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Finite element analysis of a current density–electric field formulation of Bean's model for superconductivity

C. M. ELLIOTT, D. KAY AND V. STYLES
Centre for Mathematical Analysis and its Applications, Department of Mathematics, University of Sussex, Falmer, Brighton BN1 9QH, UK

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We study a current density–electric field formulation of Bean's model for the experimental set-up of an infinitely long cylindrical superconductor subject to a transverse magnetic field. We introduce a fully practical finite-element approximation of the model and prove an error between the exact solution and the approximate solution for the current density of order $(h + \Delta t)^{1/2}$. Numerical simulations for a variety of given source currents are presented.

Keywords: finite elements; superconductivity; Bean Model; Stefan problem.

1. Introduction

In this paper we consider a critical state model for type-II superconductors formulated in terms of the current density and the electric field intensity. The physical setting is that of an infinitely long cylinder of type-II superconducting material subject to an applied transverse magnetic field. We take the cylindrical superconductor to occupy the region $D = \Omega \times \mathbb{R}$, where Ω is a bounded simply connected domain in $\mathbb{R}^2$ that denotes the cross section of the superconductor. In this set-up the current density $\boldsymbol{\mathcal{J}} = (0, 0, \mathcal{J}(\underline{x}, t))$ lies parallel to the axis of the cylinder. Surrounding the superconductor we have cylindrical sources of current $D_w = \Omega_w \times \mathbb{R}$, such that $\Omega_w = \cup_{i=1}^k \Omega_{w_i}$, where each Ω_{w_i} is a simply connected bounded domain in $\mathbb{R}^2$, see Fig. 1. In this region we apply a given source current $\mathbf{J}_s = (0, 0, J_s(\underline{x}, t))$ and outside of $\overline{D} \cup \overline{D}_w$ the current is zero.

An evolutionary variational inequality formulation of the model involving the current density $\mathcal{J}$ was derived and analysed by Prigozhin (1996a,b, 1997). In these works a numerical method was developed and computations presented. Engineering applications of this approach, relating to the modelling of superconducting induction motors, may be found in Barnes *et al.* (1999, 2000).

In a recent paper (Elliott *et al.*, 2004) we gave a finite-element approximation of the model and proved error estimates between the exact solution and the approximate solution for the current density and the magnetic field. As observed by Bossavit (1994b), Bean's critical state model can be formulated as a degenerate Stefan problem; see also Maslouh *et al.* (1997) and Prigozhin (1997). In this paper we study a Stefan problem involving the current density and the electric field equivalent to the variational inequality. We formulate the model in Section 2 and state the relationship between solutions of the model and the unique solution of the variational inequality studied in Elliott *et al.* (2004). In Sections 3 and 4 respectively we consider continuous in time and fully discrete finite-element approximations of the model. We show an error estimate between the exact solution of the model and the solution of the fully discrete model. We observe that the discretizations of the variational inequality and Stefan problems are equivalent. The error bound in this paper is for a practical fully discrete scheme involving numerical integration in the nonlinear term and this differs from the fully discrete discretization analysed in Elliott

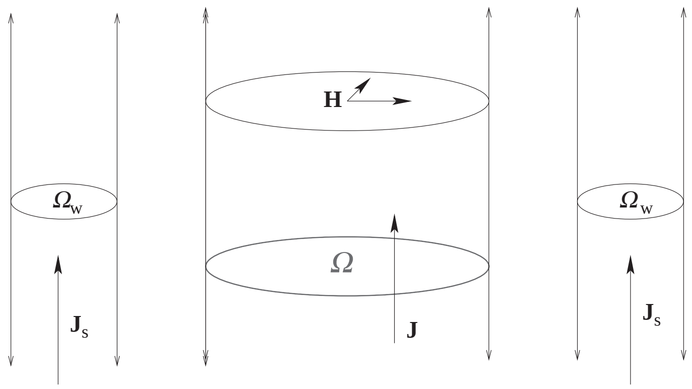

FIG. 1. Infinitely long superconducting cylinder and copper windings.

et al. (2004). In Section 5 we present a Gauss–Seidel iteration to solve the fully discrete approximation and we show the convergence of this iteration. Finally, in Section 6 we present some numerical results and in Section 7 we make some concluding remarks.

2. Formulation of the model

We use the eddy current form of Maxwell’s equations (see Bossavit, 1998) given by

$$\partial_t \mathbf{B} + \text{curl}\,\mathbf{E} = \mathbf{0} \qquad \text{in} \quad \mathbb{R}^3 \times [0, T] \tag{2.1}$$

$$\text{curl}\,\mathbf{H} = \mathcal{J} \qquad \text{in} \quad \mathbb{R}^3 \times [0, T] \tag{2.2}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{in} \quad \mathbb{R}^3 \times [0, T], \tag{2.3}$$

where $\mathbf{B}$ is the magnetic flux density, $\mathbf{E}$ is the electric field, $\mathcal{J}$ is the current and $\mathbf{H}$ is the magnetic field. We assume that the fields $\mathbf{B}$ and $\mathcal{J}$ are independent of x_3 and that $\mathbf{B} = (B_1, B_2, 0)$. Here

$$\mathcal{J} = \mathbf{J} + \mathbf{J}_s$$

where

$$\mathbf{J} = (0, 0, J) \equiv \mathbf{0} \qquad \text{in} \ \mathbb{R}^3 \backslash \overline{D} \times [0, T]$$
$$\mathbf{J}_s = (0, 0, J_s) \equiv \mathbf{0} \qquad \text{in} \ \mathbb{R}^3 \backslash \overline{D}_w \times [0, T].$$

We assume that $\mathbf{H} \sim \mathbf{0}$ as $\mathbf{x} \sim \infty$ and impose, using (2.2),

$$\int_{\mathbb{R}^2} (J + J_s) = 0.$$

Inside the superconductor the electric field intensity is related to the current density by a constitutive relation which is a critical state form of Ohm’s law:

$$\mathbf{E} = (0, 0, E) \qquad E(\underline{x}, t) \in \beta(J(\underline{x}, t)), \tag{2.4}$$

where $\beta(\cdot)$ is the multi-valued maximal monotonic mapping defined by, for $r \in [-J_c, J_c]$,

$$\beta(r) = \begin{cases} (-\infty, 0] & \text{if } r = -J_c \\ 0 & \text{if } |r| < J_c \\ [0, \infty) & \text{if } r = J_c \end{cases} \tag{2.5}$$

with J_c being the critical current magnitude.

REMARK Note that in $\mathbb{R}^3 \backslash (\overline{D} \cup \overline{D}_w)$ the conductance is taken to be zero so that $\mathcal{J} = \mathbf{0}$ in $\mathbb{R}^3 \backslash (\overline{D} \cup \overline{D}_w)$. Also, since $\mathcal{J}$ is prescribed in D_w, it is only appropriate to apply a constitutive relation (such as Ohm's law) between $\mathbf{E}$ and $\mathcal{J}$ in the superconductor, see Bossavit (1998).

We assume a linear constitutive law

$$\mathbf{B} = \mu \mathbf{H} \tag{2.6}$$

where the permeability μ is piecewise constant in space, possibly taking different values in D, D_w and $\mathbb{R}^3 \backslash (\overline{D} \cup \overline{D}_w)$.

Using (2.3) we have the existence of a magnetic potential $\mathbf{A}$ satisfying

$$\mathbf{B} = \text{curl } \mathbf{A}, \tag{2.7}$$

and recalling that $\mathbf{B} = (B_1(x_1, x_2), B_2(x_1, x_2), 0)$ we may choose $\mathbf{A} = (0, 0, A(x_1, x_2))$. Using (2.2), (2.6) and (2.7) we have

$$\mathcal{J} = \text{curl } \left(\frac{1}{\mu} \text{curl } \mathbf{A} \right) \qquad \text{in } \mathbb{R}^3 \times [0, T] \tag{2.8}$$

which implies for our choice of $\mathbf{A}$ that

$$J + J_s = -\nabla \cdot \left(\frac{1}{\mu} \nabla A \right) \qquad \text{in } \mathbb{R}^2 \times [0, T]. \tag{2.9}$$

We fix the magnetic potential A by setting

$$A = \mathcal{G}(J + J_s) \tag{2.10}$$

where the operator $\mathcal{G}$ is the unique solution operator of the following problem.

Given η with compact support and satisfying $⨍ \eta = 0$, find $\mathcal{G}\eta$ such that

$$-\nabla \cdot \left(\frac{1}{\mu} \nabla \mathcal{G}\eta \right) = \eta \qquad \text{in } \mathbb{R}^2$$

$$\nabla \mathcal{G}\eta \sim 0 \quad \text{as } \underline{x} \sim \infty, \quad \int_\Omega \mathcal{G}\eta = 0$$

and

$$⨍ \eta := \frac{1}{|\text{supp } \eta|} \int_{\mathbb{R}^2} \eta.$$

REMARK If μ is constant then

$$\mathcal{G}\eta(x) = -\frac{\mu}{2\pi}\int_{\mathbb{R}^2} \ln|x - x'|\eta(x')\mathrm{d}x' + \frac{\mu}{2\pi}⨍\int_{\mathbb{R}^2} \ln|\cdot - x'|\eta(x')\,\mathrm{d}x'. \tag{2.11}$$

From (2.1) and (2.7) we have

$$\text{curl}\,(\partial_t \mathbf{A} + \mathbf{E}) = \mathbf{0} \qquad \text{in } \mathbb{R}^3 \times [0, T] \tag{2.12}$$

$$\Rightarrow \mathbf{E} + \partial_t \mathbf{A} + \nabla\psi = \mathbf{0} \qquad \text{in } \mathbb{R}^3 \times [0, T]. \tag{2.13}$$

Using (2.13) and noting (2.4) it follows that inside the superconductor $\psi = \lambda(t)x_3$ and hence

$$E + \mathcal{G}(\partial_t J + \partial_t J_s) + \lambda(t) = 0 \qquad \text{in } \Omega \times [0, T]. \tag{2.14}$$

REMARK Equation (2.14) together with (2.4) yields a well defined problem for $\{E, J\}$ inside the superconductor, that has a unique solution for J, see Proposition 2.1.

