IMA Journal of Numerical Analysis (2007) **27**, 293−331
doi:10.1093/imanum/drl021
Advance Access publication on July 7, 2006

A finite-element analysis of critical-state models for type-II superconductivity in 3D

CHARLES M. ELLIOTT† AND YOHEI KASHIMA‡
Department of Mathematics, University of Sussex, Brighton BN1 9RF, UK

[Received on 23 October 2005; revised on 30 May 2006; accepted on 6 June 2006]

We consider the numerical analysis of evolution variational inequalities which are derived from Maxwell's equations coupled with a nonlinear constitutive relation between the electric field and the current density and governing the magnetic field around a type-II bulk superconductor located in 3D space. The nonlinear Ohm's law is formulated using the subdifferential of a convex energy so the theory is applied to the Bean critical-state model, a power law model and an extended Bean critical-state model. The magnetic field in the nonconducting region is expressed as a gradient of a magnetic scalar potential in order to handle the curl-free constraint. The variational inequalities are discretized in time implicitly and in space by Nédélec's curl-conforming finite element of lowest order. The nonsmooth energies are smoothed with a regularization parameter so that the fully discrete problem is a system of nonlinear algebraic equations at each time step. We prove various convergence results. Some numerical simulations under a uniform external magnetic field are presented.

Keywords: macroscopic models for superconductivity; variational inequality; Maxwell's equations; edge finite element; convergence; computational electromagnetism.

1. Introduction

In this paper, we propose a finite-element method to analyse critical-state problems for type-II superconductivity numerically. In particular, we are interested in analysing the situation where a bulk superconductor is located in a 3D domain. Models of type-II superconductors use the eddy current version of Maxwell's equations together with nonlinear constitutive relations between the current and the electric field such as the Bean critical-state model (Bean, 1964), the extended Bean critical-state models (Bossavit, 1994) or the power law-type relation (Rhyner, 1993) instead of the linear Ohm's law. The numerical study of the Bean critical-state model based on a variational formulation without introducing a free boundary between the region of the critical current and the subcritical current was initiated by Prigozhin (1996a,b). The approach of Prigozhin mathematically treats the electric field as a subdifferential of a critical energy density which takes the value either zero if the current density does not exceed some critical value or infinity otherwise. By analysing the subdifferential formulation, the magnetic penetration and the current distribution around the superconductor in 2D situation were intensively investigated by Prigozhin (1996b, 1997, 1998, 2004). Adopting the variational formulation by Prigozhin, Elliott *et al.* (2004) reported a numerical analysis of the Bean critical-state model modelling the magnetic field and the current density. The same authors also presented a finite-element analysis of the current density–electric field variational formulation (see Elliott *et al.*, 2005). Recently,

†Email: c.m.elliott@sussex.ac.uk
‡Corresponding author. Email: y.kashima@sussex.ac.uk

Barrett & Prigozhin (2006) derived dual variational formulations to solve the Bean critical-state model in terms of the electric field and developed the convergence theory of its finite-element approximation. Also see Barnes *et al.* (1999) for engineering application of the Bean model to modelling electrical machines containing superconductors. In all these articles, the problems are considered in 2D. The derivation of the Bean critical-state model from various models of type-II superconductivity such as the Ginzburg–Landau equations was summarized by Chapman (2000).

Bossavit (1994) extended the Bean critical-state model by allowing the current to exceed the critical value after the superconductor switched to the normal state. The numerical results using this extended Bean's model in 3D geometry were reported by Rubinacci *et al.* (2000, 2002).

As an alternative model, the power law constitutive relation $\mathbf{E} = |\mathbf{J}|^p \mathbf{J}$ for large $p > 0$ is commonly used in the modelling of type-II superconductivity (see, e.g. Rhyner, 1993, for a theory with the power law, Brandt, 1996, for 2D problems and Grilli *et al.*, 2005, for a recent engineering application of 3D model, etc). It was mathematically proved that as $p \to \infty$, the solution of the power law formulation converges to the solution of the Bean critical-state formulation (see Barrett & Prigozhin, 2000, for the 2D problem, Yin, 2001, Yin *et al.*, 2002, for 3D cases).

A software package to solve Maxwell's equations coupled with various nonlinear E–J relations modelling type-II superconductors in 3D for engineering application was developed by Pecher (2003) and Pecher *et al.* (2003).

While the numerical analysis of these critical-state models in 2D has been developed by many authors, to the best of the authors' knowledge no article tackling the numerical analysis of 3D critical-state problems is found in the mathematical literature. The purpose of this paper is to define a finite-element approximation in this setting and prove convergence. Following Prigozhin (1996a,b), we formulate the magnetic field around the bulk type-II superconductor as an unknown quantity in an evolution variational inequality obtained from the eddy current model and the subdifferential formulation of the critical-state models.

The Bean-type critical-state model requires the current density not to exceed some critical value, which is a difficult constraint to attain in 3D numerical analysis. To avoid this difficulty, we employ a penalty method which approximates the nonsmooth energy with a smooth energy so that the electric field–current relation is monotone and single valued (see Du *et al.*, 1999, for a regularization method of the inequality constraint appearing in a mean-field model for superconductivity in 2D). The curl-free constraint on the magnetic field in the nonconducting region coming from the eddy current model can be handled by introducing a scalar magnetic potential outside the superconductor. This magnetic field–scalar potential hybrid formulation is an effective method to carry out the discretization in space for eddy current problems with an unknown magnetic field (see Bermúdez *et al.*, 2002, for an application of this method), though it needs an additional treatment to ensure tangential continuity on the boundary between the conductor and the dielectric. Discretizing the problems in time variable yields an unconstrained optimization problem. The problem is then discretized in space by using a curl-conforming 'edge' element by Nédélec (1980) of lowest order on a tetrahedral mesh. The fully discrete solution consisting of the minimizers of the optimization problem is proved to converge to the unique solution of the variational inequality formulation of the Bean critical-state model. This convergence result is based on the compactness property of edge elements first proved by Kikuchi (1989) and extended by Monk (2003). The power law constitutive relation can be viewed as a penalty method for the Bean model by letting the power become arbitrarily large. We carry out a numerical analysis of both the power law and the extended Bean model in their own right and as penalty methods for the Bean model.

The outline of this paper is as follows: In Section 2, we recall the mathematical models of the eddy current problem and the critical-state constitutive laws and formulate the models as evolution variational inequalities. In Section 3, we formulate the discretization of the variational inequality formulations. In Section 4, the convergence of the sequence of approximations to the analytical solution is proved. Finally, in Section 5, we describe the implementation of the method and report some numerical results showing the behaviour of the magnetic field and the distribution of the current density flowing through a bulk cubic superconductor in an uniform applied magnetic field.

2. The models and the mathematical formulation

2.1 *The critical-state models*

We consider the problem in a convex polyhedron Ω $(\subset \mathbb{R}^3)$ with a boundary $\partial\Omega$. The bulk type-II superconductor Ω_s is a simply connected domain contained in Ω, with a connected Lipschitz boundary $\partial\Omega_s$. Let Ω_d denote the dielectric region $\Omega\backslash\overline{\Omega_s}$.

The model is based on Maxwell's equations, where the displacement current is neglected. These equations are called the eddy current model:

$$\partial_t \mathbf{B} + \mathbf{curl}\, \mathbf{E} = \mathbf{0} \text{ (Faraday's law)}, \tag{2.1}$$

$$\mathbf{curl}\, \mathbf{H} = \mathbf{J} \text{ (Ampère's law)}, \tag{2.2}$$

$$\operatorname{div} \mathbf{B} = 0 \text{ (Gauss' law)}, \tag{2.3}$$

where $\partial_t \mathbf{B}$ denotes $\partial \mathbf{B}/\partial t$, and

$$\mathbf{B}\colon \Omega \times [0, T] \to \mathbb{R}^3 \text{ denotes the magnetic flux density,}$$

$$\mathbf{E}\colon \Omega \times [0, T] \to \mathbb{R}^3 \text{ denotes the electric field intensity,}$$

$$\mathbf{H}\colon \Omega \times [0, T] \to \mathbb{R}^3 \text{ denotes the magnetic field intensity,}$$

$$\mathbf{J}\colon \Omega \times [0, T] \to \mathbb{R}^3 \text{ denotes the electric current density.}$$

We assume that the constitutive relation between $\mathbf{B}$ and $\mathbf{H}$ is

$$\mathbf{B} = \mu \mathbf{H}, \tag{2.4}$$

where the magnetic permeability is denoted by $\mu\colon \Omega \to \mathbb{R}_{>0}$, which is positive, piecewise constant and defined by

$$\mu = \begin{cases} \mu_s & \text{in } \Omega_s, \\ \mu_d & \text{in } \Omega_d, \end{cases}$$

for constants $\mu_s, \mu_d > 0$.

We assume that there are no current sources so that outside the superconductor

$$\mathbf{J} = \mathbf{0} \quad \text{in } \Omega_d. \tag{2.5}$$

We study the problem in a physical situation where an external time-dependent source magnetic field $\mathbf{H}_s$ is applied. We impose the boundary condition

$$\mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{h}_s \quad \text{on } \partial\Omega, \tag{2.6}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$ and $\mathbf{h}_s = \mathbf{H}_s|_{\partial\Omega}$. Since the source magnetic field $\mathbf{H}_s$ is induced by a generator outside the domain Ω, we extend $\mathbf{H}_s$ into Ω so that the superconductor is absent from the field $\mathbf{H}_s$ and $\mathbf{H}_s$ satisfies the curl-free condition in the domain. Using a source magnetic flux density $\mathbf{B}_s$, we suppose that the following equations hold:

$$\mathbf{curl}\,\mathbf{H}_s = \mathbf{0} \quad \text{in } \Omega, \tag{2.7}$$

$$\mathbf{B}_s = \mu_d \mathbf{H}_s \quad \text{in } \Omega, \tag{2.8}$$

$$\operatorname{div} \mathbf{B}_s = 0 \quad \text{in } \Omega, \tag{2.9}$$

$$\mathbf{n} \times \mathbf{H}_s = \mathbf{n} \times \mathbf{h}_s \quad \text{on } \partial\Omega. \tag{2.10}$$

Next we state the critical constitutive law between the electric field $\mathbf{E}$ and the supercurrent $\mathbf{J}$ in the superconductor Ω_s. In this paper, we always assume the following nonlinear constitutive law:

$$\mathbf{E} \in \partial\gamma(\mathbf{J}), \tag{2.11}$$

where $\gamma\colon \mathbb{R}^3 \to \mathbb{R} \cup \{+\infty\}$ is a convex functional and $\partial\gamma(\cdot)$ is the subdifferential of γ defined by

$$\partial\gamma(\mathbf{v}) := \{\mathbf{q} \in \mathbb{R}^3 | \langle \mathbf{q}, \mathbf{p}\rangle + \gamma(\mathbf{v}) \leqslant \gamma(\mathbf{v}+\mathbf{p})\ \forall\, \mathbf{p} \in \mathbb{R}^3\}.$$

As the convex functional γ we consider the following energy densities.

The Bean critical-state model's energy density

$$\gamma(\mathbf{v}) = \gamma^B(\mathbf{v}) := \begin{cases} 0 & \text{if } |\mathbf{v}| \leqslant \mathscr{J}_c, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.12}$$

where the positive constant $\mathscr{J}_c > 0$ is a critical current density.

The modified Bean critical-state model's energy density

$$\gamma(\mathbf{v}) = \gamma_\varepsilon^{mB}(\mathbf{v}) := \begin{cases} 0 & \text{if } |\mathbf{v}| \leqslant \mathscr{J}_c, \\ \frac{1}{2\varepsilon}(|\mathbf{v}|^2 - \mathscr{J}_c^2) & \text{otherwise,} \end{cases} \tag{2.13}$$

where $\varepsilon > 0$ is a positive constant. More generally, we consider a class of energy densities of the type $\gamma(\mathbf{v}) = g(|\mathbf{v}|)/\varepsilon$, where

$$\begin{aligned} & g\colon \mathbb{R} \to \mathbb{R} \text{ is convex,} \\ & g(x) = 0 \quad \text{if } x \leqslant \mathscr{J}_c, \qquad g(x) > 0 \quad \text{if } x > \mathscr{J}_c, \\ & A_1 x^2 - A_2 \leqslant g(x) \quad \forall\, x \in \mathbb{R}_{\geqslant 0}, \\ & g(x + \mathscr{J}_c) \leqslant A_3 x^2 + A_4 x \quad \forall\, x \in \mathbb{R}, \end{aligned} \tag{2.14}$$

where $A_i > 0$ $(i = 1, 2, 3)$ are positive constants and $A_4 \geqslant 0$ is a non-negative constant. Note that γ_ε^{mB} is one example of these $g(|\cdot|)/\varepsilon$ with $A_4 > 0$.

The power law model's energy density

$$\gamma(\mathbf{v}) = \gamma_p^P(\mathbf{v}) := \frac{\mathscr{J}_c}{p}|\mathbf{v}/\mathscr{J}_c|^p, \tag{2.15}$$

where $p \geqslant 2$.

Let us introduce a new quantity $\widehat{\mathbf{H}}$ by

$$\widehat{\mathbf{H}} = \mathbf{H} - \mathbf{H}_s. \tag{2.16}$$

Substituting (2.4), (2.7) and (2.16) into (2.1)–(2.3), we arrive at the following partial differential equations:

$$\mu\partial_t\widehat{\mathbf{H}} + \mu\partial_t\mathbf{H}_s + \mathbf{curl}\,\mathbf{E} = \mathbf{0}, \tag{2.17}$$

$$\mathbf{curl}\,\widehat{\mathbf{H}} = \mathbf{J}, \tag{2.18}$$

$$\operatorname{div}(\mu\widehat{\mathbf{H}} + \mu\mathbf{H}_s) = 0. \tag{2.19}$$

We couple the critical-state constitutive relation (2.11) with the eddy current model (2.17)–(2.19) and (2.7)–(2.10) to derive the equation for the unknown field $\widehat{\mathbf{H}}$.

To ensure the well-posedness of the model, let us give initial boundary conditions for $\widehat{\mathbf{H}}$. At the beginning of the time evolution, we assume that no source magnetic field is applied to the domain. Hence, there is no induced current in the superconductor and the initial condition of $\widehat{\mathbf{H}}$ is the zero field.

$$\widehat{\mathbf{H}}|_{t=0} = \mathbf{H}_s|_{t=0} = \mathbf{0}. \tag{2.20}$$

It follows from (2.6) and (2.10) that

$$\mathbf{n} \times \widehat{\mathbf{H}} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{2.21}$$

2.2 *Characterization of the nonlinear constitutive laws*

To see the nonlinearity of the constitutive relation (2.11) clearly, let us characterize (2.11) for each energy density.

PROPOSITION 2.1 For vectors $\mathbf{E}, \mathbf{J} \in \mathbb{R}^3$, the inclusion $\mathbf{E} \in \partial\gamma^B(\mathbf{J})$ holds if, and only if, there is a constant $\rho \geqslant 0$ such that the following relations hold:

$$\mathbf{E} = \rho\mathbf{J}, \tag{2.22}$$

$$|\mathbf{J}| \leqslant \mathscr{J}_c, \tag{2.23}$$

$$|\mathbf{J}| < \mathscr{J}_c \Longrightarrow \mathbf{E} = \mathbf{0}. \tag{2.24}$$

Proof. First note that by definition the inclusion $\mathbf{E} \in \partial\gamma^B(\mathbf{J})$ is equivalent to the inequality

$$\langle\mathbf{E}, \mathbf{p}\rangle + \gamma^B(\mathbf{J}) \leqslant \gamma^B(\mathbf{J} + \mathbf{p}) \tag{2.25}$$

for all $\mathbf{p} \in \mathbb{R}^3$.

