + 2 \int_0^t \int_{\Gamma_\tau} V_s^2 = \int_0^t \int_{\Gamma_\tau} \kappa V (2\kappa V + \kappa_s^2) \tag{6.9}$$

to get an estimate for $\int_{\Gamma_t} \kappa_s^2$ if it was initially small enough. Then one can use $\int_{\Gamma_t} \kappa_s^2$ to control $\int_{\Gamma_t} |\kappa - \kappa_{av}|$ and the rest of the proof follows the same line as the proof of Theorem 6.1.

The identities (6.8) and (6.9) can also be used to prove the equivalent of Theorem 6.4 for the flow $V = -\kappa_{ss}$.

b) To prove a version of Theorem 6.1 for the motion $V = \kappa - \kappa_{av}$ we have to replace assumption d) by a condition on the magnitude of $\int_{\Gamma_0} (\kappa(0) - \kappa_{av}(0))^2$. Then we use the identities in Remark 3.2 to show that $\int_{\Gamma_t} (\kappa - \kappa_{av})^2$ remains small. Having controlled $\int_{\Gamma_t} (\kappa - \kappa_{av})^2$ we can argue in an entirely analogous manner as in Theorem 6.1 (using Lemmas 6.2 and 6.3) to show that a global solution exists.

An equivalent of Theorem 6.4 for the conserved mean curvature flow is also true. Again we can use the identities stated in Remark 3.2 to prove this fact.

Acknowledgments: The authors want to thank Prof. S. Luckhaus for an helpful remark on regularity theory and Prof. B.D. Coleman for his useful comments about an earlier draft of this paper. The work of Harald Garcke was supported by a Research Fellowship of the European Science Foundation held at the Centre for Mathematical Analysis and its Applications (University of Sussex).

References

[1] Alikakos N.D., Bates P.W. and Chen X., *Convergence of the Cahn–Hilliard equation to the Hele–Shaw model,* Archive Rational Mech. Anal. **128** (1994), 165–205.

[2] Brakke K., *The motion of a surface by its mean curvature*, Princeton University Press (1978).

[3] Bronsard L. and Kohn R., *Motion by mean curvature as the singular limit of Ginzburg–Landau dynamics,* Journal of Differential Equations **90** (1991), 211–237.

[4] Bronsard L. and Stoth B., *Volume preserving mean curvature flow as a limit of a non-local Ginzburg–Landau equation* SIAM Journal on Mathematical Analysis **28**, No. 4 (1997), to appear.

[5] Caginalp G., *Stefan and Hele–Shaw type models as asymptotic limits of the phase–field equations,* Physical Review A **39** (1989), 5887–5896.

[6] Cahn J.W., Elliott C.M. and Novick–Cohen A., *The Cahn–Hilliard equation: Surface motion by the Laplacian of the mean curvature*, European J. Appl. Math. Vol. **7**, No.3 (1996), 287–301.

[7] Cahn J.W. and Taylor J.E., *Surface motion by surface diffusion*, Acta Metallurgica **42** (1994), 1045–1063.

[8] Cahn J.W. and Taylor J.E., *Linking anisotropic and diffusive surface motion laws via gradient flows,* J. Stat. Phys. Vol. **77** Nos 1/2 (1994), 183–197.

[9] Chen Y–G, Giga Y and Goto S., *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,* Journal of Differential Geometry **33** (1991), 749–786.

[10] Chen X., *The Hele–Shaw problem and area–preserving curve–shortening motions*, Archive Rational Mech. Anal. **123** (1993) 117–151.

[11] Coleman B.D., Falk R.S. and Moakher M., *Space–time finite element methods for surface diffusion with applications to the theory of the stability of cylinders,* SIAM Journal on Scientific Computing Vol. **17**, No. 6 (1996), 1434–1448.

[12] De Mottoni P. and Schatzman M., *Evolution geometrique d'interfaces,* C.R.A.S. **309** (1989), 453–458.

[13] Davi F. and Gurtin M., *On the motion of a phase interface by surface diffusion,* Journal of Applied Mathematics and Physics (ZAMP) Vol. **41** (1990), 782–811.

[14] Elliott C.M. and Garcke H., *On the Cahn–Hilliard equation with degenerate mobility,* SIAM Journal on Mathematical Analysis **27**, No. 2 (1996), 404–423.

[15] Evans L.C., Soner H.M. and Souganidis P.E., *Phase transitions and generalized motion by mean curvature,* Comm. Pure and Appl. Math. **XLV** (1992), 1097–1123.

[16] Evans L.C. and Spruck J., *Motion of level sets by mean curvature I*, Journal of Differential Geometry **33** (1991), 635–681.

[17] Gage M., *On an area preserving evolution equation for plane curves,* Contemporary Mathematics Vol. **51** (1986), 51–62.

[18] Gage M. and Hamilton R.S., *The heat equation shrinking convex plane curves,* Journal of Differential Geometry **23** (1986), 69–96.

[19] Gilbarg D. and Trudinger N.S., *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin, (1977).

[20] Grayson M., *The heat equation shrink embedded curves to round points,* Journal of Differential Geometry **26** (1987), 285–314.

[21] Gurtin M.E., *Thermodynamics of Evolving Phase Boundaries in the Plane*, Clarendon Press, Oxford (1993).

[22] Huisken G., *Flow by mean curvature of convex surfaces into spheres,* Journal of Differential Geometry **20** (1984), 237–266.

[23] Huisken G., *The volume preserving mean curvature flow,* J. Reine u. Angewandte Mathematik **382** (1987), 35–88.

[24] Ladyzhenskaya O.A., Solonnikov V.A. and Uralceva N.N., *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs Vol. **24**, American Mathematical Society, Providence (1968).

[25] Langer J.S., *Models of pattern formation in first–order phase transitions,* Directions in Condensed Matter Physics, World Scientific, Singapore (1986), 165–186.

[26] Mullins W.W., *Theory of thermal grooving*, J. Appl. Phys. **28** (1957), 333–339.

[27] Pego R.L., *Front migration in the nonlinear Cahn–Hilliard equation,* Proc. Roy. Soc. London A **422** (1989), 261–278.

[28] Rubinstein J. and Sternberg P., *Nonlocal reaction–diffusion equations and nucleation*, IMA J. Appl. Math. **48** (1992), 249–264.

Charles M. Elliott, Centre for Mathematical Analysis and its Applications, School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton BN1 9QH, U.K.

Harald Garcke, Institut für Angewandte Mathematik, Universität Bonn, Wegelerstr. 6, 53115 Bonn, Germany.