We now extend E so that (2.14) holds everywhere in $\mathbb{R}^2$. Hence we have that

$$\partial_t J - \text{div}\left(\frac{1}{\mu}\nabla E\right) = -\partial_t J_s \tag{2.15}$$

holds in the sense of distributions on the space–time cylinder $\mathbb{R}^2 \times (0, T)$.

This system has the initial condition

$$J(\underline{x}, 0) = J_0(\underline{x}) \quad \underline{x} \in \mathbb{R}^2 \quad \text{and} \quad ⨍(J_0(\cdot) + J_s(\cdot, 0)) = 0,$$

where $J_0(\underline{x}) = 0$ for $\underline{x} \notin \overline{\Omega}$ and we impose the boundary condition

$$\nabla E \sim 0 \quad \text{as } \underline{x} \sim \infty.$$

We suppose that

$$J_s \in H^2(0, T; L^2(\mathbb{R}^2)), \quad J_s(\underline{x}, \cdot) = 0 \quad \text{for a.e. } \underline{x} \in \mathbb{R}^2\backslash\overline{\Omega}_w, \quad \int_{\Omega_w} J_s(\cdot, t) = 0$$

and we seek a weak solution defined in the following way.

(P) Find $J \in L^\infty(\mathbb{R}^2 \times (0, T))$ and $E \in L^2(0, T; H^1_{loc}(\mathbb{R}^2))$ such that

$$\int_0^T \int_{\mathbb{R}^2} \left(-J\partial_t\eta + \frac{1}{\mu}\nabla E \cdot \nabla\eta\right) \mathrm{d}\underline{x}\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}^2} \partial_t J_s \eta\,\mathrm{d}\underline{x}\,\mathrm{d}t + \int_{\mathbb{R}^2} J_0(\underline{x})\eta(\underline{x}, 0)\,\mathrm{d}\underline{x} \tag{2.16}$$

for all

$$\eta \in \mathcal{J} := \{\eta \in H^1(0, T; L^2(\mathbb{R}^2)) : \nabla\eta \in L^2(0, T; \mathbb{R}^2),\ \eta(\cdot, T) = 0\}$$

where

$$J(\underline{x}, t) = 0 \quad \text{for a.e. } (\underline{x}, t) \notin \overline{\Omega} \times (0, T)$$

and

$$|J(\underline{x},t)| \leqslant J_c \quad \text{and} \quad E(\underline{x},t) \in \beta(J(\underline{x},t)) \;\text{ for a.e. } (\underline{x},t) \in \Omega \times (0,T). \tag{2.17}$$

We can reformulate (2.17) as

$$J(\cdot,t) \in K, \quad \int_\Omega E(\eta - J)\,\mathrm{d}\underline{x} \leqslant 0 \quad \text{for a.e. } t \in (0,T),\; \forall\, \eta \in K$$

with

$$K := \{\eta \in L^2(\Omega) : |\eta| \leqslant J_c\}.$$

REMARK Differentiating (2.8) with respect to time and using (2.12) yields an equation with the following third component:

$$\partial_t(J + J_s) = -\partial_{x_1}\left(\frac{1}{\mu}\partial_{x_3}E_1\right) + \partial_{x_1}\left(\frac{1}{\mu}\partial_{x_1}E_3\right) + \partial_{x_2}\left(\frac{1}{\mu}\partial_{x_2}E_3\right) - \partial_{x_2}\left(\frac{1}{\mu}\partial_{x_3}E_2\right).$$

If $\mathbf{E}$ is independent of x_3 we obtain (2.15). Furthermore, since $\partial_t\mathbf{B} \sim \mathbf{0}$ at ∞ we have from (2.1) that $\nabla E \sim 0$ at ∞. In this case E is the third component of the electric field outside the superconductor as well. Otherwise, the function E solving (2.15) is only the electric field inside the superconductor.

2.1 *Reduction to a bounded domain*

It is convenient to work on a bounded domain B_R which is a ball of radius R such that $\overline{\Omega} \cup \overline{\Omega}_w \subset B_R$ and μ is constant outside B_R. We observe that for v being harmonic outside B_R and $\nabla v \in L^2(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \frac{1}{\mu}\nabla v \cdot \nabla \eta = \int_{B_R} \frac{1}{\mu}\nabla v \cdot \nabla\eta + \int_{\partial B_R} \frac{1}{\mu}\mathcal{B}(v)\eta \quad \forall\, \eta \in H^1(\mathbb{R}^2)$$

where $\mathcal{B} : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the Dirichlet to Neumann map,

$$\mathcal{B}(v)\Big|_{\partial B_R} := \sum_{k=1}^{\infty} \frac{1}{\pi R} \int_0^{2\pi} \frac{\partial v_\gamma}{\partial \phi} \sin k(\phi - \theta)\,\mathrm{d}\phi$$

where v_γ is the trace of v on ∂B_R. It is useful to introduce the bilinear forms

$$(\xi,\eta) := \int_{B_R} \xi\eta, \quad a(\xi,\eta) := \left(\frac{1}{\mu}\nabla\xi, \nabla\eta\right),$$
$$b(\xi,\eta) := \int_{\partial B_R} \frac{1}{\mu}\mathcal{B}(\xi)\eta, \quad A(\xi,\eta) := a(\xi,\eta) + b(\xi,\eta).$$

For $\omega = B_R,\ \Omega$ or Ω_w we set

$$⨍_\omega \eta = \frac{1}{|\omega|}\int_\omega \eta\,\mathrm{d}\underline{x}$$

and

$$
\begin{aligned}
H_e^1(B_R) &:= \left\{\xi \in H^1(B_R) : \fint_{\Omega} \xi = 0\right\}\\
\mathcal{F} &:= (H_e^1(B_R))'\\
L_{\Omega}^2(B_R) &:= \left\{\eta \in L^2(B_R) : \eta = 0 \text{ for a.e. } \underline{x} \notin \overline{\Omega}\right\}\\
L_0^2(B_R) &:= \left\{\eta \in L^2(B_R) : \fint_{B_R} \eta = 0\right\}\\
L_{0,\Omega}^2(B_R) &:= \{\eta \in L_0^2(B_R) : \eta = 0 \text{ for a.e. } \underline{x} \notin \overline{\Omega}\}\\
L_{0,\Omega_w}^2(B_R) &:= \{\eta \in L_0^2(B_R) : \eta = 0 \text{ for a.e. } \underline{x} \notin \overline{\Omega}_w\}\\
K_{\Omega} &:= \{\eta \in L_{\Omega}^2(B_R) : |\eta| \leqslant J_c \text{ on } \Omega\}\\
K_{0,\Omega} &:= \{\eta \in L_{0,\Omega}^2(B_R) : |\eta| \leqslant J_c \text{ on } \Omega\}.
\end{aligned}
$$

Problem (**P**) may be rewritten as follows.

(**P$_\mathbf{R}$**) Find $J \in L^\infty(0, T; K_{0,\Omega}) \cap H^1(0, T; \mathcal{F})$ and $E \in L^2(0, T; H^1(B_R))$ such that

$$
\int_0^T [(-J, \partial_t \xi) + A(E, \xi)]\, \mathrm{d}t = \int_0^T (-\partial_t J_s, \xi)\, \mathrm{d}t + (J_0, \xi(\cdot, 0)) \tag{2.18}
$$

for all

$$
\xi \in \mathcal{J}_R := \{\xi \in H^1(0, T; L^2(B_R)) \cap L^2(0, T; H^1(B_R)),\ \xi(\cdot, T) = 0\}
$$

and

$$
(E, \eta - J) \leqslant 0 \quad \forall\, \eta \in K_{\Omega}, \quad \text{for a.e. } t \in (0, T). \tag{2.19}
$$

Also useful for the analysis is the Green operator

$$
G : L_0^2(B_R) \to V_0 := \left\{\xi \in V : \fint_{\Omega} \xi = 0\right\}
$$

defined as the unique solution operator of the following.

For $\eta \in \mathcal{F}$ find $G\eta \in H_e^1(B_R)$ such that

$$
A(G\eta, \xi) = \langle \eta, \xi \rangle \quad \forall\, \xi \in H^1(B_R) \tag{2.20}
$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H_e^1(B_R)$ and $(H_e^1(B_R))'$. Note that

$$
\langle \eta, \xi \rangle = (\eta, \xi) \quad \forall\, \eta \in L_0^2(B_R).
$$

If $\eta \in L_0^2(B_R)$, then $G\eta$ can be extended to all of $\mathbb{R}^2$ as the unique solution of

$$
\int_{\mathbb{R}^2} \frac{1}{\mu} \nabla G\eta \cdot \nabla \xi\, \mathrm{d}\underline{x} = \int_{B_R} \eta \xi\, \mathrm{d}\underline{x} \quad \forall\, \xi \in V \tag{2.21}
$$

where

$$
V := \left\{\eta \in L_{loc}^2(\mathbb{R}^2) : \nabla \eta \in L^2(\mathbb{R}^2)\right\}.
$$

This is the Green $\mathcal{G}$ operator defined earlier when supp η is included in B_R.