Assume (2.22)–(2.24). Fix any $\mathbf{p} \in \mathbb{R}^3$. If $|\mathbf{J}+\mathbf{p}| > \mathscr{J}_c$ holds, then the inequality (2.25) is trivial by the definition of γ^B. If $|\mathbf{J}+\mathbf{p}| \leqslant \mathscr{J}_c$ and $|\mathbf{J}| < \mathscr{J}_c$, then by the relation (2.24), the inequality (2.25) holds since in this case both sides are zero. If $|\mathbf{J}| = \mathscr{J}_c$, then the inequality $|\mathbf{J}+\mathbf{p}| \leqslant \mathscr{J}_c$ yields

$$2\langle \mathbf{J}, \mathbf{p}\rangle \leqslant -|\mathbf{p}|^2 \leqslant 0. \tag{2.26}$$

Multiplying (2.26) by $\rho/2$, we have $\langle \mathbf{E}, \mathbf{p}\rangle \leqslant 0$, which is (2.25).

Conversely, we show that the inequality (2.25) leads to the relations (2.22)–(2.24). If $|\mathbf{J}| > \mathscr{J}_c$ holds, then by substituting $\mathbf{p} = -\mathbf{J}$ into (2.25) we arrive at $+\infty \leqslant 0$, which is a contradiction. Thus, the inequality (2.23) must always hold.

Suppose $|\mathbf{J}| < \mathscr{J}_c$ and $\mathbf{E} \neq \mathbf{0}$. Then taking a large constant $C > 0$ satisfying $|\mathbf{J} + \mathbf{E}/C| \leqslant \mathscr{J}_c$ and substituting $\mathbf{p} = \mathbf{E}/C$ into (2.25), we have $|\mathbf{E}|^2/C \leqslant 0$, which is a contradiction. Therefore, the relation (2.24) is valid.

Finally, we show (2.22). Taking $\mathbf{p} = \mathbf{q} - \mathbf{J}$ for all $\mathbf{q} \in \mathbb{R}^3$ with $|\mathbf{q}| \leqslant \mathscr{J}_c$, we obtain

$$\langle \mathbf{E}, \mathbf{q} - \mathbf{J}\rangle \leqslant 0. \tag{2.27}$$

If $\mathbf{E} = \mathbf{0}$ then (2.22) is true for $\rho = 0$. Let $\mathbf{E}$ be non-zero, then by (2.24) $|\mathbf{J}| = \mathscr{J}_c$. Suppose that the vector $\mathbf{E}$ is not parallel to the vector $\mathbf{J}$. Let us consider the plane A containing the vectors $\mathbf{E}$ and $\mathbf{J}$. Draw the line L which passes through the point $\mathbf{J}$ and is perpendicular to the vector $\mathbf{E}$ on A. Then the line L divides the plane A into two domains. Take any point $\mathbf{q}$ belonging to one of these domains containing the point $\mathbf{E}$ to satisfy $|\mathbf{q}| \leqslant \mathscr{J}_c$, $\mathbf{q} \neq \mathbf{J}$ (see Fig. 1). Then we obviously see that $\langle \mathbf{E}, \mathbf{q} - \mathbf{J}\rangle > 0$, which contradicts (2.27). Thus, the vector $\mathbf{E}$ must be parallel to the vector $\mathbf{J}$ and we can write $\mathbf{E} = \rho\mathbf{J}$. If $\rho < 0$, then by taking $\mathbf{q} = \mathbf{0}$ in (2.27) we have $-\rho\mathscr{J}_c^2 \leqslant 0$, which is a contradiction. Therefore (2.22) is correct. □

REMARK 2.1 As Proposition 2.1 shows, the subdifferential formulation $\mathbf{E} \in \partial\gamma^B(\mathbf{J})$ requires the parallel condition $\mathbf{E} = \rho\mathbf{J}$ for $\rho \geqslant 0$. From a modelling perspective, this relation is accepted if the superconductor Ω_s is axially symmetric (see Prigozhin, 1996b) or a thin film (see Prigozhin, 1998) in a perpendicular external field. However, in the full 3D configuration the direction of the current flowing through the superconductor is not yet settled (see Prigozhin, 1996a,b; Chapman, 2000, where this issue is argued from the point of view of mathematical modelling). However, the power law characteristic is a popular model based on experimental measurements of superconductors (see Rhyner, 1993, and references therein). Furthermore, the Bean-type critical-state constitutive law is a limiting case of the power

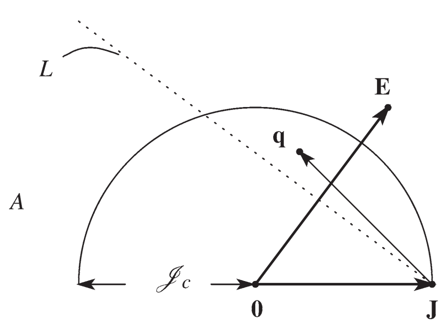

FIG. 1. The vectors on the plane A.

law even in 3D situations (see Yin, 2001; Yin *et al.*, 2002, or Proposition 2.6). Thus, it is an interesting topic to consider the numerical analysis of the Bean-type critical-state model $\mathbf{E} \in \partial\gamma^{B}(\mathbf{J})$ for a general class of 3D type-II superconductor.

PROPOSITION 2.2 For vectors $\mathbf{E}, \mathbf{J} \in \mathbb{R}^3$, the inclusion $\mathbf{E} \in \partial\gamma_{\varepsilon}^{mB}(\mathbf{J})$ holds if, and only if, the following relations hold:

$$\mathbf{E} = \begin{cases} \mathbf{0} & \text{if } |\mathbf{J}| < \mathscr{J}_c, \\ \frac{1}{\varepsilon}\mathbf{J} & \text{if } |\mathbf{J}| \geqslant \mathscr{J}_c. \end{cases} \tag{2.28}$$

Proof. This equivalence was proved by Bossavit (1994). We sketch the proof.

By definition, $\mathbf{E} \in \partial\gamma_{\varepsilon}^{mB}(\mathbf{J})$ is equivalently written as

$$\langle \mathbf{E}, \mathbf{p} \rangle + \gamma_{\varepsilon}^{mB}(\mathbf{J}) \leqslant \gamma_{\varepsilon}^{mB}(\mathbf{J} + \mathbf{p}) \tag{2.29}$$

for all $\mathbf{p} \in \mathbb{R}^3$. By elementary calculation, we can check that (2.28) yields (2.29). Let us assume (2.29). If $|\mathbf{J}| < \mathscr{J}_c$, the inequality (2.29) yields that for all $\mathbf{p} \in \mathbb{R}^3$ with $|\mathbf{p}| \leqslant \mathscr{J}_c$,

$$\langle \mathbf{E}, \mathbf{p} - \mathbf{J} \rangle \leqslant 0. \tag{2.30}$$

Taking a large $C > 0$ such that $|\mathbf{J} + \mathbf{E}/C| \leqslant \mathscr{J}_c$ and substituting $\mathbf{p} = \mathbf{J} + \mathbf{E}/C$ into (2.30), we obtain $|\mathbf{E}|^2/C \leqslant 0$, thus $\mathbf{E} = \mathbf{0}$, which is (2.28).

Assume $|\mathbf{J}| > \mathscr{J}_c$. Take any $\mathbf{p} \in \mathbb{R}^3$. Choosing a small $\delta > 0$ such that $|\mathbf{J} + \delta\mathbf{p}| > \mathscr{J}_c$ and substituting $\delta\mathbf{p}$ into (2.29), we have

$$\delta\langle \mathbf{E}, \mathbf{p} \rangle \leqslant (2\delta\langle \mathbf{J}, \mathbf{p} \rangle + \delta^2|\mathbf{p}|^2)/(2\varepsilon).$$

Dividing both sides by δ and sending $\delta \searrow 0$, we have

$$\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle \leqslant 0. \tag{2.31}$$

Similarly, taking $\delta < 0$ such that $|\mathbf{J} + \delta\mathbf{p}| > \mathscr{J}_c$, we obtain

$$\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle \geqslant 0. \tag{2.32}$$

By (2.31) and (2.32) we have $\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle = 0$ for all $\mathbf{p} \in \mathbb{R}^3$, or $\mathbf{E} = \mathbf{J}/\varepsilon$, which is (2.28).

If $|\mathbf{J}| = \mathscr{J}_c$, take any $\mathbf{p} \in \mathbb{R}^3$ such that $\langle \mathbf{p}, \mathbf{J} \rangle > 0$. Then for all $\delta > 0$, we see $|\mathbf{J} + \delta\mathbf{p}| > \mathscr{J}_c$. Thus, by substituting $\delta\mathbf{p}$ into (2.29) and sending $\delta \searrow 0$ we deduce that for all $\mathbf{p} \in \mathbb{R}^3$, with $\langle \mathbf{p}, \mathbf{J} \rangle > 0$,

$$\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle \leqslant 0.$$

This implies that there is $C \geqslant 0$ such that

$$\mathbf{E} - \mathbf{J}/\varepsilon = -C\mathbf{J}. \tag{2.33}$$

Similarly, take any $\mathbf{p} \in \mathbb{R}^3$ such that $\langle \mathbf{p}, \mathbf{J} \rangle < 0$. Then for all $\delta < 0$, we see that $|\mathbf{J} + \delta\mathbf{p}| > \mathscr{J}_c$. By substituting $\delta\mathbf{p}$ into (2.29) and sending $\delta \nearrow 0$ we have that for all $\mathbf{p} \in \mathbb{R}^3$, with $\langle \mathbf{p}, \mathbf{J} \rangle < 0$,

$$\langle \mathbf{E} - \mathbf{J}/\varepsilon, \mathbf{p} \rangle \geqslant 0,$$

which implies that there is $C' \geqslant 0$ such that

$$\mathbf{E} - \mathbf{J}/\varepsilon = C'\mathbf{J}. \tag{2.34}$$

By (2.33) and (2.34) we obtain $\mathbf{E} - \mathbf{J}/\varepsilon = \mathbf{0}$. Therefore, the relation (2.28) holds. $\square$

REMARK 2.2 The model (2.28) proposed by Bossavit (1994) is a modification of the Bean-type model (2.22)–(2.24) in the sense that if the current density $|\mathbf{J}|$ exceeds the critical value, then $\mathbf{E}-\mathbf{J}$ relation is switched to be the linear Ohm's law.

PROPOSITION 2.3 For vectors $\mathbf{E}, \mathbf{J} \in \mathbb{R}^3$, the inclusion $\mathbf{E} \in \partial\gamma_p^P(\mathbf{J})$ holds if, and only if, $\mathbf{E} = \mathscr{J}_c^{1-p}|\mathbf{J}|^{p-2}\mathbf{J}$.

Proof. If a convex function is differentiable, its subdifferential is always equal to the derivative of the function (see, e.g. Barbu & Precupanu, 1986). Since now $\gamma_p^P(\cdot)$ is differentiable, this equivalence is immediate. □

2.3 *Mathematical formulation of the magnetic field $\widehat{\mathbf{H}}$ via a variational inequality*

We formulate Faraday's law (2.17) in an integral form. Take a function $\boldsymbol{\phi}\colon \Omega \to \mathbb{R}^3$ with $\mathbf{curl}\,\boldsymbol{\phi} = \mathbf{0}$ in Ω_d and $\mathbf{n}\times\boldsymbol{\phi} = \mathbf{0}$ on $\partial\Omega$. Then (2.17) yields

$$\int_\Omega \mu\langle \partial_t\widehat{\mathbf{H}} + \partial_t\mathbf{H}_s, \boldsymbol{\phi}\rangle \mathrm{d}\mathbf{x} + \int_{\Omega_s}\langle \mathbf{E}, \mathbf{curl}\,\boldsymbol{\phi}\rangle \mathrm{d}\mathbf{x} = 0. \tag{2.35}$$

Let us combine the weak form (2.35) with the constitutive relation (2.11). Substituting $\mathbf{p} = \mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x})$ for all $\boldsymbol{\phi}\colon \Omega \to \mathbb{R}^3$ satisfying $\mathbf{curl}\,\boldsymbol{\phi} = \mathbf{0}$ in Ω_d and $\mathbf{n}\times\boldsymbol{\phi} = \mathbf{0}$ on $\partial\Omega$ into the definition of the subdifferential $\partial\gamma$ and recalling the equality (2.18), we see that

$$\begin{aligned} &-\int_\Omega \mu\langle \partial_t\widehat{\mathbf{H}}(\mathbf{x},t) + \partial_t\mathbf{H}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x})\rangle \mathrm{d}\mathbf{x} + \int_{\Omega_s}\gamma(\mathbf{curl}\,\widehat{\mathbf{H}}(\mathbf{x},t))\mathrm{d}\mathbf{x} \\ &\leqslant \int_{\Omega_s}\gamma(\mathbf{curl}\,\widehat{\mathbf{H}}(\mathbf{x},t) + \mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x}))\mathrm{d}\mathbf{x}, \end{aligned}$$

or equivalently by taking $\boldsymbol{\phi} - \widehat{\mathbf{H}}$ as $\boldsymbol{\phi}$ above, we deduce that

$$\int_\Omega \mu\langle \partial_t\widehat{\mathbf{H}}(\mathbf{x},t) + \partial_t\mathbf{H}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x}) - \widehat{\mathbf{H}}(\mathbf{x},t)\rangle \mathrm{d}\mathbf{x} + \int_{\Omega_s}\gamma(\mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x}))\mathrm{d}\mathbf{x} - \int_{\Omega_s}\gamma(\mathbf{curl}\,\widehat{\mathbf{H}}(\mathbf{x},t))\mathrm{d}\mathbf{x} \geqslant 0.$$

Thus, we formally obtained a variational inequality formulation of the unknown magnetic field $\widehat{\mathbf{H}}$.

2.3.1 *Function spaces.*

To complete the formulation, let us introduce the function spaces which are used to analyse the problem mathematically

$$H(\mathbf{curl};\Omega) := \{\boldsymbol{\phi} \in L^2(\Omega;\mathbb{R}^3) | \mathbf{curl}\,\boldsymbol{\phi} \in L^2(\Omega;\mathbb{R}^3)\}$$

with the norm $\|\boldsymbol{\phi}\|_{H(\mathbf{curl};\Omega)} := (\|\boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{curl}\,\boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)})^{1/2}$,

$$H^1(\mathbf{curl};\Omega) := \{\boldsymbol{\phi} \in H^1(\Omega;\mathbb{R}^3) | \mathbf{curl}\,\boldsymbol{\phi} \in H^1(\Omega;\mathbb{R}^3)\}$$

with the norm $\|\boldsymbol{\phi}\|_{H^1(\mathbf{curl};\Omega)} := (\|\boldsymbol{\phi}\|^2_{H^1(\Omega;\mathbb{R}^3)} + \|\mathbf{curl}\,\boldsymbol{\phi}\|^2_{H^1(\Omega;\mathbb{R}^3)})^{1/2}$, the dual space $(H(\mathbf{curl};\Omega))^*$ of $H(\mathbf{curl};\Omega)$ with respect to the inner product of $L^2(\Omega;\mathbb{R}^3)$ with the norm

$$\|\boldsymbol{\phi}\|_{(H(\mathbf{curl};\Omega))^*} = \sup_{\boldsymbol{\psi}\in H(\mathbf{curl};\Omega)} \frac{|\langle\boldsymbol{\psi},\boldsymbol{\phi}\rangle_{L^2(\Omega;\mathbb{R}^3)}|}{\|\boldsymbol{\psi}\|_{H(\mathbf{curl};\Omega)}},$$

and

$$H(\mathrm{div};\Omega) := \{\boldsymbol{\phi} \in L^2(\Omega;\mathbb{R}^3)|\mathrm{div}\boldsymbol{\phi} \in L^2(\Omega)\}$$

with the norm $\|\boldsymbol{\phi}\|_{H(\mathrm{div};\Omega)} := (\|\boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\mathrm{div}\boldsymbol{\phi}\|^2_{L^2(\Omega)})^{1/2}$.