We define the semi-norm and the norm

$$|\eta|_A^2 := A\,(\eta, \eta) \qquad \forall\, \eta \in H^1(B_R)$$
$$||\eta||_{\mathcal{F}} = ||\eta||_{A^{-1}} := |G\eta|_A \qquad \forall\, \eta \in \mathcal{F}$$

and we set

$$|\eta|_{0,\omega} := ||\eta||_{L^2(\omega)},\ |\eta|_{1,\omega} = ||\nabla\eta||_{L^2(\omega)},\ ||\eta||_{1,\omega} = ||\eta||_{H^1(\omega)}.$$

From Han & Wu (1985) we have that $A(\cdot,\cdot)$ is continuous with respect to the $H^1(B_R)$ norm

$$|A\,(\xi, \eta)| \leqslant C||\xi||_{1,B_R}||\eta||_{1,B_R}$$

and also (note that $A\,(\eta, 1) = 0,\ \ \eta \in H^1(B_R)$)

$$(\xi, \eta) = A\,(G\xi, \eta) \leqslant |G\xi|_A|\eta|_A = ||\xi||_{A^{-1}}|\eta|_A \ \ \forall\, \xi \in L_0^2(B_R),\ \eta \in H^1(B_R).$$

Henceforth, for convenience of notation we set $f := -\partial_t J_s \in H^1([0, T]; L^2_{0,\Omega_w}(B_R))$.

We introduce the following problem.

($\mathbf{Q_R}$): Find J such that

$$J \in L^\infty(0, T; K_{0,\Omega}) \cap C([0, T; \mathcal{F}]), \qquad \partial_t J \in L^2(0, T; \mathcal{F}) \tag{2.22}$$

satisfying, for any $\tau \in [0, T]$,

$$\int_0^\tau (\partial_t GJ, \eta - J)\,\mathrm{d}t \geqslant \int_0^\tau (Gf, \eta - J)\,\mathrm{d}t \qquad \forall\, \eta \in L^2(0, T; K_{0,\Omega}) \tag{2.23}$$
$$J|_{t=0} = J_0.$$

PROPOSITION 2.1 Let $J_s \in H^1(0, T; \mathcal{F})$ and $J_0 \in K_{0,\Omega}$. If (J, E) is a solution of ($\mathbf{P_R}$) then J is unique and E is unique up to an additive function of time. Furthermore, J is the unique solution of ($\mathbf{Q_R}$).

Proof. From (2.18) and (2.20) we have

$$E = G(f - \partial_t J) + \lambda(t) \qquad \text{for a.e. } t \in (0, T)$$

where $\lambda(t)$ is space independent. The variational inequality (2.23) then follows from (2.19). □

REMARK One can view (2.15) as a degenerate Stefan (or two phase Hele-Shaw) problem (Elliott & Ockendon, 1982), with the constitutive relation on Ω

$$J \in J_c \text{sign} E$$

as observed by Prigozhin (1997). Viewing it as a free boundary problem one has the decomposition of Ω, see Fig. 2:

$$\Omega^+(t) := \{\underline{x} \in \Omega : J(\underline{x}, t) = J_c\}$$
$$\Omega^-(t) := \{\underline{x} \in \Omega : J(\underline{x}, t) = -J_c\}$$
$$\Omega^o(t) := \{\underline{x} \in \Omega : |J(\underline{x}, t)| < J_c\}$$

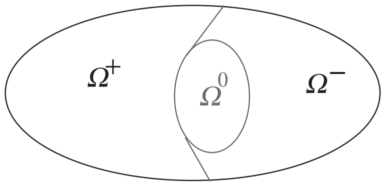

FIG. 2. Decomposition of Ω.

where

$$\begin{aligned} \Delta E = 0,\ E(\underline{x}, t) > 0 \quad &\text{in}\ \ \Omega^+(t) \\ \Delta E = 0,\ E(\underline{x}, t) < 0 \quad &\text{in}\ \ \Omega^-(t) \\ E(\underline{x}, t) = 0 \quad &\text{in}\ \ \Omega^o(t). \end{aligned}$$

Since J is unique, the sets $\Omega^+(t),\ \ \Omega^-(t)$ and $\Omega^o(t)$ are unique. Hence, if $\Omega^o(t)$ is a non-empty open set then E is unique. Furthermore, since $\int_\Omega J = 0$ it follows that even if $\Omega^o(t)$ is empty then $\Gamma(t) = \partial\Omega^+(t) \cap \partial\Omega^-(t)$ is non-empty and continuity of E would imply that E is unique. Thus one conjectures that E solving ($\mathbf{P_R}$) is unique.

3. Finite-element approximation

3.1 *Notation*

In this section we consider a finite-element approximation of ($\mathbf{P_R}$). We make the following assumptions on the partitioning.

Let T^h be a quasi-uniform partitioning of B_R into disjoint open simplicial elements $\kappa \in T^h$ such that $\cup_{\kappa \in T^h} \overline{\kappa} = \overline{B}_R$ and h is the largest diameter of the elements κ in T^h. Furthermore if $\bar{\kappa} \cap (\partial\Omega \cup \partial\Omega_w \cup \partial B_R)$ is non-empty then the intersection consists of either one vertex of κ or one curved edge of κ. There exist subsets $T^h_\omega \subset T^h$ such that $\cup_{\kappa \in T^h_\omega} \bar{\kappa} = \bar{\omega}$ with $\omega = B_R,\ \Omega$ or Ω_w.

Associated with T^h are the finite-element spaces

$$\begin{aligned} S^h(\equiv S^h_{B_R}) &:= \left\{ \eta \in C(\overline{B_R}) : \eta|_\kappa \ \text{ is linear } \forall\, \kappa \in T^h \right\} \\ S^h_0 &:= \{\eta \in S^h : (\eta, 1)^h = 0\} \\ S^h_e &:= \left\{ \eta \in S^h : \int_\Omega \eta = 0 \right\} \\ S^h_\Omega &:= \{\eta \in L^2(B_R) : \eta \in C(\overline{\Omega}) : \eta|_\kappa \ \text{ is linear } \forall\, \kappa \in T^h_\Omega,\ \eta|_{B_R \setminus \overline{\Omega}} = 0\} \\ S^h_{\Omega_w} &:= \{\eta \in L^2(B_R) : \eta \in C(\overline{\Omega}_w) : \eta|_\kappa \ \text{ is linear } \forall\, \kappa \in T^h_{\Omega_w},\ \eta|_{B_R \setminus \overline{\Omega}_w} = 0\} \\ S^h_{0,\Omega} &:= \left\{ \eta \in S^h_\Omega : \fint_{B_R} \eta = 0 \right\} \\ S^h_{0,\Omega_w} &:= \left\{ \eta \in S^h_{\Omega_w} : \fint_{B_R} \eta = 0 \right\} \end{aligned}$$

$$K^h_\Omega := \{\eta \in S^h_\Omega : |\eta| \leqslant J_c\}$$
$$K^h_{0,\Omega} := \{\eta \in S^h_{0,\Omega} : |\eta| \leqslant J_c\}$$

where

$$(\eta_1, \eta_2)^h := \sum_{\kappa \in T^h_\omega} \int_\kappa \Pi^h(\eta_1\eta_2)\,\mathrm{d}\underline{x}, \quad \eta_i \in S^h_\omega := S^h_{B_R}, S^h_\Omega, S^h_{\Omega_w}$$

and Π^h is the standard linear interpolant which interpolates continuous functions at the vertices of κ. Observe that $S^h_e \subset H^1_e(B_R)$ since

$$(\eta, 1)^h = (\eta, 1) \qquad \forall\, \eta \in S^h \cup S^h_\Omega \cup S^h_{\Omega_w}.$$

For $\xi, \eta \in S^h_\omega$ we define

$$I^h(\xi, \eta) := (\xi, \eta) - (\xi, \eta)^h$$

and we note the well known result

$$|I^h(\xi, \eta)| = |(\xi, \eta) - (\xi, \eta)^h| \leqslant Ch^2|\xi|_{1,\omega}|\eta|_{1,\omega} \leqslant Ch|\xi|_{1,\omega}|\eta|_{0,\omega}. \tag{3.1}$$

Analogous to (2.20) it is convenient to introduce the operator $G^h : L^2_0(B_R) \to S^h_e$ such that for any $\xi \in L^2_0(B_R)$, $G^h\xi \in S^h_e$ is the unique solution of

$$A(G^h\xi, \psi) = (\xi, \psi) \quad \forall\, \psi \in S^h. \tag{3.2}$$

For $\eta \in L^2_0(B_R)$ we set

$$\|\eta\|_{A^{-h}} := \left\|G^h\eta\right\|_A = (\eta, G^h\eta)^{1/2} \quad \forall \eta \in L^2_0(B_R)$$

and we note that

$$(G^h\xi, \psi) = (\xi, G^h\psi) \qquad \forall\, \xi, \psi \in L^2_0(B_R) \tag{3.3}$$

and that

$$(\xi, \psi) \;\leqslant\; C\,\|\xi\|_{A^{-h}}\,|\psi|_A \qquad \forall\, \psi \in S^h,\ \xi \in L^2_0(B_R). \tag{3.4}$$

Standard finite-element estimates, see Han & Wu (1985), yield

$$\left|(G - G^h)\eta\right|_{0,B_R} + h\left|(G - G^h)\eta\right|_{1,B_R} \;\leqslant\; Ch^2|\eta|_{0,B_R} \qquad \forall\, \eta \in L^2_0(B_R). \tag{3.5}$$

It is easy to see that

$$|G^h\eta|_A \leqslant |G\eta|_A \qquad \forall\, \eta \in L^2_0(B_R) \tag{3.6}$$

and since

$$|\eta|^2_{0,B_R} = A(G\eta, \eta) \leqslant C|G\eta|_A||\eta||_{1,B_R} \qquad \forall\, \eta \in S^h_0$$

a standard inverse inequality yields

$$|\eta|_{0,B_R} \leqslant Ch^{-1}|G\eta|_A \quad \forall\, \eta \in S^h_0, \tag{3.7}$$

which implies, using the error bound (3.5),

$$|G\eta|_A \leqslant C|G^h\eta|_A \qquad \forall\, \psi \in S^h_0. \tag{3.8}$$

3.2 *Continuous in time discretization*

We now introduce a continuous in time finite-element approximation of ($\mathbf{P_R}$):

For $J_h^0 \in K^h_{0,\Omega}$ and $f_h(\cdot, t) \in S^h_{0,\Omega_w}$ satisfying

$$\int_0^T \left| G^h f_h \right|_A^2 \leqslant C \tag{3.9}$$

we have the following.

($\mathbf{P_R^h}$) Find for $t \in (0, T]$, $J_h(\cdot, t) \in K^h_{0,\Omega}$ and $E_h(\cdot, t) \in S^h$ such that

$$(\partial_t J_h, \psi) + A(E_h, \psi) = (f_h, \psi) \quad \forall\, \psi \in S^h \tag{3.10}$$

$$J_h(\underline{x}, 0) = J_0^h(\underline{x}) \qquad \forall\, \underline{x} \in \Omega \tag{3.11}$$

$$(E_h(\cdot, t), \eta - J_h(\cdot, t)) \leqslant 0 \quad \forall\, \eta \in K^h_\Omega,\ \forall\, t \in (0, T]. \tag{3.12}$$

For $\chi \in S_0^h$, setting $\psi = G^h \chi$ in (3.10) and noting (3.12) yields the following variational formulation of ($\mathbf{P_R^h}$).

($\mathbf{Q_R^h}$) Find $J_h \in L^\infty(0, T; K^h_{0,\Omega})$ such that

$$\left(\partial_t G^h J_h, \chi - J_h\right) \geqslant \left(G^h f_h, \chi - J_h\right) \quad \forall\, \chi \in K^h_{0,\Omega}. \tag{3.13}$$

PROPOSITION 3.1 If (J_h, E_h) is a solution of ($\mathbf{P_R^h}$) then J_h is unique and E_h is unique up to an additive function of time. Furthermore, J_h is the unique solution of ($\mathbf{Q_R^h}$).

Proof. This follows by using the arguments presented in the proof of Proposition 2.1. □

For the forthcoming error analysis we require

$$\int_0^T || f_h - f ||^2_{A^{-1}}\, \mathrm{d}t \leqslant Ch. \tag{3.14}$$

REMARK Taking $f_h \in S^h_{\Omega_w}$ to be the solution of

$$(f_h(t), \chi) = (f, \chi) \qquad \forall\, \chi \in S^h_{\Omega_w}$$

yields (3.14).

LEMMA 3.1 The unique solutions of ($\mathbf{Q_R}$) and ($\mathbf{Q_R^h}$) satisfy

$$||J - J_h||_{L^\infty(0,T;A^{-1})} \leqslant C(T)h^{1/2}. \tag{3.15}$$

Proof. See the proof of Lemma 3.2 in Elliott *et al.* (2004). □

4. Fully discrete model

In this section we consider a fully discrete discretization of $(\mathbf{P_R})$. We set $N\Delta t = T$, $t_n := n\Delta t$ for $n = 0 \to N$, $f_h^n \in S^h_{0,\Omega_w}$ to be an approximation to $f(\cdot, t_n)$ and for any $g_h \in S^h$ we define

$$\delta_t g_h^n = \frac{g_h^n - g_h^{n-1}}{\Delta t}.$$

We introduce the operator $\hat{G}^h : (S^h_{0,\Omega}, S^h_{0,\Omega_w}) \to S^h_e$ such that for any $\xi \in (S^h_{0,\Omega}, S^h_{0,\Omega_w})$, $\hat{G}^h\xi \in S^h_e$ is the unique solution of

$$A(\hat{G}^h\xi, \psi) = (\xi, \psi)^h \quad \forall\, \psi \in S^h. \tag{4.1}$$

We set

$$||\eta||^2_{\hat{A}^{-h}} := |\hat{G}^h\eta|^2_A = (\hat{G}^h\eta, \eta)^h \qquad \forall\, \eta \in S^h_{0,\Omega} \cup S^h_{0,\Omega_w}$$

and we note using (3.1) that

$$|(G^h - \hat{G}^h)\eta|_A \leqslant Ch|\eta|_{0,B_R} \qquad \forall\, \eta \in S^h_{0,\Omega} \cup S^h_{0,\Omega_w} \tag{4.2}$$

and hence from (3.6) we have that

$$||\eta||^2_{\hat{A}^{-h}} \leqslant ||\eta||^2_{A^{-1}} + Ch^2|\eta|^2_{0,B_R} \qquad \forall\, \eta \in S^h_{0,\Omega} \cup S^h_{0,\Omega_w}. \tag{4.3}$$

Furthermore, from (3.2) and (4.1) we note that

$$(\xi, G^h\psi)^h = (\hat{G}^h\xi, \psi) \qquad \forall\, \xi, \psi \in S^h_{0,\Omega} \cup S^h_{0,\Omega_w}. \tag{4.4}$$

We consider the following fully discrete discretization of $(\mathbf{P_R})$.

$(\mathbf{P_R^{h,\Delta t}})$ Find $\{J_h^n, E_h^n\}_{n\geqslant 1} \in K^h_{0,\Omega} \times S^h$ such that

$$(\delta_t J_h^n, \psi)^h + A(E_h^n, \psi) = (f_h^n, \psi)^h \quad \forall\, \psi \in S^h \tag{4.5}$$

$$J_h^0(\underline{x}) = J_0^h(\underline{x}) \quad \forall\, \underline{x} \in \Omega \tag{4.6}$$

$$(E_h^n, \eta - J_h^n)^h \leqslant 0 \quad \forall\, \eta \in K^h_\Omega. \tag{4.7}$$

For $\chi \in S^h_0$, setting $\psi = \hat{G}^h\chi$ in (4.5) and noting (4.7) yields the following variational formulation of $(\mathbf{P_R^{h,\Delta t}})$.

$(\mathbf{Q_R^{h,\Delta t}})$ Find $\{J_h^n\}_{n\geqslant 1} \in K^h_{0,\Omega}$ such that

$$\left(\hat{G}^h(\delta_t J_h^n), \chi - J_h^n\right)^h \geqslant \left(\hat{G}^h f_h^n, \chi - J_h^n\right)^h \qquad \forall\, \chi \in K^h_{0,\Omega}. \tag{4.8}$$

PROPOSITION 4.1 Let $\Delta t = Ch$ and $\sum_{n=1}^N \Delta t|f_h^n|_h^2 \leqslant C$. Then there exists a solution pair $\{J_h^n, E_h^n\}_{n\geqslant 0}$ to $(\mathbf{P_R^{h,\Delta t}})$ such that

$$\sum_{n=1}^N \Delta t \left\|\delta_t J_h^n\right\|^2_{\hat{A}^{-h}} + h\sum_{n=1}^N \Delta t|\delta_t J_h^n|^2_{0,B_R} + \Delta t\sum_{n=1}^N |E_h^n|^2_A \leqslant C. \tag{4.9}$$

Also, for each n, J_h^n is unique and E_h^n is unique up to an additive constant. Furthermore, $\{J_h^n\}_{n\geqslant 0}$ is the unique solution of $(\mathbf{Q_R^{h,\Delta t}})$.

Proof. Let I^{Ω} be the index set of triangle vertices $\underline{x}_i \in \overline{\Omega}$. Let $E_i^n := E_h^n(\underline{x}_i)$ and $J_i^n := J_h^n(\underline{x}_i)$ for $i \in I^{\Omega}$. It is easy to see that (4.7) is equivalent to

$$|J_i^n| \leqslant J_c \text{ and } E_i^n(\psi - J_i^n) \leqslant 0 \quad \forall\, |\psi| \leqslant J_c. \tag{4.10}$$

Elementary calculations yield the equivalence of (4.10) with

$$J_c(|\psi| - |E_i^n|) \geqslant J_i^n(\psi - E_i^n) \quad \forall\, |\psi| \leqslant J_c \tag{4.11}$$

and also the equivalence with

$$J_i^n \in J_c \operatorname{sign} E_i^n. \tag{4.12}$$

It follows that (4.5) and (4.7) are equivalent to

$$J_c \int_{\Omega} \Pi^h(|\psi| - |E_h^n|)\,\mathrm{d}\underline{x} + \Delta t\, A(E_h^n, \psi - E_h^n) \geqslant (J_h^{n-1} + \Delta t\, f_h^n, \psi - E_h^n)^h \qquad \forall\, \psi \in S^h.$$

This is a necessary condition for E_h^n to be a solution of the minimization problem

$$\mathcal{F}(E_h^n) := \min_{\psi \in S^h} \mathcal{F}(\psi)$$
$$\mathcal{F}(\psi) := J_c \int_{\Omega} \Pi^h(|\psi|)\,\mathrm{d}\underline{x} + \frac{\Delta t}{2} A(\psi, \psi) - (J_h^{n-1} + \Delta t\, f_h^n, \psi)^h.$$

Since $\mathcal{F}$ is continuous and bounded below (using the fact that $A(\cdot,\cdot)$ is positive definite on S_0^h) there exists a minimizer E_h^n, and hence by (4.5) there also exists a $J_h^n \in S^h$. Furthermore, by the above equivalences it follows that for $J_h^n \in S_{0,\Omega}^h$ we have also $J_h^n \in K_{0,\Omega}^h$ and that (4.7) holds. Hence we have existence of a solution pair $\{J_h^n, E_h^n\}$.

Suppose $\{J_h^n, E_h^n\}$ and $\{\tilde{J}_h^n, \tilde{E}_h^n\}$ are two separate solution pairs. It follows from (4.5) and (4.7) that

$$(J_h^n - \tilde{J}_h^n, \psi)^h + \Delta t\, A(E_h^n - \tilde{E}_h^n, \psi) = 0 \qquad \forall\, \psi \in S^h$$

and

$$(E_h^n - \tilde{E}_h^n, J_h^n - \tilde{J}_h^n) \geqslant 0.$$

This immediately implies that J_h^n is unique and that E_h^n is unique up to an additive constant. Furthermore, it follows from (4.1) and (4.5) that

$$E_h^n = \hat{G}^h(f_h^n - \delta_t J_h^n) + \lambda_h^n$$

for a scalar λ_h^n. By considering (4.7) for $\eta \in K_{0,\Omega}^h$ we obtain (4.8) which implies that J_h^n is the unique solution of $(\mathbf{Q}_{\mathbf{R}}^{\mathbf{h},\Delta\mathbf{t}})$.