Next we define the traces of functions in $H(\mathbf{curl};\Omega)$ and $H(\mathrm{div};\Omega)$. Note that for all $\boldsymbol{\phi} \in H^1(\Omega;\mathbb{R}^N)$ $(N = 1, 3)$ $\boldsymbol{\phi}|_{\partial\Omega} \in H^{1/2}(\partial\Omega;\mathbb{R}^N)$, where $H^{1/2}(\partial\Omega;\mathbb{R}^N)$ is a Sobolev space with the norm

$$\|\boldsymbol{\phi}\|_{H^{1/2}(\partial\Omega;\mathbb{R}^N)} := \left(\|\boldsymbol{\phi}\|^2_{L^2(\partial\Omega;\mathbb{R}^N)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|\boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\phi}(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}A(\mathbf{x})\mathrm{d}A(\mathbf{y})\right)^{1/2}.$$

Let $H^{-1/2}(\partial\Omega;\mathbb{R}^N)$ be the dual space of $H^{1/2}(\partial\Omega;\mathbb{R}^N)$ with respect to the inner product of $L^2(\partial\Omega;\mathbb{R}^N)$ with the norm

$$\|\boldsymbol{\phi}\|_{H^{-1/2}(\partial\Omega;\mathbb{R}^N)} := \sup_{\boldsymbol{\psi}\in H^{1/2}(\partial\Omega;\mathbb{R}^N)} \frac{|\langle\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2(\partial\Omega;\mathbb{R}^N)}|}{\|\boldsymbol{\psi}\|_{H^{1/2}(\partial\Omega;\mathbb{R}^N)}}.$$

For all $\boldsymbol{\phi} \in H(\mathbf{curl};\Omega)$, the trace $\mathbf{n}\times\boldsymbol{\phi}$ on $\partial\Omega$ is well-defined in $H^{-1/2}(\partial\Omega;\mathbb{R}^3)$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, in the sense that

$$\langle\mathbf{n}\times\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2(\partial\Omega;\mathbb{R}^3)} := \langle\mathbf{curl}\,\boldsymbol{\phi},\boldsymbol{\psi}\rangle_{L^2(\Omega;\mathbb{R}^3)} - \langle\boldsymbol{\phi},\mathbf{curl}\,\boldsymbol{\psi}\rangle_{L^2(\Omega;\mathbb{R}^3)}$$

for all $\boldsymbol{\psi} \in H^1(\Omega;\mathbb{R}^3)$. For all $\boldsymbol{\phi} \in H(\mathrm{div};\Omega)$, the trace $\mathbf{n}\cdot\boldsymbol{\phi}$ on $\partial\Omega$ is well-defined in $H^{-1/2}(\partial\Omega)$ in the sense that

$$\langle\mathbf{n}\cdot\boldsymbol{\phi}, f\rangle_{L^2(\partial\Omega)} := \langle\mathrm{div}\,\boldsymbol{\phi}, f\rangle_{L^2(\Omega)} + \langle\boldsymbol{\phi},\nabla f\rangle_{L^2(\Omega;\mathbb{R}^3)}$$

for all $f \in H^1(\Omega)$.

Define the subspace $V(\Omega)$ of $H(\mathbf{curl};\Omega)$ by

$$V(\Omega) := \{\boldsymbol{\phi} \in H(\mathbf{curl};\Omega)|\mathbf{curl}\,\boldsymbol{\phi} = \mathbf{0} \text{ in } \Omega_d,\ \mathbf{n}\times\boldsymbol{\phi} = \mathbf{0} \text{ on } \partial\Omega\}.$$

The subspace $V_p(\Omega)$ of $V(\Omega)$ $(p \geqslant 2)$ is defined by

$$V_p(\Omega) := \{\boldsymbol{\phi} \in V(\Omega)|\mathbf{curl}\,\boldsymbol{\phi}|_{\Omega_s} \in L^p(\Omega_s;\mathbb{R}^3)\}.$$

The subset S of $V(\Omega)$ is defined by

$$S := \{\boldsymbol{\phi} \in V(\Omega)||\mathbf{curl}\,\boldsymbol{\phi}| \leqslant \mathscr{J}_c \text{ a.e. in } \Omega_s\}.$$

The subspace $X^{(\mu)}(\Omega)$ of $H(\mathbf{curl};\Omega)$ consisting of divergence-free functions for the magnetic permeability μ is defined by

$$X^{(\mu)}(\Omega) := \{\boldsymbol{\phi} \in H(\mathbf{curl};\Omega)|\mathrm{div}(\mu\boldsymbol{\phi}) = 0 \text{ in } \mathcal{D}'(\Omega)\},$$

where $\mathcal{D}'(\Omega)$ denotes the space of Schwartz distributions.

The spaces $L^q(0,T;B)$ $(q = 2 \text{ or } \infty)$, $H^1(0,T;B)$, $C([0,T];B)$ and $C^{1,1}([0,T];B)$ for a Banach space B are defined in the usual way.

2.3.2 *External magnetic field.* In this section, we discuss the external magnetic field $\mathbf{H}_s$ solving (2.7)–(2.10). We assume that the boundary value $\mathbf{h}_s \colon \partial\Omega \times [0, T] \to \mathbb{R}^3$ satisfies

$$\mathbf{h}_s \in C^{1,1}([0, T]; H^{1/2}(\partial\Omega; \mathbb{R}^3)) \quad \text{and} \quad \mathbf{h}_s(0) = \mathbf{0}, \tag{2.36}$$

and, for all $t \in [0, T]$ and $\boldsymbol{\phi} \in H(\mathbf{curl}; \Omega)$ with $\mathbf{curl}\,\boldsymbol{\phi} = \mathbf{0}$,

$$\int_{\partial\Omega} \langle \mathbf{h}_s(t), \mathbf{n} \times \boldsymbol{\phi} \rangle \mathrm{d}A = 0. \tag{2.37}$$

LEMMA 2.1 Under Assumptions (2.36) and (2.37), there exists a unique function $\mathbf{H}_s \in C^{1,1}([0, T]; H^1(\mathbf{curl}; \Omega))$ such that $\mathbf{H}_s$ satisfies System (2.7)–(2.10) in the weak sense for all $t \in [0, T]$ and $\mathbf{H}_s(0) = \mathbf{0}$. Moreover, the following inequalities hold for all $t \in [0, T]$:

$$\|\mathbf{H}_s(t)\|_{H^1(\mathbf{curl};\Omega)} \leqslant C \|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)}, \tag{2.38}$$

$$\|\partial_t \mathbf{H}_s(t)\|_{H^1(\mathbf{curl};\Omega)} \leqslant C \|\partial_t \mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)}. \tag{2.39}$$

Proof. The proof of existence of a unique solution follows Auchmuty & Alexander (2005), where the unique solvability theory for general div–**curl** systems assuming a boundary of C^2 class was developed. Fix any $t \in [0, T]$. We use the following Helmholtz decomposition (see, e.g. Cessenat, 1996, Theorem 10′, Chapter 2):

$$L^2(\Omega; \mathbb{R}^3) = \nabla H_0^1(\Omega) \oplus \mathbf{curl}\, H^1(\Omega; \mathbb{R}^3), \tag{2.40}$$

$$L^2(\Omega; \mathbb{R}^3) = \nabla H^1(\Omega) \oplus \mathbf{curl}\, H_0^1(\Omega; \mathbb{R}^3). \tag{2.41}$$

We will find $\mathbf{H}_s(t) \in L^2(\Omega; \mathbb{R}^3)$ such that

$$\mathbf{curl}\,\mathbf{H}_s(t) = \mathbf{0} \quad \text{in } \Omega, \tag{2.42}$$

$$\operatorname{div} \mathbf{H}_s(t) = 0 \quad \text{in } \Omega, \tag{2.43}$$

$$\mathbf{n} \times \mathbf{H}_s(t) = \mathbf{n} \times \mathbf{h}_s(t) \quad \text{on } \partial\Omega. \tag{2.44}$$

By the decomposition (2.40), we can write $\mathbf{H}_s(t) = \nabla f + \mathbf{curl}\,\mathbf{H}_1$ with $f \in H_0^1(\Omega)$ and $\mathbf{H}_1 \in H^1(\Omega; \mathbb{R}^3)$. Condition (2.43) implies $f \equiv 0$. Thus, our problem is equivalent to finding $\mathbf{H}_1 \in H^1(\Omega; \mathbb{R}^3)$ such that

$$\mathbf{curl}\,(\mathbf{curl}\,\mathbf{H}_1) = \mathbf{0} \quad \text{in } \Omega, \tag{2.45}$$

$$\mathbf{n} \times \mathbf{curl}\,\mathbf{H}_1 = \mathbf{n} \times \mathbf{h}_s(t) \quad \text{on } \partial\Omega. \tag{2.46}$$

The weak form of (2.45)–(2.46) is

$$\int_\Omega \langle \mathbf{curl}\,\mathbf{H}_1, \mathbf{curl}\,\boldsymbol{\phi} \rangle \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \langle \mathbf{n} \times \mathbf{h}_s(t), \boldsymbol{\phi} \rangle \mathrm{d}A = 0, \tag{2.47}$$

for all $\boldsymbol{\phi} \in H^1(\Omega; \mathbb{R}^3)$.

For all $\boldsymbol{\phi} \in H^1(\Omega; \mathbb{R}^3)$, the decomposition (2.41) implies that there exist unique $\hat{f} \in H^1(\Omega)$ and $\mathbf{H}_2 \in \mathbf{curl}\, H_0^1(\Omega; \mathbb{R}^3)$ such that $\boldsymbol{\phi} = \nabla \hat{f} + \mathbf{H}_2$. Note that $\mathbf{H}_2 \cdot \mathbf{n} = \mathbf{0}$ on $\partial\Omega$. Therefore, by Assumption (2.37), Problem (2.47) is equivalent to the problem: find $\mathbf{H}_2 \in X_1$ such that

$$\int_\Omega \langle \mathbf{curl}\, \mathbf{H}_2, \mathbf{curl}\, \boldsymbol{\phi} \rangle \mathrm{d}\mathbf{x} - \int_{\partial\Omega} \langle \mathbf{h}_s(t), \mathbf{n} \times \boldsymbol{\phi} \rangle \mathrm{d}A = 0, \tag{2.48}$$

for all $\boldsymbol{\phi} \in X_1$, where the space X_1 is defined by

$$X_1 := \{\boldsymbol{\phi} \in H(\mathbf{curl}; \Omega) | \mathrm{div}\boldsymbol{\phi} = 0 \text{ in } \Omega,\ \mathbf{n} \cdot \boldsymbol{\phi} = 0 \text{ on } \partial\Omega\}$$

equipped with the norm of $H(\mathbf{curl}; \Omega)$.

Let us define a functional $F\colon X_1 \to \mathbb{R}$ by

$$F(\boldsymbol{\phi}) := \frac{1}{2} \int_\Omega |\mathbf{curl}\, \boldsymbol{\phi}|^2\, \mathrm{d}\mathbf{x} - \int_{\partial\Omega} \langle \mathbf{h}_s(t), \mathbf{n} \times \boldsymbol{\phi} \rangle \mathrm{d}A.$$

Then we see that

$$\begin{aligned} F(\boldsymbol{\phi}) &\geqslant \frac{1}{2} \|\mathbf{curl}\, \boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)} - \|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)} \|\mathbf{n} \times \boldsymbol{\phi}\|_{H^{-1/2}(\partial\Omega;\mathbb{R}^3)} \\ &\geqslant \frac{1}{2} \|\mathbf{curl}\, \boldsymbol{\phi}\|^2_{L^2(\Omega;\mathbb{R}^3)} - C \|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)} \|\mathbf{curl}\, \boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)}, \end{aligned} \tag{2.49}$$

where we have used the fact that the map $\boldsymbol{\phi} \mapsto \mathbf{n} \times \boldsymbol{\phi}\colon H(\mathbf{curl}; \Omega) \to H^{-1/2}(\partial\Omega; \mathbb{R}^3)$ is continuous together with the Friedrichs inequality (see Girault & Raviart, 1986)

$$\|\boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant C \|\mathbf{curl}\, \boldsymbol{\phi}\|_{L^2(\Omega;\mathbb{R}^3)}$$

for all $\boldsymbol{\phi} \in X_1$. Therefore, by noting the convexity of F and (2.49), we can show the existence of a unique $\mathbf{H}_2 \in X_1$ satisfying

$$F(\mathbf{H}_2) = \min_{\boldsymbol{\phi} \in X_1} F(\boldsymbol{\phi}), \tag{2.50}$$

which is equivalent to Problem (2.48). Hence, the existence of a unique solution to (2.42)–(2.44) has been proved.

Next we will show that $\mathbf{H}_s \in C^{1,1}([0, T]; H^1(\mathbf{curl}; \Omega))$ and the inequalities (2.38) and (2.39) hold. Fix $t \in [0, T]$. Let $\boldsymbol{\xi}^t \in H^1(\Omega; \mathbb{R}^3)$ be a weak solution to the following elliptic problem:

$$\begin{aligned} \Delta \boldsymbol{\xi}^t &= \mathbf{0} \quad \text{in } \Omega, \\ \boldsymbol{\xi}^t &= \mathbf{h}_s(t) \quad \text{on } \partial\Omega. \end{aligned} \tag{2.51}$$

Then, we have

$$\|\boldsymbol{\xi}^t\|_{H^1(\Omega;\mathbb{R}^3)} \leqslant C \|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)}. \tag{2.52}$$

Since Ω is convex, the space X_2 defined by

$$X_2 := \{\boldsymbol{\phi} \in H(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega) | \mathbf{n} \times \boldsymbol{\phi} = \mathbf{0} \text{ on } \partial\Omega\},$$

equipped with the inner product

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{X_2} := \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle \mathbf{curl}\, \boldsymbol{\phi}, \mathbf{curl}\, \boldsymbol{\psi} \rangle_{L^2(\Omega;\mathbb{R}^3)} + \langle \mathrm{div}\, \boldsymbol{\phi}, \mathrm{div}\, \boldsymbol{\psi} \rangle_{L^2(\Omega)},$$

is continuously imbedded into $H^1(\Omega;\mathbb{R}^3)$ (see Amrouche *et al.*, 1998, Proposition 2.17). Noting that $\mathbf{H}_s(t) - \boldsymbol{\xi}^t \in X_2$, we have

$$\begin{aligned}
\|\mathbf{H}_s(t)\|_{H^1(\Omega;\mathbb{R}^3)} &\leqslant \|\mathbf{H}_s(t) - \boldsymbol{\xi}^t\|_{H^1(\Omega;\mathbb{R}^3)} + \|\boldsymbol{\xi}^t\|_{H^1(\Omega;\mathbb{R}^3)} \\
&\leqslant C_1(\|\mathbf{H}_s(t) - \boldsymbol{\xi}^t\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{curl}\, \boldsymbol{\xi}^t\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathrm{div}\, \boldsymbol{\xi}^t\|_{L^2(\Omega)}) + \|\boldsymbol{\xi}^t\|_{H^1(\Omega;\mathbb{R}^3)} \\
&\leqslant C_2\|\mathbf{n} \times \mathbf{h}_s(t)\|_{L^2(\partial\Omega;\mathbb{R}^3)} + C_3\|\boldsymbol{\xi}^t\|_{H^1(\Omega;\mathbb{R}^3)} \\
&\leqslant C_4\|\mathbf{h}_s(t)\|_{H^{1/2}(\partial\Omega;\mathbb{R}^3)},
\end{aligned}$$

where we have used the inequality (2.52) and the Friedrichs inequality (see Girault & Raviart, 1986)

$$\|\mathbf{H}_s(t)\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant C\|\mathbf{n} \times \mathbf{h}_s(t)\|_{L^2(\partial\Omega;\mathbb{R}^3)}.$$

Since $\mathbf{h}_s\colon [0,T] \to H^{1/2}(\partial\Omega;\mathbb{R}^3)$ is continuous, we have $\mathbf{H}_s \in C([0,T]; H^1(\Omega;\mathbb{R}^3))$. By repeating the same argument for $\partial_t\mathbf{H}_s$, we can show that $\partial_t\mathbf{H}_s\colon [0,T] \to H^1(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and the inequality (2.39) holds. □

From now on, the magnetic field $\mathbf{H}_s$ is the one whose existence was proved in Lemma 2.1 under Assumptions (2.36) and (2.37).