Taking $\chi = J_h^{n-1}$ in (4.8) we obtain

$$|\hat{G}^h \delta_t J_h^n|_A^2 \leqslant (\hat{G}^h f_h^n, \delta_t J_h^n)^h$$

and so

$$||\delta_t J_h^n||_{\hat{A}^{-h}} \leqslant ||f_h^n||_{\hat{A}^{-h}}.$$

Setting $\tilde{J}_h^n \in S^h$ to be the interpolant of J_h^n we observe that, see Elliott (1987),

$$A(E_h^n, \tilde{J}_h^n) = \int_{B_R} \frac{1}{\mu} \nabla E_h^n \nabla \tilde{J}_h^n \geqslant 0$$

and hence it follows from (4.5) that

$$|\tilde{J}_h^n|_h^2 - |\tilde{J}_h^{n-1}|_h^2 + |\tilde{J}_h^n - \tilde{J}_h^{n-1}|_h^2 \leqslant 2\Delta t (f_h^n, \tilde{J}_h^n)^h \leqslant C \Delta t |f_h^n|_h.$$

By elementary calculations we have that the required bounds hold. □

Before we derive an error bound on the solutions of ($\mathbf{P_R^h}$) and ($\mathbf{P_R^{h,\Delta t}}$) we introduce some useful notation. For $n \geqslant 1$ we set

$$J_{h,\Delta t}(t) := \frac{t - t_{n-1}}{\Delta t} J_h^n + \frac{t_n - t}{\Delta t} J_h^{n-1}, \quad f_{h,\Delta t}(t) := \frac{t - t_{n-1}}{\Delta t} f_h^n + \frac{t_n - t}{\Delta t} f_h^{n-1} \ \ \forall\, t \in [t_{n-1}, t_n], \tag{4.13}$$

and

$$\hat{J}_{h,\Delta t}(t) := J_h^n, \quad \hat{f}_{h,\Delta t}(t) := f_h^n \quad \forall\, t \in (t_{n-1}, t_n]. \tag{4.14}$$

From (4.13) and (4.14) it follows that for a.e. $t \in (0, T)$

$$J_{h,\Delta t} - \hat{J}_{h,\Delta t} = -(t_n - t)\partial_t J_{h,\Delta t}. \tag{4.15}$$

For the forthcoming error analysis we require that

$$\int_0^T || f_h - \hat{f}_{h,\Delta t} ||_{\hat{A}^{-h}}^2 \, \mathrm{d}t \leqslant C \Delta t. \tag{4.16}$$

REMARK Taking

$$f_h^n := \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f_h(t) \, \mathrm{d}t$$

yields (4.16).

LEMMA 4.1 For $\Delta t = Ch$ the unique solutions of ($\mathbf{Q_R^h}$) and ($\mathbf{Q_R^{h,\Delta t}}$) satisfy

$$\| J_h - J_{h,\Delta t} \|_{A^{-1}} \leqslant C(T)(h + \Delta t)^{1/2}. \tag{4.17}$$

Proof. Setting $\chi = J_{h,\Delta t}$ in (3.13) and noting (4.4) we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \| J_h - J_{h,\Delta t} \|_{A^{-h}}^2 &= \Big(\partial_t G^h J_h, J_h - J_{h,\Delta t}\Big) - \Big(\partial_t G^h J_{h,\Delta t}, J_h - J_{h,\Delta t}\Big) \\
&\leqslant \Big(G^h f_h, J_h - J_{h,\Delta t}\Big) - \Big(\partial_t G^h J_{h,\Delta t}, J_h - J_{h,\Delta t}\Big) \\
&= I^h(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}) + \Big(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}\Big)^h \\
&= I^h(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}) + \Big(f_h - \partial_t J_{h,\Delta t}, \hat{G}^h(J_h - J_{h,\Delta t})\Big) \\
&= I^h(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}) + I^h(f_h - \partial_t J_{h,\Delta t}, \hat{G}^h(J_h - J_{h,\Delta t})) \\
&\quad + \Big(\hat{G}^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}\Big)^h.
\end{aligned} \tag{4.18}$$

Setting $\chi = J_h$ in (4.8) we have

$$\begin{aligned}\left(\partial_t \hat{G}^h J_{h,\Delta t}\ ,\ J_h - J_{h,\Delta t}\right)^h &= \left(\partial_t \hat{G}^h J_{h,\Delta t}, J_h - \hat{J}_{h,\Delta t}\right)^h + \left(\partial_t \hat{G}^h J_{h,\Delta t}, \hat{J}_{h,\Delta t} - J_{h,\Delta t}\right)^h \\ &\geqslant \left(\hat{G}^h \hat{f}_{h,\Delta t}, J_h - \hat{J}_{h,\Delta t}\right)^h + \left(\partial_t \hat{G}^h J_{h,\Delta t}, \hat{J}_{h,\Delta t} - J_{h,\Delta t}\right)^h \\ &= \left(\hat{G}^h \hat{f}_{h,\Delta t}, J_h - J_{h,\Delta t}\right)^h + \left(\hat{G}^h (\partial_t J_{h,\Delta t} - \hat{f}_{h,\Delta t}), \hat{J}_{h,\Delta t} - J_{h,\Delta t}\right)^h. \end{aligned} \tag{4.19}$$

From (4.18), (4.19), (3.1) and Propositions 3.1 and 4.1 we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||J_h - J_{h,\Delta t}||^2_{A^{-h}} &\leqslant I^h(G^h(f_h - \partial_t J_{h,\Delta t}), J_h - J_{h,\Delta t}) + I^h(f_h - \partial_t J_{h,\Delta t}, \hat{G}^h(J_h - J_{h,\Delta t})) \\ &\quad + \left(\hat{G}^h(f_h - \hat{f}_{h,\Delta t}), J_h - J_{h,\Delta t}\right)^h - \left(\hat{G}^h(\partial_t J_{h,\Delta t} - \hat{f}_{h,\Delta t}), \hat{J}_{h,\Delta t} - J_{h,\Delta t}\right)^h \\ &\leqslant Ch\left|G^h(f_h - \partial_t J_{h,\Delta t})\right|_{1,B_R}\left|J_h - J_{h,\Delta t}\right|_{0,B_R} + Ch\left|f_h - \partial_t J_{h,\Delta t}\right|_{0,B_R}\left|\hat{G}^h(J_h - J_{h,\Delta t})\right|_{1,B_R} \\ &\quad + \left\| f_h - \hat{f}_{h,\Delta t}\right\|_{\hat{A}^{-h}} \left\|J_h - J_{h,\Delta t}\right\|_{\hat{A}^{-h}} + \left\|\partial_t J_{h,\Delta t} - \hat{f}_{h,\Delta t}\right\|_{\hat{A}^{-h}} \left\|\hat{J}_{h,\Delta t} - J_{h,\Delta t}\right\|_{\hat{A}^{-h}} \\ &\leqslant Ch + Ch\left|f_h - \partial_t J_{h,\Delta t}\right|_{0,B_R} \left\|J_h - J_{h,\Delta t}\right\|_{\hat{A}^{-h}} + \left\| f_h - \hat{f}_{h,\Delta t}\right\|_{\hat{A}^{-h}} \left\|J_h - J_{h,\Delta t}\right\|_{\hat{A}^{-h}} \\ &\quad + \left\|\partial_t J_{h,\Delta t} - \hat{f}_{h,\Delta t}\right\|_{\hat{A}^{-h}} \left\|\hat{J}_{h,\Delta t} - J_{h,\Delta t}\right\|_{\hat{A}^{-h}}. \end{aligned} \tag{4.20}$$

Using (4.20), (3.8), (4.3), (4.15), (3.7) and Young's inequality we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|J_h - J_{h,\Delta t}\right\|^2_{A^{-1}} &\leqslant Ch + Ch^2\left|f_h - \partial_t J_{h,\Delta t}\right|^2_{0,B_R} + C\left\|J_h - J_{h,\Delta t}\right\|^2_{A^{-1}} + C\left\|f_h - \hat{f}_{h,\Delta t}\right\|^2_{\hat{A}^{-h}} \\ &\quad + C\Delta t\left\|\partial_t J_{h,\Delta t} - \hat{f}_{h,\Delta t}\right\|_{\hat{A}^{-h}} \left\|\partial_t J_{h,\Delta t}\right\|_{A^{-1}}. \end{aligned}$$

The result follows using a Grönwall inequality, (4.9) and (4.16). □

Finally from Lemmas 3.1 and 4.1 we have our main result.

THEOREM 4.1 The unique solutions of ($\mathbf{Q}_\mathbf{R}^{\mathbf{h},\Delta \mathbf{t}}$) and ($\mathbf{Q}_\mathbf{R}$) satisfy

$$\left\|J - J_{h,\Delta t}\right\|_{L^\infty(0,T;A^{-1})} \leqslant C(T)(h + \Delta t)^{1/2}.$$

5. The Gauss–Seidel iteration

It is easy to see that the fully discrete scheme ($\mathbf{P}_\mathbf{R}^{\mathbf{h},\Delta \mathbf{t}}$) yields the following algebraic problem.

Find $(\mathbf{J}, \mathbf{E}) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ such that

$$\begin{aligned} M\mathbf{J} + A\mathbf{E} - \mathbf{b} &= \mathbf{0}, \\ J_i &= 0, \qquad i \notin I^\Omega, \\ J_i &\in J_c \operatorname{sign} E_i, \qquad i \in I^\Omega. \end{aligned}$$

Here $\mathbf{J}$ and $\mathbf{E}$ are the nodal values of J_h^n and E_h^n at the vertices of the triangulation according to some ordering. We denote by I^{Ω} the set of vertices on $\overline{\Omega}$ and set $J_i = 0$ for all $i \notin I^{\Omega}$. The diagonal mass matrix M^{Ω} is defined by

$$M_{ii}^{\Omega} = \begin{cases} \int_{\Omega} \chi_i \, \mathrm{d}\underline{x} & i \in I^{\Omega} \\ 0 & i \notin I^{\Omega}, \end{cases}$$

where χ_i is the basis function associated with node i. A is the symmetric positive semi-definite matrix defined by

$$\boldsymbol{\xi}^T A \boldsymbol{\psi} = A(\boldsymbol{\xi}, \boldsymbol{\psi}) \qquad \xi, \psi \in S^h$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\psi}$ are the nodal values of ξ and ψ. It follows that

$$A\mathbf{e} = 0,$$

and

$$\boldsymbol{\xi}^T A \boldsymbol{\xi} \geqslant C_A ||\boldsymbol{\xi}||^2 \quad \forall\, \boldsymbol{\xi} \text{ such that } \boldsymbol{\xi}^T \mathbf{e} = 0$$

where $\{\mathbf{e}\}_j = 1$ for all j and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{\Lambda}$. The right-hand side $\mathbf{b}$ is defined by

$$\mathbf{b}^T \psi = (J_h^{n-1} + \Delta t\, f_h^n, \psi)^h \quad \psi \in S^h$$

and

$$\mathbf{b}^T \mathbf{e} = 0,$$

since $J_h^{n-1} \in S_{0,\Omega}^h$ and $f_h^n \in S_{0,\Omega_w}^h$. We set $|\mathbf{v}| := (|v_1|, |v_2|, \ldots, |v_{\Lambda}|)^T$ and $\mathbf{v_p} := \mathbf{v} - \frac{1}{\Lambda}\mathbf{e}^T \mathbf{v}\mathbf{e}$.

In order to solve this problem we set out a version of the Gauss–Seidel iteration formulated by Elliott (1987), for the enthalpy method for the Stefan problem.

Gauss–Seidel Iteration

Given $\mathbf{E}^0$, for $k \geqslant 1$, $\{\mathbf{E}^k, \mathbf{J}^k\}$ are defined as follows.

For $i = 1 \to \Lambda$, (J_i^{k+1}, E_i^{k+1}) are the unique solutions of

$$(A\mathbf{E}^{i-1,k+1} - \mathbf{b})_i + A_{ii}(E_i^{k+1} - E_i^k) + M_{ii} J_i^{k+1} = 0 \tag{5.1}$$

$$J_i^{k+1} = 0 \qquad i \notin I^{\Omega} \tag{5.2}$$

$$J_i^{k+1} \in J_c \operatorname{sign} E_i^{k+1} \qquad i \in I^{\Omega}, \tag{5.3}$$

where

$$\mathbf{E}^{i,k+1} := (E_1^{k+1}, E_2^{k+1}, \ldots, E_i^{k+1}, E_{i+1}^k, \ldots, E_{\Lambda}^k)^T \qquad i = 0 \to \Lambda.$$

As noted in the proof of existence in Proposition 4.1, this problem is associated with energy minimization.

We set

$$\begin{aligned} \mathcal{F}(\mathbf{E}) &:= J_c (M^{\Omega}\mathbf{e})^T |\mathbf{E}| + \frac{1}{2}\mathbf{E}^T A\mathbf{E} - \mathbf{b}^T \mathbf{E} \\ &= J_c (M^{\Omega}\mathbf{e})^T |\mathbf{E}| + \frac{1}{2}\mathbf{E}_\mathbf{p}^T A\mathbf{E}_\mathbf{p} - \mathbf{b}^T \mathbf{E}_\mathbf{p} \\ &\geqslant J_c \mathbf{e}^T M^{\Omega} |\mathbf{E}| + \frac{1}{2} C_{A,b} ||\mathbf{E}_\mathbf{p}||^2 - \hat{C}_{A,b}. \end{aligned}$$

Hence $\mathcal{F}(\mathbf{E})$ is bounded below and $||\mathbf{E}|| \leqslant C(\mathcal{F}(\mathbf{E}), A, \mathbf{b})$.

We define

$$\mathcal{F}_i^k(z) := \mathcal{F}(E_1^{k+1}, \ldots, E_{i-1}^{k+1}, z, E_{i+1}^k, \ldots, E_\Lambda^k).$$

Clearly,

$$\mathcal{F}_i^k(E_i^{k+1}) = \mathcal{F}(\mathbf{E}^{i,k+1}) \quad \text{and} \quad \mathcal{F}_i^k(E_i^k) = \mathcal{F}(\mathbf{E}^{i-1,k+1}).$$

Furthermore,

$$\sum_{i=1}^{\Lambda} \left(\mathcal{F}_i^k(E_i^{k+1}) - \mathcal{F}_i^k(E_i^k) \right) = \mathcal{F}(\mathbf{E}^{\Lambda,k+1}) - \mathcal{F}(\mathbf{E}^{0,k+1})$$
$$= \mathcal{F}(\mathbf{E}^{k+1}) - \mathcal{F}(\mathbf{E}^k). \tag{5.4}$$

LEMMA 5.1 The above iteration satisfies

$$\mathcal{F}(\mathbf{E}^{k+1}) - \mathcal{F}(\mathbf{E}^k) \leqslant -C_A \left\| \mathbf{E}^{k+1} - \mathbf{E}^k \right\|^2$$

and

$$\left\| \mathbf{J}^k \right\|_\infty \leqslant J_c,$$

for all $k \geqslant 0$.

Proof. A straightforward calculation gives

$$\delta_i^k := \mathcal{F}_i^k(E_i^{k+1}) - \mathcal{F}_i^k(E_i^k) = \frac{1}{2} A_{ii} (E_i^{k+1} - E_i^k)^2 + (A\mathbf{E}^{i-1,k+1} - \mathbf{b})_i (E_i^{k+1} - E_i^k)$$
$$+ J_c M_{ii} (|E_i^{k+1}| - |E_i^k|).$$

From (5.1) we have

$$\delta_i^k = -\frac{1}{2} A_{ii} (E_i^{k+1} - E_i^k)^2 + M_{ii} \left(J_c |E_i^{k+1}| - J_i^{k+1} E_i^{k+1} + J_i^{k+1} E_i^k - J_c |E_i^k| \right).$$

Since $E_i^{k+1} = 0$ if $J_i^{k+1} = 0$ from (5.2) and (5.3) we have

$$J_c |E_i^{k+1}| - J_i^{k+1} E_i^{k+1} = 0 \text{ and } J_i^{k+1} E_i^k - J_c |E_i^k| \leqslant 0 \qquad \text{for } i = 1 \to \Lambda.$$

Noting that $A_{ii} > 0$ for $i = 1 \to \Lambda$ and using (5.4) we have

$$\sum_{i=1}^{\Lambda} \delta_i^k = \mathcal{F}(\mathbf{E}^{k+1}) - \mathcal{F}(\mathbf{E}^k) \leqslant -C_A \left\| \mathbf{E}^{k+1} - \mathbf{E}^k \right\|^2.$$

The bound on $\mathbf{J}^k$ follows directly from (5.2) and (5.3). □

THEOREM 5.1 The Gauss–Seidel iteration is globally convergent.

Proof. By Lemma 5.1 we have

$$\mathcal{F}(\mathbf{E}^k) + C_A \sum_{l=0}^{k-1} ||\mathbf{E}^{l+1} - \mathbf{E}^l||^2 \leqslant \mathcal{F}(\mathbf{E}^0).$$

Hence for $k \geqslant 1$,

$$||\mathbf{E}^k|| \leqslant C, \quad \max_{i \in I^\Omega} |J_i^k| \leqslant C, \quad J_i^k = 0 \;\; i \notin I^\Omega, \quad \sum_{l=0}^{k-1} ||\mathbf{E}^{l+1} - \mathbf{E}^l||^2 \leqslant C,$$

where the constants C depend on $\mathbf{E}^0$. It follows that there is a subsequence labelled $\{\mathbf{E}^{k_p}\}$ such that as $k_p \to \infty$

$$\mathbf{E}^{k_p} \to \mathbf{E}^*, \quad \mathbf{E}^{k_p+1} - \mathbf{E}^{k_p} \to 0, \quad \mathbf{J}^{k_p} \to \mathbf{J}^*.$$

Clearly

$$J_i^* = 0 \;\; i \notin I^\Omega, \quad |J_i^*| \leqslant J_c \quad i \in I^\Omega, \quad \mathbf{e}^T M \mathbf{J}^* = 0.$$

Observe that

$$A\mathbf{E}^{i-1,k_p+1} = A\mathbf{E}^{k_p} + A(\mathbf{E}^{i-1,k_p+1} - \mathbf{E}^{k_p}).$$

Since $||\mathbf{E}k_p + 1 - \mathbf{E}^{k_p}|| \to 0$ it then follows by passing to the limit in (5.1) for $k = k_p$,

$$A\mathbf{E}^* - \mathbf{b} + M\mathbf{J}^* = 0.$$

From the equivalence of (4.10)–(4.12) we have

$$(\mathbf{E}^{k_p})^T M^\Omega(\boldsymbol{\eta} - \mathbf{J}^{k_p}) \leqslant 0 \quad \forall\, \boldsymbol{\eta}, \; |\eta_i| \leqslant J_c,$$

and passing to the limit we have

$$(\mathbf{E}^*)^T M^\Omega(\boldsymbol{\eta} - \mathbf{J}^*) \leqslant 0 \quad \forall\, \boldsymbol{\eta}, \; |\eta_i| \leqslant J_c.$$

Hence $\mathbf{J}^*$, $\mathbf{E}^*$ solve our problem and since $\mathbf{J}^* = \mathbf{J}$ is unique, the whole sequence $\{\mathbf{J}^k\}$ converges to $\mathbf{J}$. □

REMARK If there exists a node i in $\overline{\Omega}$ where $|J_i| < J_c$ then at this node $E_i = 0$ and we have uniqueness of $\mathbf{E}$ and E_h^n. However, it may be possible to have a computation where $|J_h^n| = J_c$ for all nodes in $\overline{\Omega}$. In this case it may be possible to identify a node where $E_h^n = 0$, in which case we again have uniqueness of E_h^n. Otherwise, there will be an indeterminacy of E_h^n and there will exist triangles at which E_h^n is not zero at any node but changes sign. It follows that there exists a largest number which can be added or subtracted from E_h^n such that the sign of E_h^n at the nodes does not change. The size of this number should depend on the discretization error.