2.3.3 *Variational inequality formulations.* Now we are ready to propose our mathematical formulation of (2.17)–(2.21) coupled with the nonlinear constitutive law (2.11) as the initial-value problem for the evolution variational inequality for the unknown $\widehat{\mathbf{H}}$. The first one is the formulation with Bean's model

($\mathbf{P}^B\mathbf{1}$) Find $\widehat{\mathbf{H}} \in H^1(0,T; L^2(\Omega;\mathbb{R}^3))$ such that $\widehat{\mathbf{H}}(t) \in S$ for a.e. $t \in [0,T]$,

$$\int_\Omega \mu\langle \partial_t\widehat{\mathbf{H}}(\mathbf{x},t) + \partial_t\mathbf{H}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x}) - \widehat{\mathbf{H}}(\mathbf{x},t)\rangle \mathrm{d}\mathbf{x} \geqslant 0, \qquad \text{for a.e. } t \in (0,T], \tag{2.53}$$

holds for all $\boldsymbol{\phi} \in S$ and $\widehat{\mathbf{H}}(\mathbf{x},0) = \mathbf{0}$ in Ω.

PROPOSITION 2.4 The solution $\widehat{\mathbf{H}}$ of ($\mathbf{P}^B\mathbf{1}$) uniquely exists. Moreover, the solution $\widehat{\mathbf{H}}(t)\colon [0,T] \to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and satisfies $\widehat{\mathbf{H}}(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$.

Proof. The proof essentially follows Prigozhin (1996a, Theorem 2), where the magnetic permeability μ was assumed to be constant and the problem was formulated in the whole space $\mathbb{R}^3$. Let $L^2_\mu(\Omega;\mathbb{R}^3)$ denote the Hilbert space $L^2(\Omega;\mathbb{R}^3)$ equipped with the inner product $\langle \mu\cdot,\cdot\rangle_{L^2(\Omega;\mathbb{R}^3)}$. Problem ($\mathbf{P}^B\mathbf{1}$) becomes an evolution problem in $L^2_\mu(\Omega;\mathbb{R}^3)$ as follows:

$$\begin{cases} d_t\widehat{\mathbf{H}}(t) + \partial_t\mathbf{H}_s(t) \in -\partial E(\widehat{\mathbf{H}}(t)) \ \text{ a.e. } t \in (0,T], \\ \widehat{\mathbf{H}}(0) = \mathbf{0}, \end{cases} \tag{2.54}$$

where the energy functional $E\colon L^2_\mu(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ is an indicator functional of the non-empty closed convex set $S \subset L^2_\mu(\Omega;\mathbb{R}^3)$ so that

$$E(\boldsymbol{\phi}) := \begin{cases} 0 & \text{if } \boldsymbol{\phi} \in S, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.55}$$

Since E is convex and lower semicontinuous, not identically $+\infty$, its subdiffrential ∂E is a maximal monotone operator in $L^2_\mu(\Omega;\mathbb{R}^3)$. Therefore, by the Lipschitz continuity of the given data $\partial_t \mathbf{H}_s(t)$, a standard theorem from nonlinear semigroup theory (see, e.g. Brezis, 1971, Theorem 21) ensures the existence of a unique $\widehat{\mathbf{H}} \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ satisfying $\widehat{\mathbf{H}}(t) \in S$ for all $t \in [0,T]$, (2.54) and the Lipschitz continuity on $[0,T]$.

We show that $\widehat{\mathbf{H}}(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$. Take any $f \in \mathcal{D}(\Omega)$ and any $\delta \in \mathbb{R}$. Substituting $\delta\nabla f + \widehat{\mathbf{H}}(t) \in S$ into (2.53), we obtain

$$\delta \int_\Omega \mu\langle \partial_t \widehat{\mathbf{H}}(t) + \partial_t \mathbf{H}_s(t), \nabla f\rangle \mathrm{d}\mathbf{x} \geqslant 0.$$

By separately taking positive and negative δ, we have

$$\int_\Omega \mu\langle \partial_t \widehat{\mathbf{H}}(t) + \partial_t \mathbf{H}_s(t), \nabla f\rangle \mathrm{d}\mathbf{x} = 0 \tag{2.56}$$

for a.e. $t \in (0,T]$. Since now $\widehat{\mathbf{H}}+\mathbf{H}_s$ is Lipschitz continuous, by integrating (2.56) over $[0,t]$ we deduce that $\langle \mu(\widehat{\mathbf{H}}(t) + \mathbf{H}_s(t)), \nabla f\rangle_{L^2(\Omega;\mathbb{R}^3)} = 0$ for all $t \in [0,T]$. □

Formulating the modified Bean model with the energy density $g(|\cdot|)/\varepsilon$ in the same way as $(\mathbf{P}^B\mathbf{1})$ leads to the initial-value problem $(\mathbf{P}^{mB}_\varepsilon\mathbf{1})$

$(\mathbf{P}^{mB}_\varepsilon\mathbf{1})$ Find $\widehat{\mathbf{H}}_\varepsilon \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ such that $\widehat{\mathbf{H}}_\varepsilon(t) \in V(\Omega)$ for all $t \in [0,T]$,

$$\int_\Omega \mu\langle \partial_t \widehat{\mathbf{H}}_\varepsilon(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x}) - \widehat{\mathbf{H}}_\varepsilon(\mathbf{x},t)\rangle \mathrm{d}\mathbf{x} + \frac{1}{\varepsilon}\int_\Omega g(|\mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x})|)\mathrm{d}\mathbf{x}$$
$$- \tfrac{1}{\varepsilon}\int_\Omega g(|\mathbf{curl}\,\widehat{\mathbf{H}}_\varepsilon(\mathbf{x},t)|)\mathrm{d}\mathbf{x} \geqslant 0, \qquad \text{for a.e. } t \in (0,T], \tag{2.57}$$

holds for all $\boldsymbol{\phi} \in V(\Omega)$ and $\widehat{\mathbf{H}}_\varepsilon(\mathbf{x},0) = \mathbf{0}$ in Ω.

PROPOSITION 2.5 The solution $\widehat{\mathbf{H}}_\varepsilon$ of $(\mathbf{P}^{mB}_\varepsilon\mathbf{1})$ exists and is unique. The solution $\widehat{\mathbf{H}}_\varepsilon\colon [0,T] \to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and satisfies $\widehat{\mathbf{H}}_\varepsilon(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$. Moreover, the following convergence properties to the solution $\widehat{\mathbf{H}}$ of $(\mathbf{P}^B\mathbf{1})$ hold. As $\varepsilon \searrow 0$,

$$\begin{aligned} &\widehat{\mathbf{H}}_\varepsilon \to \widehat{\mathbf{H}} \quad \text{strongly in } C([0,T];L^2(\Omega;\mathbb{R}^3)), \\ &\partial_t\widehat{\mathbf{H}}_\varepsilon \to \partial_t\widehat{\mathbf{H}} \quad \text{strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \\ &\frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_\varepsilon(t)|)\mathrm{d}\mathbf{x} \to 0 \quad \text{uniformly in } [0,T]. \end{aligned} \tag{2.58}$$

Proof. Define $E_\varepsilon(\cdot)\colon L^2_\mu(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ by

$$E_\varepsilon(\boldsymbol{\phi}) := \begin{cases} \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\boldsymbol{\phi}|)\mathrm{d}\mathbf{x} & \text{if } \boldsymbol{\phi}\in V(\Omega), \\ +\infty & \text{otherwise} . \end{cases}$$

Then Problem $(\mathbf{P}^{mB}_\varepsilon\mathbf{1})$ is written as an evolution equation

$$d_t\widehat{\mathbf{H}}_\varepsilon(t) + \partial_t\mathbf{H}_s(t) \in -\partial E_\varepsilon(\widehat{\mathbf{H}}_\varepsilon(t)) \tag{2.59}$$

for a.e $t\in(0,T]$ and $\widehat{\mathbf{H}}_\varepsilon(0)=\mathbf{0}$.

The energy E_ε is convex and not identically $+\infty$. By noting the property (2.14) of g, we can show that E_ε is lower semicontinuous. The existence of a unique solution $\widehat{\mathbf{H}}_\varepsilon$ is, thus, proved in the same way as Proposition 2.4.

We will prove the convergence properties (2.58). Take any sequence $\{\varepsilon_i\}_{i=1}^\infty$ satisfying $\varepsilon_i \searrow 0$ as $i\to+\infty$. By Attouch (1978, Theorem 2.1), it is sufficient to prove that the sequence of energies E_{ε_i} converges to the energy E defined in (2.55) in the sense of Mosco (see, e.g. Attouch (1984)) as $i\to+\infty$, i.e.

(i) If $\boldsymbol{\phi}_{\varepsilon_i} \rightharpoonup \boldsymbol{\phi}$ weakly in $L^2_\mu(\Omega;\mathbb{R}^3)$ as $i\to+\infty$, $E(\boldsymbol{\phi}) \leqslant \liminf_{i\to+\infty} E_{\varepsilon_i}(\boldsymbol{\phi}_{\varepsilon_i})$ holds.

(ii) For any $\boldsymbol{\phi}\in L^2_\mu(\Omega;\mathbb{R}^3)$ with $E(\boldsymbol{\phi}) < +\infty$, there exists a sequence $\{\boldsymbol{\phi}_{\varepsilon_i}\}_{i=1}^\infty$ such that $\boldsymbol{\phi}_{\varepsilon_i}\to\boldsymbol{\phi}$ strongly in $L^2_\mu(\Omega;\mathbb{R}^3)$ and $E_{\varepsilon_i}(\boldsymbol{\phi}_{\varepsilon_i}) \to E(\boldsymbol{\phi})$ as $i\to+\infty$.

Since $E_{\varepsilon_i}(\boldsymbol{\phi}) = E(\boldsymbol{\phi}) = 0$ for all $\boldsymbol{\phi}\in S$ and all i, (ii) is true. Assume $\boldsymbol{\phi}_{\varepsilon_i} \rightharpoonup \boldsymbol{\phi}$ weakly in $L^2_\mu(\Omega;\mathbb{R}^3)$ and $E_{\varepsilon_i}(\boldsymbol{\phi}_{\varepsilon_i}) \leqslant \lambda$ for all $i\in\mathbb{N}$. By convexity of $g(\cdot)$ and Fatou's lemma, we see that by taking a subsequence and its convex combination denoted by $\{\sum_{i=1}^n \boldsymbol{\phi}_{\varepsilon_i}/n\}_{n=1}^\infty$

$$\int_{\Omega_s} g(|\mathbf{curl}\,\boldsymbol{\phi}|)\mathrm{d}\mathbf{x} \leqslant \liminf_{n\to+\infty}\int_{\Omega_s} g\left(\left|\frac{1}{n}\sum_{i=1}^n \mathbf{curl}\,\boldsymbol{\phi}_{\varepsilon_i}\right|\right)\mathrm{d}\mathbf{x} \leqslant \liminf_{n\to+\infty}\frac{1}{n}\sum_{i=1}^n \varepsilon_i\lambda = 0,$$

which yields $|\mathbf{curl}\boldsymbol{\phi}(\mathbf{x})| \leqslant \mathscr{J}_c$ a.e. in Ω_s. Therefore, $\boldsymbol{\phi}\in S$ and $E(\boldsymbol{\phi}) = 0 \leqslant \lambda$, which means that (i) is correct. The desired convergence properties are proved by applying Attouch (1978, Theorem 2.1). □

The variational inequality formulation with the power law constitutive relation $\mathbf{E}\in\partial\gamma_p^P(\mathbf{J})$ is stated as follows:

$(\mathbf{P}^P_p\mathbf{1})$ Find $\widehat{\mathbf{H}}_p \in H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ such that $\widehat{\mathbf{H}}_p(t)\in V_p(\Omega)$ for all $t\in[0,T]$,

$$\int_\Omega \mu\langle\partial_t\widehat{\mathbf{H}}_p(\mathbf{x},t) + \partial_t\mathbf{H}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x}) - \widehat{\mathbf{H}}_p(\mathbf{x},t)\rangle\mathrm{d}\mathbf{x} + \int_\Omega \gamma_p^P(\mathbf{curl}\boldsymbol{\phi}(\mathbf{x}))\mathrm{d}\mathbf{x}$$
$$-\int_\Omega \gamma_p^P(\mathbf{curl}\,\widehat{\mathbf{H}}_p(\mathbf{x},t))\mathrm{d}\mathbf{x} \geqslant 0, \qquad \text{for a.e. } t\in(0,T], \tag{2.60}$$

holds for all $\boldsymbol{\phi}\in V_p(\Omega)$ and $\widehat{\mathbf{H}}_p(\mathbf{x},0) = \mathbf{0}$ in Ω.

PROPOSITION 2.6 The solution $\widehat{\mathbf{H}}_p$ of $(\mathbf{P}^P_p\mathbf{1})$ exists and is unique. The solution $\widehat{\mathbf{H}}_p\colon[0,T]\to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and satisfies $\widehat{\mathbf{H}}_p(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t\in[0,T]$. Moreover, the following convergence properties to the solution $\widehat{\mathbf{H}}$ of $(\mathbf{P}^B\mathbf{1})$ hold. As $p \to +\infty$,

$$\begin{aligned} &\widehat{\mathbf{H}}_p \to \widehat{\mathbf{H}} \quad \text{strongly in } C([0,T]; L^2(\Omega;\mathbb{R}^3)),\\ &\partial_t\widehat{\mathbf{H}}_p \to \partial_t\widehat{\mathbf{H}} \quad \text{strongly in } L^2(0,T; L^2(\Omega;\mathbb{R}^3)),\\ &\int_{\Omega_s} \gamma_p^P(\mathbf{curl}\,\widehat{\mathbf{H}}_p(t))\mathrm{d}\mathbf{x} \to 0 \quad \text{uniformly in } [0,T]. \end{aligned} \tag{2.61}$$

Proof. Let us define the energy functional $E_p\colon L^2_\mu(\Omega;\mathbb{R}^3) \to \mathbb{R}\cup\{+\infty\}$ by

$$E_p(\boldsymbol{\phi}) := \begin{cases} \int_{\Omega_s} \gamma_p^P(\mathbf{curl}\,\boldsymbol{\phi})\mathrm{d}\mathbf{x} & \text{if } \boldsymbol{\phi} \in V_p(\Omega),\\ +\infty & \text{otherwise.} \end{cases}$$

The convexity of E_p is obvious. Similarly as in Proposition 2.4, we can show the existence of a unique solution $\widehat{\mathbf{H}}_p$ with the desired properties.