6. Numerical results

In this section we report on numerical computations associated with a particular geometric configuration. We suppose that Ω is the interior of a circle of radius 0·5 that is set in an annular region Ω_I with inner radius 0·55 and outer radius 1. Contained in Ω_I are 12 symmetrically arranged components Ω_{w_i} of Ω_w. Each Ω_{w_i} is a section of an annular region with inner radius 0·55 and outer radius 0·8 subtending an angle $\pi/12$, see Fig. 3. This geometric configuration can be used to model superconducting induction

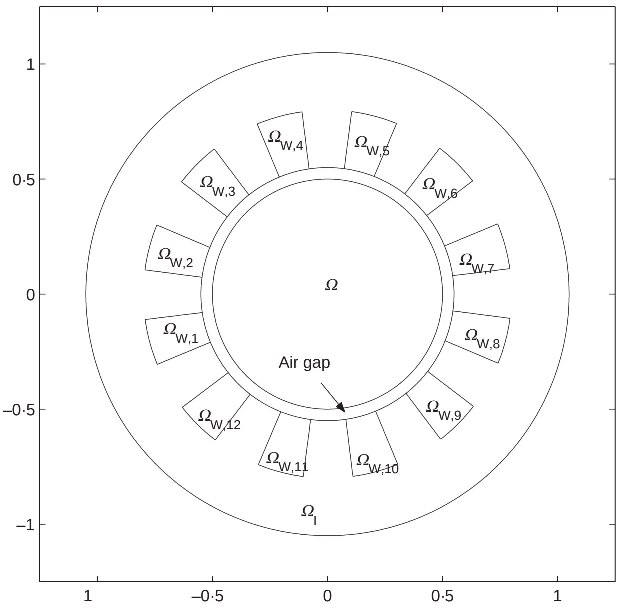

FIG. 3. Geometric configuration.

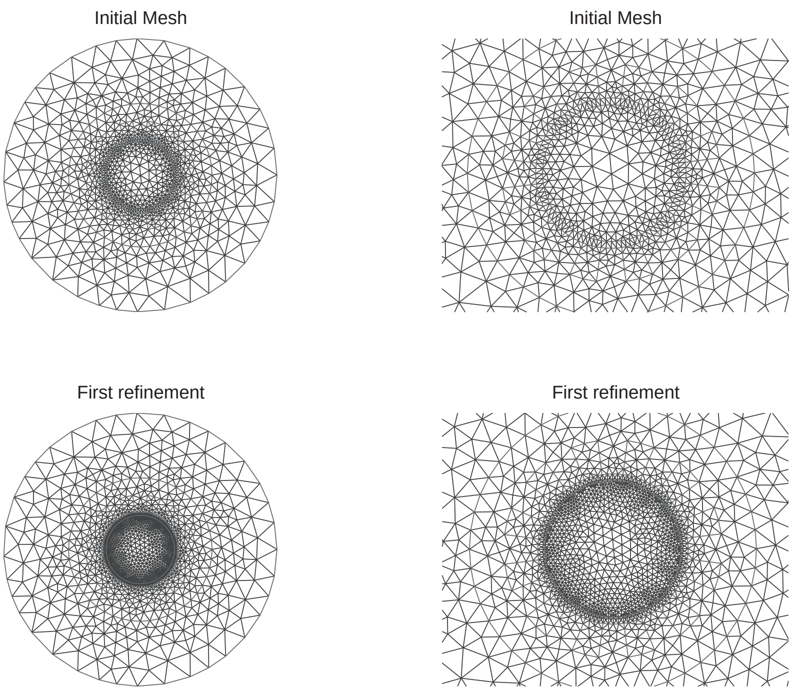

FIG. 4. Initial mesh and first refinement in the superconductor.

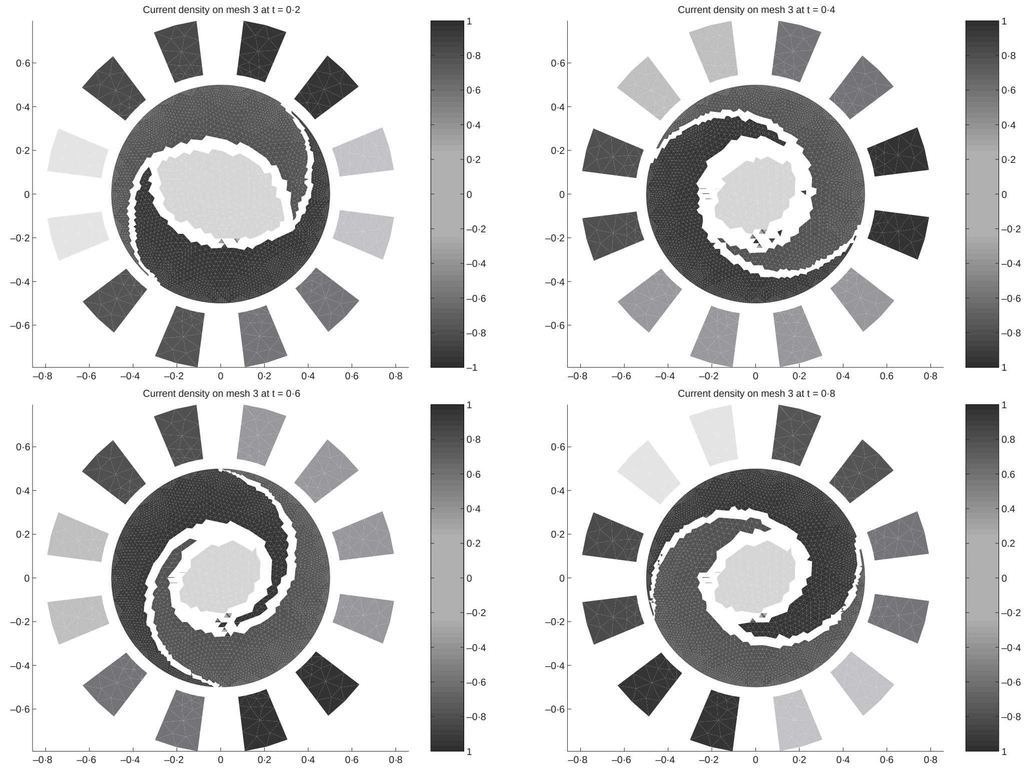

FIG. 5. Current density with $\mu = 1$.

motors by viewing Ω_w as modelling the effect of the copper windings set in an annular laminated iron region Ω_I with a thin air gap separating Ω and Ω_I. Note that because of the horizontal lamination of the iron cylinder $\Omega_I \times \mathbb{R}$ we can assume that current is zero in Ω_I.

The applied source current J_s is given by

$$J_s|_{\Omega_{w,n+1} \cup \Omega_{w,n+2}}(t) = \min(5t, 1) \cos(4t + n\pi/3),$$

for $n = 0, 2, 4,$ and

$$J_s|_{\Omega_{w,n+1} \cup \Omega_{w,n+2}}(t) = -\min(5t, 1) \cos(4t + n\pi/3),$$

for $n = 6, 8, 10$. In all computations B_R has radius 2, and the critical current density $J_c = 1$.

6.1 *Constant magnetic permeability*

Some computations were performed for $\mu = 1$ everywhere in order to test the rate of convergence. Since an exact solution is not known in Table 1 the results on coarser meshes are compared with the solution on a fine mesh with a mesh size $h_{\max} \leqslant 1/128$. Typical meshes are shown in Fig. 4 and the results of some computations are displayed in Figs 5 and 6.