To show the convergence properties (2.61), we show that E_{p_i} converges to E in the sense of Mosco as $i \to +\infty$ for any sequence $\{p_i\}_{i=1}^{\infty} \subset \mathbb{R}_{\geqslant 2}$ satisfying $p_i \nearrow +\infty$. Let us check Condition (ii) of Mosco convergence stated in the proof of Proposition 2.5 first. Take any $\boldsymbol{\phi} \in S$. We see that

$$0 \leqslant E_{p_i}(\boldsymbol{\phi}) - E(\boldsymbol{\phi}) \leqslant \mathscr{J}_c|\Omega_s|/p_i \to 0$$

as $i \to +\infty$. Thus, (ii) holds. To show (i), assume that $\boldsymbol{\phi}_i \rightharpoonup \boldsymbol{\phi}$ weakly in $L^2_\mu(\Omega;\mathbb{R}^3)$ as $i \to +\infty$ and $E_{p_i}(\boldsymbol{\phi}_i) \leqslant \lambda$ for any $i \in \mathbb{N}$, i.e.

$$\frac{\mathscr{J}_c}{p_i}\int_{\Omega_s} |\mathbf{curl}\,\boldsymbol{\phi}_i/\mathscr{J}_c|^{p_i}\,\mathrm{d}\mathbf{x} \leqslant \lambda. \tag{2.62}$$

Fix p_i and take $q \in [2, p_i]$. Then, by applying Hölder's inequality to (2.62) we have

$$\int_{\Omega_s} |\mathbf{curl}\,\boldsymbol{\phi}_i/\mathscr{J}_c|^q\,\mathrm{d}\mathbf{x} \leqslant \left(\int_{\Omega_s} |\mathbf{curl}\,\boldsymbol{\phi}_i/\mathscr{J}_c|^{p_i}\,\mathrm{d}\mathbf{x}\right)^{q/p_i} |\Omega_s|^{1-q/p_i} \leqslant (\lambda p_i/\mathscr{J}_c)^{q/p_i}|\Omega_s|^{1-q/p_i}. \tag{2.63}$$

By taking $q = 2$ in (2.63), we obtain

$$\int_{\Omega_s} |\mathbf{curl}\,\boldsymbol{\phi}_i/\mathscr{J}_c|^2\,\mathrm{d}\mathbf{x} \leqslant (\lambda p_i/\mathscr{J}_c)^{2/p_i}|\Omega_s|^{1-2/p_i}. \tag{2.64}$$

Since $\lim_{i\to\infty}(\lambda p_i/\mathscr{J}_c)^{2/p_i}|\Omega_s|^{1-2/p_i} = |\Omega_s|$, (2.64) implies that $\{\mathbf{curl}\,\boldsymbol{\phi}_i\}_{i=1}^{\infty}$ is bounded in $L^2(\Omega;\mathbb{R}^3)$. Therefore, by extracting a subsequence still denoted by $\{\mathbf{curl}\,\boldsymbol{\phi}_i\}_{i=1}^{\infty}$, we observe that $\boldsymbol{\phi}_i$ weakly converges to $\boldsymbol{\phi}$ in $H(\mathbf{curl};\Omega)$ as $i \to \infty$ and $\boldsymbol{\phi} \in V(\Omega)$. We show that $|\mathbf{curl}\,\boldsymbol{\phi}| \leqslant \mathscr{J}_c$ in Ω_s. We can choose a subsequence of $\{\boldsymbol{\phi}_i\}_{i=1}^{\infty}$ so that its convex combination denoted by $\{\sum_{j=1}^{i}\boldsymbol{\phi}_j/i\}_{i=1}^{\infty}$ strongly converges to $\boldsymbol{\phi}$ in $H(\mathbf{curl};\Omega)$ as $i \to +\infty$. Thus, if necessary by taking a subsequence, we see that $\mathbf{curl}(\sum_{j=1}^{i}\boldsymbol{\phi}_j(\mathbf{x})/i)$ converges to $\mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x})$ a.e. in Ω as $i \to +\infty$. By applying Fatou's lemma to (2.63), we have

$$\int_{\Omega_s} |\mathbf{curl}\,\boldsymbol{\phi}/\mathscr{J}_c|^q\,\mathrm{d}\mathbf{x} \leqslant \liminf_{i\to+\infty}\left(\sum_{j=1}^{i}(\lambda p_j/\mathscr{J}_c)^{q/p_j}|\Omega_s|^{1-q/p_j}/i\right) = |\Omega_s|,$$

or $\|\mathbf{curl}\,\boldsymbol{\phi}/\mathscr{J}_c\|_{L^q(\Omega_s;\mathbb{R}^3)} \leqslant |\Omega_s|^{1/q}$. By sending $q \to \infty$ we obtain $\|\mathbf{curl}\,\boldsymbol{\phi}/\mathscr{J}_c\|_{L^\infty(\Omega_s;\mathbb{R}^3)} \leqslant 1$. Therefore, $\boldsymbol{\phi} \in S$ and $E(\boldsymbol{\phi}) = 0 \leqslant \lambda$. Thus, (ii) has been proved. This Mosco convergence immediately shows the desired convergence properties by Attouch (1978, Theorem 2.1). □

We will use the following statement, which can be proved in the same way as the proof above, in Section 4.

COROLLARY 2.1 Let $\{p_n\}_{n=1}^\infty$ be a sequence satisfying $p_n \geqslant 2$ for any $n \in \mathbb{N}$ and $p_n \to +\infty$ as $n \to +\infty$.

1. If a sequence $\{\boldsymbol{\psi}_n\}_{n=1}^\infty \subset L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ satisfies that, for a.e. $t \in (0, T)$ and any $n \in \mathbb{N}$,

$$\frac{1}{p_n}\int_\Omega |\boldsymbol{\psi}_n(t)|^{p_n}\,\mathrm{d}\mathbf{x} \leqslant \lambda,$$

   where $\lambda \geqslant 0$, then $\{\boldsymbol{\psi}_n\}_{n=1}^\infty$ is bounded in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$.
2. For a sequence $\{\boldsymbol{\phi}_n\}_{n=1}^\infty \subset L^2(0, T; L^2(\Omega; \mathbb{R}^3))$ with $\boldsymbol{\phi}_n \to \boldsymbol{\phi}$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$, assume

$$\frac{1}{p_n}\int_0^T\int_{\Omega_s} |\mathbf{curl}\,\boldsymbol{\phi}_n/\mathscr{J}_c|^{p_n}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \leqslant \lambda$$

   for all $n \in \mathbb{N}$. Then $|\mathbf{curl}\,\boldsymbol{\phi}(\mathbf{x}, t)| \leqslant \mathscr{J}_c$ a.e. in $\Omega_s \times (0, T)$.

2.4 *Magnetic field—magnetic scalar potential hybrid formulation*

The curl-free constraint in the nonconducting region Ω_d can be enforced by expressing the magnetic field as a magnetic scalar potential. This hybrid formulation was recently applied to time-harmonic eddy current models with input current intensities on the boundary of the domain in Bermúdez *et al.* (2002). We adopt this method to rewrite the variational inequality formulation $(\mathbf{P}^B\mathbf{1})$ in an equivalent form without the constraint.

Let us prepare some notations. For $\mathbf{u_1} \in L^2(\Omega_s; \mathbb{R}^3)$ and $\mathbf{u_2} \in L^2(\Omega_d; \mathbb{R}^3)$, $(\mathbf{u_1}|\mathbf{u_2}) \in L^2(\Omega; \mathbb{R}^3)$ is defined by

$$(\mathbf{u_1}|\mathbf{u_2}) := \begin{cases} \mathbf{u_1} & \text{in } \Omega_s, \\ \mathbf{u_2} & \text{in } \Omega_d. \end{cases}$$

We define a linear space $W(\Omega)$ and its subspace $W_p(\Omega)$ $(p \geqslant 2)$ by

$$\begin{aligned} W(\Omega) &:= \{(\boldsymbol{\phi}|\nabla v) \in L^2(\Omega; \mathbb{R}^3) | (\boldsymbol{\phi}, v) \in L^2(\Omega_s; \mathbb{R}^3) \\ &\qquad \times H^1(\Omega_d), (\boldsymbol{\phi}|\nabla v) \in H(\mathbf{curl}; \Omega), v = 0 \text{ on } \partial\Omega\}. \\ W_p(\Omega) &:= \{(\boldsymbol{\phi}|\nabla v) \in W(\Omega) | \mathbf{curl}\,\boldsymbol{\phi} \in L^p(\Omega_s; \mathbb{R}^3)\}. \end{aligned}$$

The space $W(\Omega)$ is endowed with the inner product of $H(\mathbf{curl}; \Omega)$.

PROPOSITION 2.7 The space $W(\Omega)$ is isomorphic to $V(\Omega)$ as a Hilbert space.

Proof. For any $\mathbf{H} \in V(\Omega)$, there exists a scalar potential $v_H \in H^1(\Omega_d)$ such that $\mathbf{H}|_{\Omega_d} = \nabla v_H$ and v_H is unique up to an additive constant since $\mathbf{curl}\,\mathbf{H} = \mathbf{0}$ in a simply connected domain Ω_d (see, e.g. Monk, 2003, Theorem 3.37). The boundary condition $\mathbf{n} \times \mathbf{H} = \mathbf{0}$ on $\partial\Omega$ implies that the surface gradient of v_H

on $\partial\Omega$ is zero, and therefore v_H is constant on $\partial\Omega$. By choosing v_H to be zero on $\partial\Omega$, we can uniquely determine v_H satisfying $\mathbf{H}|_{\Omega_d} = \nabla v_H$. The linear map $\mathbf{H} \mapsto (\mathbf{H}|_{\Omega_s}|\nabla v_H)$ from $V(\Omega)$ to $W(\Omega)$ is thus well-defined and gives the desired isomorphism. □

This proposition allows us to reformulate Problem ($\mathbf{P}^B\mathbf{1}$) as a problem where the curl-free constraint imposed on test functions is eliminated. Define a convex set $R \subset W(\Omega)$ by

$$R := \{(\boldsymbol{\phi}|\nabla v) \in W(\Omega) \| \mathbf{curl}\,\boldsymbol{\phi}| \leqslant \mathscr{J}_c \text{ a.e. in } \Omega_s\}.$$

The hybrid problem ($\mathbf{P}^B\mathbf{2}$) is stated as follows.

($\mathbf{P}^B\mathbf{2}$) Find $\boldsymbol{\psi}$: $[0,T] \to H(\mathbf{curl};\Omega_s)$ and u: $[0,T] \to H^1(\Omega_d)$ such that $(\boldsymbol{\psi}|\nabla u) \in H^1(0,T; L^2(\Omega;\mathbb{R}^3))$, $(\boldsymbol{\psi}|\nabla u)(t) \in R$ for all $t \in [0,T]$,

$$\int_{\Omega_s} \mu_s \langle \partial_t \boldsymbol{\psi}(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \boldsymbol{\phi}(\mathbf{x}) - \boldsymbol{\psi}(\mathbf{x},t)\rangle \mathrm{d}\mathbf{x} + \int_{\Omega_d} \mu_d \langle \partial_t \nabla u(\mathbf{x},t) + \partial_t \mathbf{H}_s(\mathbf{x},t), \nabla v(\mathbf{x}) - \nabla u(\mathbf{x},t)\rangle \mathrm{d}\mathbf{x} \geqslant 0, \quad \text{for a.e. } t \in (0,T], \tag{2.65}$$

holds for all $(\boldsymbol{\phi}|\nabla v) \in R$ and $(\boldsymbol{\psi}|\nabla u)(\mathbf{x},0) = \mathbf{0}$ in Ω.

By the equivalence between $V(\Omega)$ and $W(\Omega)$ and Proposition 2.4, the existence of a unique solution $(\boldsymbol{\psi}|\nabla u)$ of ($\mathbf{P}^B\mathbf{2}$) such that $(\boldsymbol{\psi}|\nabla u)$: $[0,T] \to L^2(\Omega;\mathbb{R}^3)$ is Lipschitz continuous and $(\boldsymbol{\psi}|\nabla u)(t) + \mathbf{H}_s(t) \in X^{(\mu)}(\Omega)$ for all $t \in [0,T]$ follows immediately. It is also possible to rewrite Problems ($\mathbf{P}_\varepsilon^{mB}\mathbf{1}$) and ($\mathbf{P}_p^P\mathbf{1}$) as hybrid problems with the scalar magnetic potential.

3. Discretization

In this section, we discretize our variational inequality formulations ($\mathbf{P}_\varepsilon^{mB}\mathbf{1}$) and ($\mathbf{P}_p^P\mathbf{1}$) to construct discrete solutions converging to the analytical solutions of ($\mathbf{P}^B\mathbf{1}$) and ($\mathbf{P}^B\mathbf{2}$). Let us precisely define the geometry. The domain Ω ($\subset\mathbb{R}^3$) is a convex polyhedron. The bulk type-II superconductor Ω_s ($\subset\Omega$) is a simply connected polyhedral domain with a connected boundary $\partial\Omega_s$ satisfying $\partial\Omega_s \cap \partial\Omega = \emptyset$. Moreover, we assume that the domain Ω_s is starshaped for a point $y_0 \in \Omega_s$ in the sense that

$$\text{for any } z \in \overline{\Omega_s}, \quad \alpha(z - y_0) + y_0 \in \Omega_s \quad \forall \alpha \in [0,1). \tag{3.1}$$

Let Ω_d denote the nonconducting region $\Omega\backslash\overline{\Omega_s}$. Note that in this situation Ω_d ($=\Omega\backslash\overline{\Omega_s}$) is simply connected, and Ω and Ω_s can be meshed by tetrahedra (see Fig. 2).

3.1 *Finite-element approximation*

Let τ_h be a tetrahedral mesh covering Ω, satisfying $h = \max\{h_K | K \in \tau_h\}$, where h_K is the diameter of the smallest sphere containing $\overline{K}$. The mesh τ_h is assumed to be regular in the sense that there are constants $C > 0$ and $h_0 > 0$ such that

$$h_K/\rho_K \leqslant C \quad \forall K \in \tau_h, \quad 0 < \forall h \leqslant h_0, \tag{3.2}$$

where ρ_K is the diameter of the largest sphere contained in $\overline{K}$. Moreover, the mesh τ_h is quasi-uniform on $\partial\Omega$ in the sense that there is a constant $C' > 0$ such that

$$h/h_f \leqslant C' \quad \text{for any face } f \subset \partial\Omega \text{ and } 0 < \forall h \leqslant h_0, \tag{3.3}$$

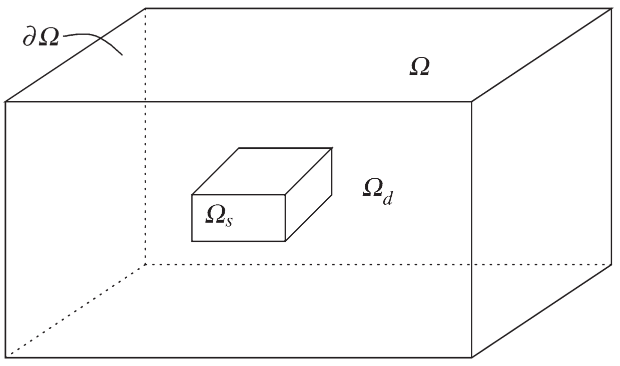

FIG. 2. Domain of the problem.

where h_f is the diameter of the smallest circle containing f (see Monk, 2003). We assume that each element $K \in \tau_h$ belongs either to $\overline{\Omega_s}$ or to $\overline{\Omega_d}$.

Set the space R_1 of vector polynomials of degree 1 by

$$R_1 := \{\mathbf{a} + \mathbf{b} \times \mathbf{x} | \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}.$$

The curl-conforming finite-element space $U_h(\Omega)$ by Nédélec (1980) of the lowest order on tetrahedral mesh is defined by

$$U_h(\Omega) := \{\boldsymbol{\phi}_h \in H(\mathbf{curl}; \Omega) | \boldsymbol{\phi}_h|_K \in R_1 \; \forall K \in \tau_h\},$$

with the degrees of freedom

$$M_e(\boldsymbol{\phi}_h) := \int_e \langle \boldsymbol{\phi}_h, \boldsymbol{\tau} \rangle \mathrm{d}s,$$

where e is an edge of $K \in \tau_h$ and $\boldsymbol{\tau}$ is a unit tangent to e. The interpolant $r_h(\boldsymbol{\phi}) \in U_h(\Omega)$ of a sufficiently smooth function $\boldsymbol{\phi}$ is defined by $M_e(\boldsymbol{\phi} - r_h(\boldsymbol{\phi})) = 0$ for all edges e. For more details on the edge element, see Girault & Raviart (1986) or Monk (2003). To make the argument clear, let us state a lemma proved in Girault & Raviart (1986, Chapter III, Lemma 5.7) and Monk (2003, Lemma 5.35).

LEMMA 3.1 For $\boldsymbol{\phi}_h \in U_h(\Omega)$ and a face $f \subset K$ $(K \in \tau_h)$, the tangential component of $\boldsymbol{\phi}_h$ on f is zero if, and only if, $M_{e_i}(\boldsymbol{\phi}_h) = 0$ $(i = 1, 2, 3)$, where e_i $(i = 1, 2, 3)$ are the edges of f.

We define the finite-dimensional subspace $V_h(\Omega)$ of $V(\Omega)$ by

$$V_h(\Omega) := \{\boldsymbol{\phi}_h \in U_h(\Omega) | \mathbf{curl}\, \boldsymbol{\phi}_h = \mathbf{0} \text{ in } \Omega_d, \; \mathbf{n} \times \boldsymbol{\phi}_h = \mathbf{0} \text{ on } \partial\Omega\}.$$

Note that the boundary condition $\mathbf{n} \times \boldsymbol{\phi}_h = \mathbf{0}$ on $\partial\Omega$ is attained by taking all the degrees of freedom associated with the edges on $\partial\Omega$ to be zero by Lemma 3.1.

To define a discrete space satisfying a discrete divergence-free condition and a discrete subspace of the space $W(\Omega)$, we need to use the standard H^1-conforming finite-element space $Z_h(\Omega)$ of the lowest order on a tetrahedral mesh

$$Z_h(\Omega) := \{ f_h \in H^1(\Omega) | f_h|_K \in P_1 \; \forall K \in \tau_h\},$$

where $P_1 := \{a_0 + a_1 x + a_2 y + a_3 z | a_i \in \mathbb{R}, i = 0, 1, 2, 3\}$. The degrees of freedom $m_\upsilon(f_h)$ of $Z_h(\Omega)$ are defined by

$$m_\upsilon(f_h) := f_h(\mathbf{x}_\upsilon),$$

where $\mathbf{x}_\upsilon$ ($\in \mathbb{R}^3$) is the coordinate of the vertex υ. Similarly, let us define the finite-element space $Z_{0,h}(\Omega)$ by

$$Z_{0,h}(\Omega) := \{f_h \in Z_h(\Omega) | f_h|_{\partial\Omega} = 0\}.$$

The boundary condition $f_h|_{\partial\Omega} = 0$ is attained by taking $m_\upsilon(f_h)$ for each vertex υ on $\partial\Omega$ to be zero.

The space of discrete divergence-free functions $X_h^{(\mu)}(\Omega)$ is defined by

$$X_h^{(\mu)}(\Omega) := \{\boldsymbol{\phi}_h \in U_h(\Omega) \mid \langle \mu\boldsymbol{\phi}_h, \nabla f_h \rangle_{L^2(\Omega;\mathbb{R}^3)} = 0 \ \forall\ f_h \in Z_{0,h}(\Omega)\}.$$

The discrete subspace $W_h(\Omega)$ of $W(\Omega)$ is defined by

$$W_h(\Omega) := \{(\boldsymbol{\phi}_h | \nabla u_h) \in L^2(\Omega;\mathbb{R}^3) | (\boldsymbol{\phi}_h, u_h) \in U_h(\Omega_s) \times Z_h(\Omega_d),$$

$$(\boldsymbol{\phi}_h | \nabla u_h) \in U_h(\Omega),\ u_h|_{\partial\Omega} = 0\},$$

where $U_h(\Omega_s) := \{\boldsymbol{\phi}_h|_{\Omega_s} | \boldsymbol{\phi}_h \in U_h(\Omega)\}$ and $Z_h(\Omega_d) := \{u_h|_{\Omega_d} | u_h \in Z_h(\Omega)\}$.

The following proposition is the discrete analogue of Proposition 2.7.

PROPOSITION 3.1 The space $W_h(\Omega)$ is isomorphic to $V_h(\Omega)$ as a Hilbert space.

Proof. Take any $\boldsymbol{\phi}_h \in V_h(\Omega)$. Similarly as in Proposition 2.7, there uniquely exists $\upsilon_{\phi_h} \in H^1(\Omega_d)$ such that $\boldsymbol{\phi}_h|_{\Omega_d} = \nabla\upsilon_{\phi_h}$ and $\upsilon_{\phi_h} = 0$ on $\partial\Omega$. We will show that $\upsilon_{\phi_h} \in Z_h(\Omega_d)$.

Take any $K \in \tau_h$ with $K \subset \overline{\Omega_d}$. We can write

$$\boldsymbol{\phi}_h|_K = \mathbf{a} + \mathbf{b} \times \mathbf{x} \quad (\mathbf{a} = (a_1, a_2, a_3)^{\mathrm{T}},\ \mathbf{b} \in \mathbb{R}^3).$$

The condition $\mathbf{curl}\,\boldsymbol{\phi}_h|_K = \mathbf{0}$ and an explicit calculation lead to $\mathbf{b} = \mathbf{0}$. Therefore, we see that $\boldsymbol{\phi}_h|_K = \nabla\upsilon_{\phi_h}\big|_K = \mathbf{a}$, or

$$\upsilon_{\phi_h}\big|_K = \text{constant} + a_1 x + a_2 y + a_3 z \in R_1,$$

which means that $\upsilon_{\phi_h} \in Z_h(\Omega_d)$.

Thus, the linear map $\boldsymbol{\phi}_h \mapsto (\boldsymbol{\phi}_h|_{\Omega_s} | \nabla\upsilon_{\phi_h})$ from $V_h(\Omega)$ to $W_h(\Omega)$ is well-defined. This map gives the desired isomorphism. □

Let Λ denote a bounded subset of $\mathbb{R}_{>0}$ which has the only accumulation point 0. Our assumptions on μ, Ω, τ_h enable us to apply the following discrete compactness result proved in Monk (2003, Chapter 7). In particular, the quasi-uniformity property (3.3) of τ_h on $\partial\Omega$ is assumed only to apply to this lemma.

LEMMA 3.1 Let $\{\boldsymbol{\phi}_h\}_{h\in\Lambda}$ satisfy $\boldsymbol{\phi}_h \in X_h^{(\mu)}(\Omega)$ for all $h \in \Lambda$. The following statements hold.

(i) If $\|\boldsymbol{\phi}_h\|_{H(\mathbf{curl};\Omega)} \leqslant C$ for all $h \in \Lambda$, there exist a subsequence $\{\boldsymbol{\phi}_{h_n}\}_{n=1}^{\infty} \subset \{\boldsymbol{\phi}_h\}_{h\in\Lambda}$ and $\boldsymbol{\phi} \in X^{(\mu)}(\Omega)$ such that as $n \to +\infty$

$$\boldsymbol{\phi}_{h_n} \to \boldsymbol{\phi} \text{ strongly in } L^2(\Omega;\mathbb{R}^3),$$

$$\boldsymbol{\phi}_{h_n} \rightharpoonup \boldsymbol{\phi} \text{ weakly in } H(\mathbf{curl};\Omega).$$

(ii) There is a constant $\hat{C} > 0$ such that for any $h \in \Lambda$,

$$\|\boldsymbol{\phi}_h\|_{L^2(\Omega;\mathbb{R}^3)} \leqslant \hat{C}(\|\mathbf{curl}\,\boldsymbol{\phi}_h\|_{L^2(\Omega;\mathbb{R}^3)} + \|\mathbf{n} \times \boldsymbol{\phi}_h\|_{L^2(\partial\Omega;\mathbb{R}^3)}).$$

The following lemma is from Girault & Raviart (1986, Chapter III, Theorem 5.4) and Monk (2003, Theorem 5.41).

LEMMA 3.2 There is a constant $C > 0$ such that

$$\|\boldsymbol{\phi} - r_h(\boldsymbol{\phi})\|_{H(\mathbf{curl};\Omega)} \leqslant Ch\|\boldsymbol{\phi}\|_{H^1(\mathbf{curl};\Omega)},$$

for any $\boldsymbol{\phi} \in H^1(\mathbf{curl}; \Omega)$.

By a similar argument as in Girault & Raviart (1986, Chapter III, Theorem 5.4) and Monk (2003, Theorem 5.41), we can prove the following estimates.

LEMMA 3.3 There exists a constant $C > 0$ depending only on the constant appearing in (3.2) such that

$$\|\boldsymbol{\phi} - r_h(\boldsymbol{\phi})\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqslant Ch\|\nabla\boldsymbol{\phi}\|_{L^\infty(\Omega;\mathbb{R}^9)},$$

$$\|\mathbf{curl}\,\boldsymbol{\phi} - \mathbf{curl}\,r_h(\boldsymbol{\phi})\|_{L^\infty(\Omega;\mathbb{R}^3)} \leqslant Ch\|\nabla\mathbf{curl}\,\boldsymbol{\phi}\|_{L^\infty(\Omega;\mathbb{R}^9)},$$

for any $\boldsymbol{\phi} \in C^2(\overline{\Omega}; \mathbb{R}^3)$.

We need one more lemma where Assumption (3.1) is used. Let $W^{p,q}(\Omega; \mathbb{R}^3)$ $(p \in \mathbb{N} \cup \{0\},\ 1 \leqslant q \leqslant +\infty)$ denote a Sobolev space defined as usual.

LEMMA 3.4 For any $\boldsymbol{\phi} \in C([0, T]; L^2(\Omega; \mathbb{R}^3))$ with $\boldsymbol{\phi}(t) \in S$ for all $t \in [0, T]$, there exists a sequence $\{\boldsymbol{\phi}_l\}_{l=1}^\infty \subset C([0, T]; W^{p,q}(\Omega; \mathbb{R}^3))$ for all $p \in \mathbb{N} \cup \{0\}$ and $1 \leqslant q \leqslant +\infty$ with $\boldsymbol{\phi}_l(t) \in S \cap C_0^\infty(\Omega; \mathbb{R}^3)$ for all $t \in [0, T]$ such that

$$\boldsymbol{\phi}_l \to \boldsymbol{\phi} \quad \text{strongly in } L^2(0, T; H(\mathbf{curl}; \Omega)) \tag{3.4}$$

as $l \to +\infty$.

Proof. Take any $\boldsymbol{\phi} \in C([0, T]; L^2(\Omega; \mathbb{R}^3))$ with $\boldsymbol{\phi}(t) \in S$ for all $t \in [0, T]$. Fix any $t \in [0, T]$. Noting $\mathbf{n} \times \boldsymbol{\phi} = \mathbf{0}$ on $\partial\Omega$, define $\widetilde{\boldsymbol{\phi}}(t) \in H(\mathbf{curl}; \mathbb{R}^3)$ by

$$\widetilde{\boldsymbol{\phi}}(t) := \begin{cases} \boldsymbol{\phi}(t) & \text{in } \Omega, \\ \mathbf{0} & \text{in } \mathbb{R}^3 \backslash \Omega. \end{cases}$$

For $\theta \in (0, 1)$, define $\widetilde{\boldsymbol{\phi}}_\theta(t) \in H(\mathbf{curl}; \mathbb{R}^3)$ by

$$\widetilde{\boldsymbol{\phi}}_\theta(\mathbf{x}, t) := \theta\widetilde{\boldsymbol{\phi}}\left(\frac{\mathbf{x} - \mathbf{y}_0}{\theta} + \mathbf{y}_0, t\right),$$

where $\mathbf{y}_0 \in \Omega_s$ is the point appearing in Assumption (3.1). Then we see that $\operatorname{supp}(\mathbf{curl}\,\widetilde{\boldsymbol{\phi}}_\theta(t)) \subset \Omega_s$. Indeed, if $\operatorname{supp}(\mathbf{curl}\,\widetilde{\boldsymbol{\phi}}_\theta(t)) \neq \emptyset$, for any $\hat{\mathbf{x}} \in \operatorname{supp}(\mathbf{curl}\,\widetilde{\boldsymbol{\phi}}_\theta(t))$ there is a sequence $\{\mathbf{x}_n\}_{n=1}^\infty \subset \mathbb{R}^3$ such that $\mathbf{x}_n \to \hat{\mathbf{x}}$ as $n \to +\infty$ and $\mathbf{curl}\,\widetilde{\boldsymbol{\phi}}_\theta(\mathbf{x}_n, t) \neq \mathbf{0}$. By the definition of $\widetilde{\boldsymbol{\phi}}_\theta$, we obtain

$$\frac{\mathbf{x}_n - \mathbf{y}_0}{\theta} + \mathbf{y}_0 \in \Omega_s.$$

By sending $n \to +\infty$, we have

$$\frac{\hat{\mathbf{x}} - \mathbf{y}_0}{\theta} + \mathbf{y}_0 \in \overline{\Omega_s}.$$

Assumption (3.1) yields

$$\hat{\mathbf{x}} = \theta \left(\frac{\hat{\mathbf{x}} - \mathbf{y}_0}{\theta} + \mathbf{y}_0 - \mathbf{y}_0 \right) + \mathbf{y}_0 \in \Omega_s.$$

Since Ω is convex we can similarly show $\mathrm{supp}(\widetilde{\boldsymbol{\phi}}_\theta(t)) \subset \Omega$, which implies $\mathbf{n} \times \widetilde{\boldsymbol{\phi}}_\theta(t) = \mathbf{0}$ on $\partial\Omega$. Moreover, the inequality $|\mathbf{curl}\,\widetilde{\boldsymbol{\phi}}_\theta(\mathbf{x}, t)| \leqslant \mathscr{J}_c$ holds a.e. in Ω_s.