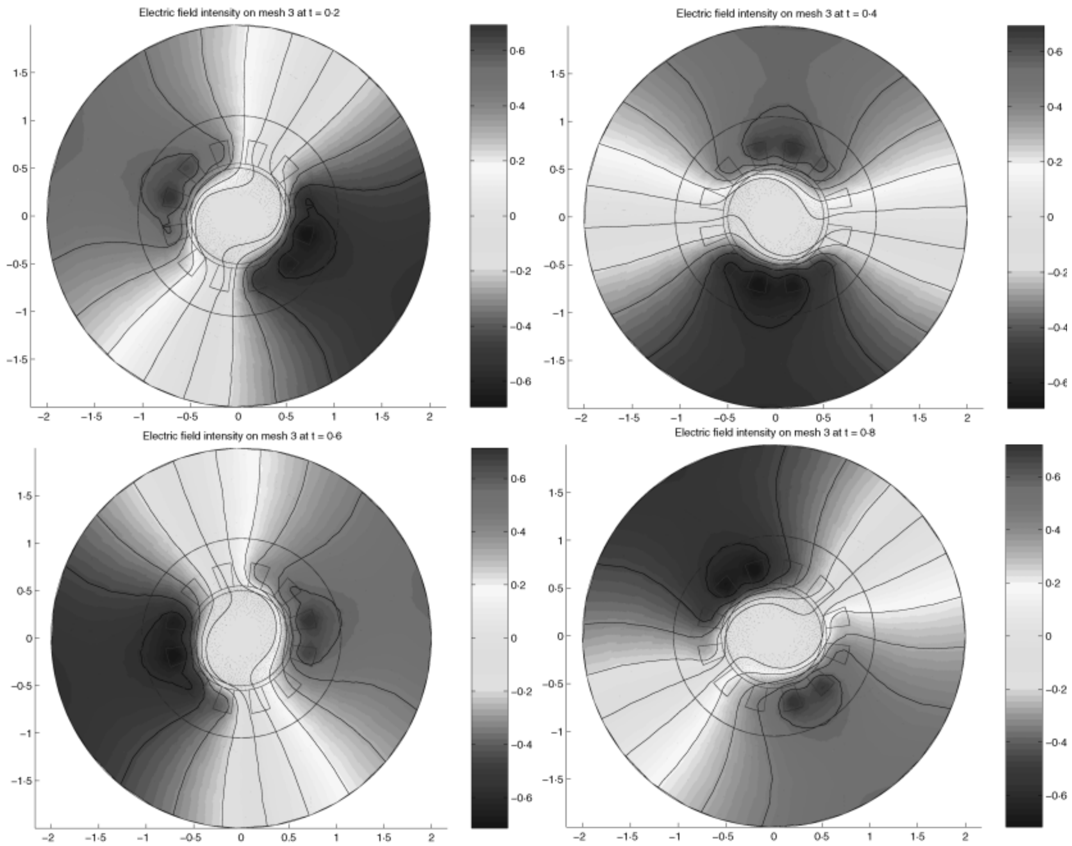

FIG. 6. Electric field intensity with $\mu = 1$.

TABLE 1 $H^{-1}(\Omega)$ *errors for current density*

	$t = 0{\cdot}2$	$t = 0{\cdot}4$	$t = 0{\cdot}6$	$t = 0{\cdot}8$
$h_{ave} \approx 1/8,\ \Delta t = 1/25$	0·0292	0·0278	0·0304	0·0319
$h_{ave} \approx 1/16,\ \Delta t = 1/50$	0·0160	0·0159	0·0166	0·0176
$h_{ave} \approx 1/32,\ \Delta t = 1/100$	0·0066	0·0075	0·0079	0·0080

REMARK From the discrete version of the J–E relationship in the superconductor, (4.12), we note that in our numerical approximations the discrete zero current core coincides with the discrete zero electric field core.

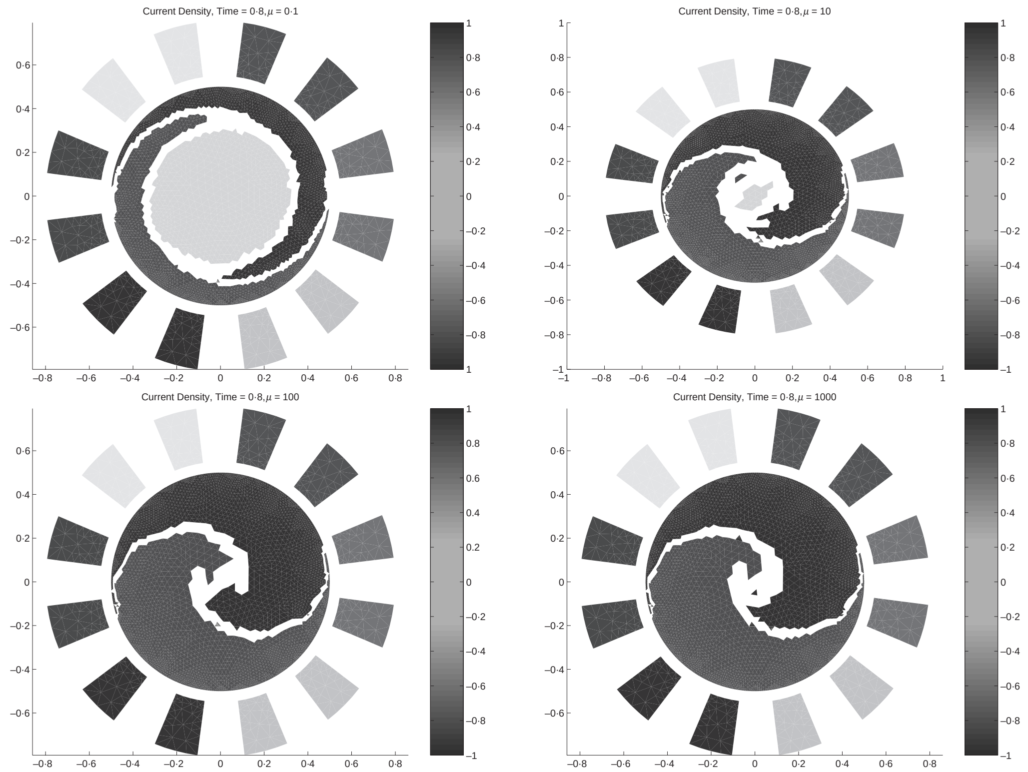

FIG. 7. Current Density at time $t = 0{\cdot}8$ with $\mu = 0{\cdot}1$, 10, 100 and 1000.

6.2 *Piecewise constant permeability*

In order to simulate the high magnetic permeability in the annular iron region Ω_I we set

$$\mu = \begin{cases} 1 & \text{in } \mathbb{R}^2 \backslash \overline{\Omega}_I \\ 10^3 & \text{in } \Omega_I. \end{cases}$$

In Figs 7 and 8 we can clearly see the effect on the amount of current in the superconductor and the insulation of the electric field as μ in the iron increases, as expected.

Note that a larger current density in the superconductor leads to a stronger magnetic field and thus a much more powerful motor.

7. Concluding remarks

In order to compute the current density within the superconductor several approaches have been considered. A common approach is to reformulate (2.14) along with the constitutive relationship (2.4) into an obstacle problem for J. In Prigozhin (1996b) this method is applied when μ is a uniform constant and the non-local operator $\mathcal{G}$ is used explicitly in the form (2.11). The discretization leads to a quadratic programming problem with a dense matrix of size the number of degrees of freedom associated with the

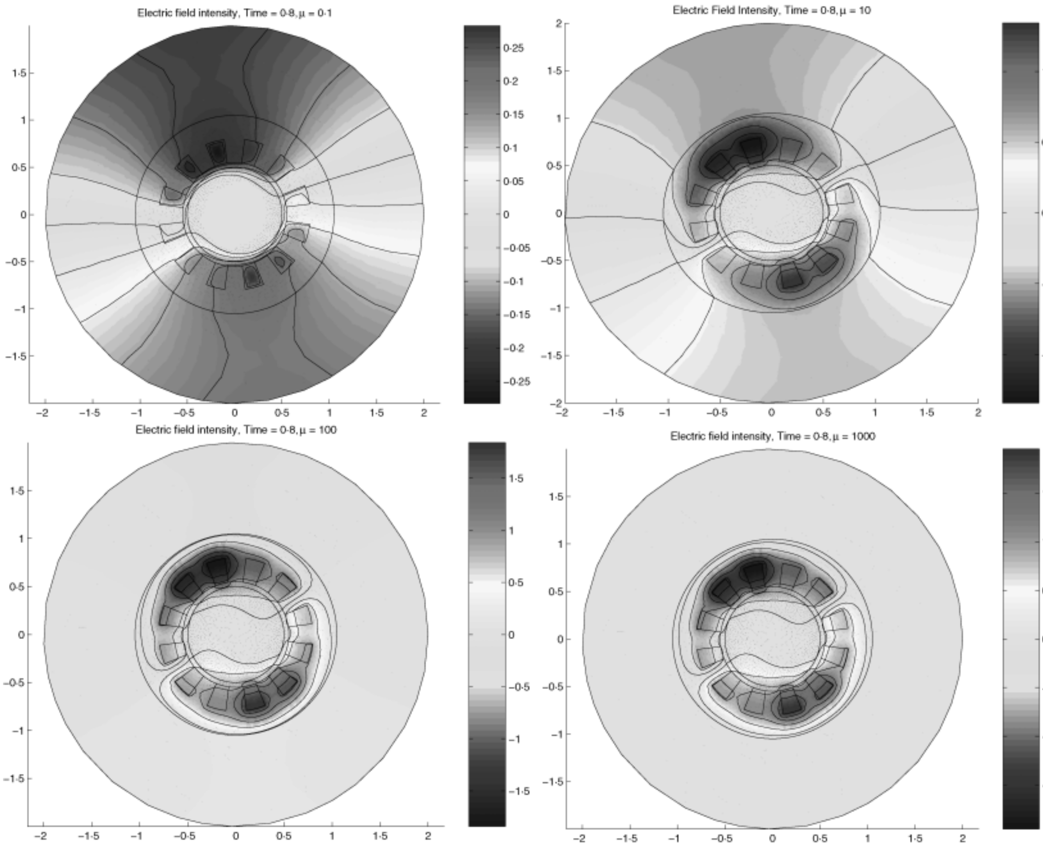

FIG. 8. Electric field intensity at time $t = 0{\cdot}8$ with $\mu = 0{\cdot}1$, 10, 100 and 1000.

domain Ω. This can be solved by projected SOR. The same approach is used in Barnes *et al.* (1999), except a finite-element method is used to form an approximation to $\mathcal{G}$ explicitly by a matrix inversion. The result of the approximation is the current density J and then $E + \lambda$ can be found by (2.14). If E is zero at a node in Ω then, as described in the remark at the end of Section 2, this may be used to identify λ. In Elliott *et al.* (2004) the discretization of (2.9) is indirectly formulated using a discrete Laplacian on B_R. An operator splitting algorithm combined with a nonlinear projection is used to solve the resulting system without explicitly using an approximation of $\mathcal{G}$.

In this paper we have proposed a method that calculates the current density using a Gauss–Seidel iteration for the solution of (2.15). Approximations to J and E are computed simultaneously. This scheme requires no non-local operators and no operator splitting, leading to an efficient scheme.

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