For any $\theta \in (0, 1)$, we can choose $\varepsilon = \varepsilon(\theta) > 0$ sufficiently small so that we have $\rho_\varepsilon * \widetilde{\boldsymbol{\phi}}_\theta(t)|_\Omega \in S \cap C_0^\infty(\Omega; \mathbb{R}^3)$, where $\rho_\varepsilon \in C_0^\infty(\mathbb{R}^3)$ is a mollifier. By standard properties of a mollifier, it is seen that $\rho_\varepsilon * \widetilde{\boldsymbol{\phi}}_\theta|_\Omega \to \boldsymbol{\phi}$ strongly in $L^2(0, T; H(\mathbf{curl}; \Omega))$ as $\theta \nearrow 1$, $\varepsilon(\theta) \searrow 0$.

For any multi-index $\alpha \in (\mathbb{N} \cup \{0\})^3$

$$|\partial\mathbf{x}^\alpha(\rho_\varepsilon * \widetilde{\boldsymbol{\phi}}_\theta)(\mathbf{x}, t)| \leqslant C(\varepsilon, \alpha)\|\widetilde{\boldsymbol{\phi}}_\theta(t)\|_{L^2(\Omega;\mathbb{R}^3)} = C(\varepsilon, \alpha)\theta^{5/2}\|\boldsymbol{\phi}(t)\|_{L^2(\Omega;\mathbb{R}^3)},$$

which implies that $\rho_\varepsilon * \widetilde{\boldsymbol{\phi}}_\theta|_\Omega \in C([0, T]; W^{p,q}(\Omega; \mathbb{R}^3))$ for all $p \in \mathbb{N} \cup \{0\}$ and $1 \leqslant q \leqslant +\infty$. □

Take $N \in \mathbb{N}$ and set $\Delta t := T/N$. By using a function $\boldsymbol{\phi}_l$ from Lemma 3.4, we define a piecewise constant in time function $\overline{\boldsymbol{\phi}_{l,h}}$: $[0, T] \to V_h(\Omega)$ by

$$\overline{\boldsymbol{\phi}_{l,h}}(t) := \begin{cases} r_h(\boldsymbol{\phi}_l(\Delta t i)) & \text{in } (\Delta t(i-1), \Delta t i]\ (i = 1, \ldots, N), \\ r_h(\boldsymbol{\phi}_l(0)) & \text{on } \{t = 0\}. \end{cases}$$

The following properties will be useful in Section 4.

COROLLARY 3.1 There is a constant $C > 0$, independent of $l \in \mathbb{N}$, $h \in \Lambda$ and Δt, such that

$$\|\mathbf{curl}\,\overline{\boldsymbol{\phi}_{l,h}}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} \leqslant Ch\|\nabla\mathbf{curl}\,\boldsymbol{\phi}_l\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))} + \mathscr{J}_c. \tag{3.5}$$

Moreover, assume that the time step size Δt depends on h and satisfies $\lim_{h\searrow 0, h\in\Lambda} \Delta t(h) = 0$. Then the following convergence properties hold as $h \searrow 0$:

$$\overline{\boldsymbol{\phi}_{l,h}} \to \boldsymbol{\phi}_l \quad \text{strongly in } L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^3)), \tag{3.6}$$

$$\mathbf{curl}\,\overline{\boldsymbol{\phi}_{l,h}} \to \mathbf{curl}\,\boldsymbol{\phi}_l \quad \text{strongly in } L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^3)). \tag{3.7}$$

Proof. These statements can be proved by noting Lemmas 3.3 and 3.4. We only give the proof of (3.5).

$$\begin{aligned} \|\mathbf{curl}\,\overline{\boldsymbol{\phi}_{l,h}}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} &\leqslant \|\mathbf{curl}\,\overline{\boldsymbol{\phi}_{l,h}} - \mathbf{curl}\,\overline{\boldsymbol{\phi}_l}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} + \|\mathbf{curl}\,\overline{\boldsymbol{\phi}_l}\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^3))} \\ &\leqslant Ch\|\nabla\mathbf{curl}\,\boldsymbol{\phi}_l\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))} + \mathscr{J}_c. \end{aligned}$$

□

3.2 *Full discretization of the evolution problem*

Now we shall discretize the problems implicitly in time and in space by the finite element introduced in Section 3.1.

We need to introduce additional notation. Let $\mathbf{H}_{s,n}$ denote $\mathbf{H}_s(\Delta t n)$ and let $\mathbf{H}_{s,h,n}$ denote $r_h(\mathbf{H}_{s,n})$ for $n = 0, 1, \ldots, N$ $(=T/\Delta t)$. Note that since $\mathbf{H}_{s,n} \in H^1(\mathbf{curl}; \Omega)$ by Lemma 2.1, the interpolation is well-defined. Define the functional $F_{h,n,\varepsilon}$ $(n = 1, \ldots, N)$ on the fully discrete space $U_h(\Omega)$ by

$$
\begin{aligned}
&F_{h,n,\varepsilon}(\boldsymbol{\phi}_h) \\
&\quad := \frac{1}{2\Delta t}\int_\Omega \mu|\boldsymbol{\phi}_h|^2\,\mathrm{d}\mathbf{x} + \frac{1}{\Delta t}\int_\Omega \mu\langle -\widehat{\mathbf{H}}_{h,n-1,\varepsilon} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1}, \boldsymbol{\phi}_h\rangle \mathrm{d}\mathbf{x} + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\boldsymbol{\phi}_h|)\mathrm{d}\mathbf{x},
\end{aligned}
$$

where $\widehat{\mathbf{H}}_{h,0,\varepsilon} = r_h(\widehat{\mathbf{H}}_0) = \mathbf{0}$.

We consider the following optimization problems over the finite-dimensional space.

$(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1})$ For $n = 1 \to N$, find $\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ such that

$$
F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon}) = \min_{\boldsymbol{\phi}_h \in V_h(\Omega)} F_{h,n,\varepsilon}(\boldsymbol{\phi}_h),
$$

where $\widehat{\mathbf{H}}_{h,0,\varepsilon} = \mathbf{0}$.

Equivalently,

$(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ For $n = 1 \to N$, find $(\boldsymbol{\psi}_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon}) \in W_h(\Omega)$ such that

$$
F_{h,n,\varepsilon}((\boldsymbol{\psi}_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon})) = \min_{(\boldsymbol{\phi}_h|\nabla v_h) \in W_h(\Omega)} F_{h,n,\varepsilon}((\boldsymbol{\phi}_h|\nabla v_h)),
$$

where $(\boldsymbol{\psi}_{h,0,\varepsilon}|\nabla u_{h,0,\varepsilon}) = \mathbf{0}$.

PROPOSITION 3.2 There exists a unique minimizer $\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ of $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1})$. Moreover, $\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ satisfies the discrete divergence-free condition

$$
\widehat{\mathbf{H}}_{h,n,\varepsilon} + \mathbf{H}_{s,h,n} \in X_h^{(\mu)}(\Omega) \tag{3.8}
$$

and the discrete variational inequality

$$
\begin{aligned}
&\int_\Omega \mu\langle(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1})/\Delta t, \boldsymbol{\phi}_h - \widehat{\mathbf{H}}_{h,n,\varepsilon}\rangle \mathrm{d}\mathbf{x} \\
&\qquad + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\boldsymbol{\phi}_h|)\mathrm{d}\mathbf{x} - \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,n,\varepsilon}|)\mathrm{d}\mathbf{x} \geqslant 0
\end{aligned} \tag{3.9}
$$

for all $\boldsymbol{\phi}_h \in V_h(\Omega)$.

Proof. The existence of a unique minimizer $\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ is standard. We show (3.8).

Since $\nabla Z_{0,h}(\Omega) \subset V_h(\Omega)$ (see Monk, 2003), for any $w_h \in Z_{0,h}(\Omega)$ and $\delta > 0$, $\widehat{\mathbf{H}}_{h,n,\varepsilon} + \delta\nabla w_h \in V_h(\Omega)$ and

$$\begin{aligned}
&\lim_{\delta\downarrow 0}\{(F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon} + \delta\nabla w_h) - F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon}))/\delta\} \\
&\quad = \frac{1}{\Delta t}\int_\Omega \mu\langle \widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1}, \nabla w_h\rangle \mathrm{d}\mathbf{x} \\
&\quad = \frac{1}{\Delta t}\int_\Omega \mu\langle \widehat{\mathbf{H}}_{h,n,\varepsilon} + \mathbf{H}_{s,h,n}, \nabla w_h\rangle \mathrm{d}\mathbf{x} \geqslant 0. \qquad (3.10)
\end{aligned}$$

Here we have used the assumption $\widehat{\mathbf{H}}_{h,n-1,\varepsilon} + \mathbf{H}_{s,h,n-1} \in X_h^{(\mu)}(\Omega)$. Similarly by calculating $\lim_{\delta\downarrow 0}\{(F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \delta\nabla w_h) - F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon}))/\delta\}$, we have

$$\frac{1}{\Delta t}\int_\Omega \mu\langle \widehat{\mathbf{H}}_{h,n,\varepsilon} + \mathbf{H}_{s,h,n}, \nabla w_h\rangle \mathrm{d}\mathbf{x} \leqslant 0. \qquad (3.11)$$

Combining (3.10) with (3.11) we obtain (3.8).

We derive (3.9). The inequality $F_{h,n,\varepsilon}(\widehat{\mathbf{H}}_{h,n,\varepsilon}) \leqslant F_{h,n,\varepsilon}(\boldsymbol{\phi}_h)$ is equivalent to the inequality

$$\begin{aligned}
&\frac{1}{2\Delta t}\int_\Omega \mu|\boldsymbol{\phi} - \widehat{\mathbf{H}}_{h,n,\varepsilon}|^2\,\mathrm{d}\mathbf{x} + \frac{1}{\Delta t}\int_\Omega \mu\langle \widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1}, \boldsymbol{\phi}_h - \widehat{\mathbf{H}}_{h,n,\varepsilon}\rangle \mathrm{d}\mathbf{x} \\
&\qquad + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\boldsymbol{\phi}_h|)\mathrm{d}\mathbf{x} - \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,n,\varepsilon}|)\mathrm{d}\mathbf{x} \geqslant 0. \qquad (3.12)
\end{aligned}$$

Take any $\boldsymbol{\psi}_h \in V_h(\Omega)$ and $\alpha \in (0,1)$. Substituting $\boldsymbol{\phi}_h = \alpha\boldsymbol{\psi}_h + (1-\alpha)\widehat{\mathbf{H}}_{h,n,\varepsilon} \in V_h(\Omega)$ into (3.12), dividing both sides by α and sending $\alpha \downarrow 0$, we obtain the inequality (3.9). □

By Proposition 3.1 we immediately deduce the following statement.

COROLLARY 3.2 There exists a unique minimizer $(\boldsymbol{\psi}_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon}) \in W_h(\Omega)$ of $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$. Moreover $(\boldsymbol{\psi}_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon}) + \mathbf{H}_{s,h,n} \in X_h^{(\mu)}(\Omega)$ and the inequality (3.9) holds for $\widehat{\mathbf{H}}_{h,n,\varepsilon} = (\boldsymbol{\psi}_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon})$.

Similarly, we define the functional $G_{h,n,p}$ on $U_h(\Omega)$ by

$$\begin{aligned}
G_{h,n,p}(\boldsymbol{\phi}_h) &:= \frac{1}{2\Delta t}\int_\Omega \mu|\boldsymbol{\phi}_h|^2\,\mathrm{d}\mathbf{x} + \frac{1}{\Delta t}\int_\Omega \mu\langle -\widehat{\mathbf{H}}_{h,n-1,p} + \mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1}, \boldsymbol{\phi}_h\rangle \mathrm{d}\mathbf{x} \\
&\quad + \frac{\mathscr{J}_c}{p}\int_{\Omega_s} |\mathbf{curl}\,\boldsymbol{\phi}_h/\mathscr{J}_c|^p\,\mathrm{d}\mathbf{x},
\end{aligned}$$

where $\widehat{\mathbf{H}}_{h,0,p} = r_h(\widehat{\mathbf{H}}_0) = \mathbf{0}$.

The fully discrete formulation of $(\mathbf{P}^P_p\mathbf{1})$ is defined as follows.

$(\mathbf{P}^P_{h,\Delta t,p}\mathbf{1})$ For $n = 1 \to N$, find $\widehat{\mathbf{H}}_{h,n,p} \in V_h(\Omega)$ such that

$$G_{h,n,p}(\widehat{\mathbf{H}}_{h,n,p}) = \min_{\boldsymbol{\phi}_h \in V_h(\Omega)} G_{h,n,p}(\boldsymbol{\phi}_h),$$

where $\widehat{\mathbf{H}}_{h,0,p} = \mathbf{0}$.

Equivalently, we can define the full discretization of $(\mathbf{P}^P_p\mathbf{2})$ as follows.

($\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2}$) For $n = 1 \to N$, find $(\boldsymbol{\psi}_{h,n,p}|\nabla u_{h,n,p}) \in W_h(\Omega)$ such that

$$G_{h,n,p}((\boldsymbol{\psi}_{h,n,p}|\nabla u_{h,n,p})) = \min_{(\boldsymbol{\phi}_h|\nabla v_h)\in W_h(\Omega)} G_{h,n,p}((\boldsymbol{\phi}_h|\nabla v_h)),$$

where $(\boldsymbol{\psi}_{h,0,p}|\nabla u_{h,0,p}) = \mathbf{0}$.

The existence of unique minimizers of Problems ($\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{1}$) and ($\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2}$) can be stated in the same way as in Proposition 3.2 and Corollary 3.2. Note that the hybrid problems ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$) and ($\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2}$) are quite useful for practical computation since the curl-free constraint is automatically fulfilled by the scalar potential.

4. Convergence of discrete solutions

In this section, we will show the convergence of the discrete solutions, constructed by using the minimizers of the optimization problems proposed in Section 3, to the unique solution of the evolution variational inequality formulation.

4.1 *Convergence of the discrete solutions solving ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1}$) and ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$)*

We will show that the discrete solutions defined as the minimizers of ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1}$) and ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$) converge to the solution of ($\mathbf{P}^{B}\mathbf{1}$) and ($\mathbf{P}^{B}\mathbf{2}$), respectively. We define the piecewise linear in-time functions $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}$, $\widehat{\mathbf{H}}_{s,h,\Delta t}$, and the piecewise constant in-time functions $\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}$, $\overline{\mathbf{H}}_{s,h,\Delta t}$ by

$$\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) := \frac{t-\Delta t(n-1)}{\Delta t}\widehat{\mathbf{H}}_{h,n,\varepsilon} + \frac{\Delta tn - t}{\Delta t}\widehat{\mathbf{H}}_{h,n-1,\varepsilon} \quad \text{in } [\Delta t(n-1), \Delta tn],$$

$$\widehat{\mathbf{H}}_{s,h,\Delta t}(t) := \frac{t-\Delta t(n-1)}{\Delta t}\mathbf{H}_{s,h,n} + \frac{\Delta tn - t}{\Delta t}\mathbf{H}_{s,h,n-1} \quad \text{in } [\Delta t(n-1), \Delta tn],$$

$$\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) := \begin{cases} \widehat{\mathbf{H}}_{h,n,\varepsilon} & \text{in } (\Delta t(n-1), \Delta tn], \\ \widehat{\mathbf{H}}_{h,0,\varepsilon} & \text{on } \{t=0\}, \end{cases} \qquad \overline{\mathbf{H}}_{s,h,\Delta t}(t) := \begin{cases} \mathbf{H}_{s,h,n} & \text{in } (\Delta t(n-1), \Delta tn], \\ \mathbf{H}_{s,h,0} & \text{on } \{t=0\}, \end{cases}$$

for $n = 1, \ldots, N$, where $\widehat{\mathbf{H}}_{h,n,\varepsilon}$ are the minimizers of ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{1}$) and $\widehat{\mathbf{H}}_{h,0,\varepsilon} = \mathbf{0}$.

By definition, we easily see that $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}, \widehat{\mathbf{H}}_{s,h,\Delta t} \in C([0,T]; H(\mathbf{curl};\Omega))$, $\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}, \overline{\mathbf{H}}_{s,h,\Delta t} \in L^{\infty}(0,T; H(\mathbf{curl};\Omega))$ and $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t), \overline{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) \in V_h(\Omega)$ for all $t \in [0,T]$. The discrete analogue of (2.19) holds in the sense that $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) + \widehat{\mathbf{H}}_{s,h,\Delta t}(t)$, $\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) + \overline{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ for all $t \in [0,T]$ by (3.8).

LEMMA 4.1 The following estimates hold:

$$\|\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon} - \overline{\mathbf{H}}_{h,\Delta t,\varepsilon}\|_{L^{\infty}(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant \Delta t\|\partial_t\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}, \tag{4.1}$$

$$\begin{aligned}\|\partial_t\widehat{\mathbf{H}}_{s,h,\Delta t}\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} &\leqslant Ch\|\partial_t\mathbf{h}_s\|_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} \\ &\quad + C\|\mathbf{n}\times\partial_t\mathbf{h}_s\|_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}.\end{aligned} \tag{4.2}$$

The following convergence properties also hold as $h \searrow 0$ and $\Delta t \searrow 0$:

$$\widehat{\mathbf{H}}_{s,h,\Delta t} \to \mathbf{H}_s \quad \text{strongly in } C([0,T]; L^2(\Omega;\mathbb{R}^3)), \tag{4.3}$$

$$\overline{\mathbf{H}}_{s,h,\Delta t} \to \mathbf{H}_s \quad \text{strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)), \tag{4.4}$$

$$\partial_t\widehat{\mathbf{H}}_{s,h,\Delta t} \to \partial_t\mathbf{H}_s \quad \text{strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)). \tag{4.5}$$

Proof. To show (4.1), (4.3)–(4.5) is standard. We only give a proof for (4.2). By using Lemma 3.2, we observe that

$$\begin{aligned}
&\|\partial_t\widehat{\mathbf{H}}_{s,h,\Delta t}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\\
&\quad\leqslant \sum_{i=1}^{N}\frac{1}{\Delta t}\|\mathbf{H}_{s,h,i}-\mathbf{H}_{s,h,i-1}\|^2_{L^2(\Omega;\mathbb{R}^3)}\\
&\quad\leqslant 2\sum_{i=1}^{N}\frac{1}{\Delta t}\|r_h(\mathbf{H}_{s,i}-\mathbf{H}_{s,i-1})-\mathbf{H}_{s,i}+\mathbf{H}_{s,i-1}\|^2_{L^2(\Omega;\mathbb{R}^3)}+2\sum_{i=1}^{N}\frac{1}{\Delta t}\|\mathbf{H}_{s,i}-\mathbf{H}_{s,i-1}\|^2_{L^2(\Omega;\mathbb{R}^3)}\\
&\quad\leqslant Ch^2\sum_{i=1}^{N}\frac{1}{\Delta t}\|\mathbf{H}_{s,i}-\mathbf{H}_{s,i-1}\|^2_{H^1(\Omega;\mathbb{R}^3)}+2\sum_{i=1}^{N}\frac{1}{\Delta t}\|\mathbf{H}_{s,i}-\mathbf{H}_{s,i-1}\|^2_{L^2(\Omega;\mathbb{R}^3)}\\
&\quad\leqslant Ch^2\sum_{i=1}^{N}\int_{\Delta t(i-1)}^{\Delta t i}\|\partial_t\mathbf{H}_s(t)\|^2_{H^1(\Omega;\mathbb{R}^3)}\,\mathrm{d}t+2\sum_{i=1}^{N}\int_{\Delta t(i-1)}^{\Delta t i}\|\partial_t\mathbf{H}_s(t)\|^2_{L^2(\Omega;\mathbb{R}^3)}\,\mathrm{d}t\\
&\quad\leqslant Ch^2\|\partial_t\mathbf{H}_s\|^2_{L^2(0,T;H^1(\Omega;\mathbb{R}^3))}+2\|\partial_t\mathbf{H}_s\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}.
\end{aligned}$$

By combining this inequality with (2.39) and the Friedrichs inequality

$$\|\partial_t\mathbf{H}_s\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\leqslant C\|\mathbf{n}\times\partial_t\mathbf{h}_s\|_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))},$$

we obtain (4.2). □

Moreover, we have the following result.

PROPOSITION 4.1 Take any $\tau\in(0,1)$. The following bounds hold. For any $h\in\Lambda$, $\varepsilon>0$, $\Delta t\in(0,\tau]$,

$$\begin{aligned}
&\|\partial_t\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\\
&\quad\leqslant C\frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}}\big(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))}+\|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}\big),
\end{aligned}\tag{4.6}$$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in[0,T]}\left\{\int_{\Omega_s}g(|\mathbf{curl}\,\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}(t)|)\mathrm{d}\mathbf{x}\right\}\\
&\quad\leqslant C\varepsilon\max\{\mu_d,\mu_s\}\big(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))}+\|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}\big),
\end{aligned}\tag{4.7}$$

$$\|\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \leqslant \frac{C}{1-\tau}\mathrm{e}^{T/(1-\tau)}\frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}}\big(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} + \|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}\big), \tag{4.8}$$

where $C > 0$ is a positive constant independent of h, ε, Δt and μ.

Proof. By substituting $\boldsymbol{\phi}_h = \widehat{\mathbf{H}}_{h,n-1,\varepsilon}$ into (3.9), we have

$$\begin{aligned}
&\Delta t\int_\Omega \mu|(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon})/\Delta t|^2\,\mathrm{d}\mathbf{x} + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,n,\varepsilon}|)\mathrm{d}\mathbf{x} - \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,n-1,\varepsilon}|)\mathrm{d}\mathbf{x}\\
&\quad\leqslant \int_\Omega \mu\langle(\mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1})/\Delta t, \widehat{\mathbf{H}}_{h,n-1,\varepsilon} - \widehat{\mathbf{H}}_{h,n,\varepsilon}\rangle\mathrm{d}\mathbf{x}\\
&\quad\leqslant \frac{\Delta t}{2}\int_\Omega \mu|(\mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1})/\Delta t|^2\,\mathrm{d}\mathbf{x} + \frac{\Delta t}{2}\int_\Omega \mu|(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon})/\Delta t|^2\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\frac{\min\{\mu_d,\mu_s\}}{2}\int_{\Delta t(n-1)}^{\Delta tn}\int_\Omega |\partial_t\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,n,\varepsilon}|)\mathrm{d}\mathbf{x}\\
&\quad - \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,n-1,\varepsilon}|)\mathrm{d}\mathbf{x} \leqslant \frac{\max\{\mu_d,\mu_s\}}{2}\int_{\Delta t(n-1)}^{\Delta tn}\int_\Omega |\partial_t\widehat{\mathbf{H}}_{s,h,\Delta t}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned} \tag{4.9}$$

Summing (4.9) over $n = 1 \to m$ $(\leqslant N)$, we obtain

$$\begin{aligned}
&\frac{\min\{\mu_d,\mu_s\}}{2}\int_0^{\Delta tm}\int_\Omega |\partial_t\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \frac{1}{\varepsilon}\int_{\Omega_s} g(|\mathbf{curl}\,\widehat{\mathbf{H}}_{h,m,\varepsilon}|)\mathrm{d}\mathbf{x}\\
&\quad\leqslant \frac{\max\{\mu_d,\mu_s\}}{2}\int_0^{\Delta tm}\int_\Omega |\partial_t\widehat{\mathbf{H}}_{s,h,\Delta t}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned} \tag{4.10}$$

Combining the inequality (4.10) with (4.2) we obtain (4.6) and (4.7).

On the other hand, substituting $\boldsymbol{\phi}_h = \mathbf{0}$ into (3.9) and noting the equality $\langle\mathbf{p}-\mathbf{q},\mathbf{p}\rangle = |\mathbf{p}-\mathbf{q}|^2/2 + (|\mathbf{p}|^2 - |\mathbf{q}|^2)/2$, we have

$$\begin{aligned}
&\frac{\Delta t}{2}\int_\Omega \mu|(\widehat{\mathbf{H}}_{h,n,\varepsilon} - \widehat{\mathbf{H}}_{h,n-1,\varepsilon})/\Delta t|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2\Delta t}\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2\,\mathrm{d}\mathbf{x} - \frac{1}{2\Delta t}\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n-1,\varepsilon}|^2\,\mathrm{d}\mathbf{x}\\
&\quad\leqslant \frac{1}{2}\int_\Omega \mu|(\mathbf{H}_{s,h,n} - \mathbf{H}_{s,h,n-1})/\Delta t|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{4.11}$$

Multiplying (4.11) by Δt and summing over $n = 1 \to m$ $(\leqslant N)$, we have

$$\int_\Omega \mu|\widehat{\mathbf{H}}_{h,m,\varepsilon}|^2\,\mathrm{d}\mathbf{x} \leqslant \int_0^{\Delta tm}\int_\Omega \mu|\partial_t\mathbf{H}_{s,h}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \sum_{n=0}^m \Delta t\int_\Omega \mu|\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2\,\mathrm{d}\mathbf{x},$$

which is equivalent to

$$\int_{\Omega} \mu |\widehat{\mathbf{H}}_{h,m,\varepsilon}|^2 \,\mathrm{d}\mathbf{x} \leqslant \frac{1}{1-\Delta t} \int_0^{\Delta t m} \int_{\Omega} \mu |\partial_t \mathbf{H}_{s,h}|^2 \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \sum_{n=0}^{m-1} \frac{\Delta t}{1-\Delta t} \int_{\Omega} \mu |\widehat{\mathbf{H}}_{h,n,\varepsilon}|^2 \,\mathrm{d}\mathbf{x}. \tag{4.12}$$

By applying the discrete Gronwall inequality (see, e.g. Thomée, 1997, Lemma 10.5) to (4.12) and combining (4.2) we obtain (4.8). □

To reduce the parameters, we assume that Δt and ε are positive functions of h satisfying

$$\sup_{h \in \Lambda} \Delta t(h) < 1, \qquad \lim_{h \searrow 0, h \in \Lambda} \Delta t(h) = \lim_{h \searrow 0, h \in \Lambda} \varepsilon(h) = \lim_{h \searrow 0, h \in \Lambda} \frac{h^{2-\operatorname{sgn} A_4}}{\varepsilon(h)} = 0, \tag{4.13}$$

where $A_4 \geqslant 0$ is a constant in Assumption (2.14) and $\operatorname{sgn} A_4 = 0$ if $A_4 = 0$, 1 if $A_4 > 0$.

We are now ready to state the convergence result.

THEOREM 4.1 The piecewise linear in-time approximation $\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)}$ and the piecewise constant in-time approximation $\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)}$ converge to the unique solution $\widehat{\mathbf{H}}$ of $(\mathbf{P}^B\mathbf{1})$ in the following sense.

$$\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \to \widehat{\mathbf{H}} \quad \text{strongly in } C([0,T]; L^2(\Omega;\mathbb{R}^3)), \tag{4.14}$$

$$\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \rightharpoonup \widehat{\mathbf{H}} \quad \text{weak} * \text{ in } L^\infty(0,T; H(\mathbf{curl};\Omega)), \tag{4.15}$$

$$\partial_t\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \rightharpoonup \partial_t\widehat{\mathbf{H}} \quad \text{weakly in } L^2(0,T; L^2(\Omega;\mathbb{R}^3)), \tag{4.16}$$

$$\mathbf{curl}\,\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \to \mathbf{curl}\,\widehat{\mathbf{H}} \quad \text{strongly in } C([0,T]; (H(\mathbf{curl};\Omega))^*), \tag{4.17}$$

$$\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \to \widehat{\mathbf{H}} \quad \text{strongly in } L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)), \tag{4.18}$$

$$\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \rightharpoonup \widehat{\mathbf{H}} \quad \text{weak} * \text{ in } L^\infty(0,T; H(\mathbf{curl};\Omega)), \tag{4.19}$$

$$\mathbf{curl}\,\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} \to \mathbf{curl}\,\widehat{\mathbf{H}} \quad \text{strongly in } L^\infty(0,T; (H(\mathbf{curl};\Omega))^*), \tag{4.20}$$

as $h \searrow 0$, $h \in \Lambda$.

Proof. To simplify the notation let $\widehat{\mathbf{H}}_h$, $\overline{\mathbf{H}}_h$, $\widehat{\mathbf{H}}_{s,h}$ denote $\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)}$, $\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)}$, $\widehat{\mathbf{H}}_{s,h,\Delta t(h)}$, respectively.

Step 1: We show that there exist subsequences $\{\widehat{\mathbf{H}}_{h_n}\}_{n=1}^\infty$ and $\{\overline{\mathbf{H}}_{h_n}\}_{n=1}^\infty$ of $\{\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ and $\{\overline{\mathbf{H}}_h\}_{h\in\Lambda}$, respectively, and $\widehat{\mathbf{H}} \in L^\infty(0,T; H(\mathbf{curl};\Omega)) \cap H^1(0,T; L^2(\Omega;\mathbb{R}^3))$ with $\widehat{\mathbf{H}}(t) \in V(\Omega)$ for all $t \in [0,T]$ such that the convergence properties (4.14)–(4.20) hold for $\widehat{\mathbf{H}}_{h_n}$, $\overline{\mathbf{H}}_{h_n}$ and $\widehat{\mathbf{H}}$ as $n \to +\infty$.

By (4.7) and (4.8), we see that $\{\overline{\mathbf{H}}_h\}_{h\in\Lambda}$ is bounded in $L^\infty(0,T; H(\mathbf{curl};\Omega))$. Thus, so is $\{\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ in $L^\infty(0,T; H(\mathbf{curl};\Omega))$ by definition. Moreover, by (4.6) $\{\partial_t\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ is bounded in $L^2(0,T; L^2(\Omega;\mathbb{R}^3))$. Therefore, by extracting subsequences $\{\widehat{\mathbf{H}}_{h_n}\}_{n=1}^\infty$, $\{\overline{\mathbf{H}}_{h_n}\}_{n=1}^\infty$ of $\{\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ and $\{\overline{\mathbf{H}}_h\}_{h\in\Lambda}$, respectively, we observe the weak(∗) convergences (4.15), (4.16) and (4.19) to some $\widehat{\mathbf{H}}$ $(\in L^\infty(0,T; H(\mathbf{curl};\Omega)) \cap H^1(0,T; L^2(\Omega;\mathbb{R}^3)))$ with $\widehat{\mathbf{H}}(t) \in V(\Omega)$ a.e. $t \in (0,T)$.

We show the strong convergences (4.14) and (4.18). Fix any $t \in [0, T]$. Since $\{\widehat{\mathbf{H}}_{h_n}(t) + \widehat{\mathbf{H}}_{s,h_n}(t)\}_{n=1}^{\infty}$ is bounded in $H(\mathbf{curl}; \Omega)$ and $\widehat{\mathbf{H}}_{h_n}(t) + \widehat{\mathbf{H}}_{s,h_n}(t) \in X_{h_n}^{(\mu)}(\Omega)$ for any $n \in \mathbb{N}$, we can apply Lemma 3.1 (1) to see that $\{\widehat{\mathbf{H}}_{h_n}(t) + \widehat{\mathbf{H}}_{s,h_n}(t)\}_{n=1}^{\infty}$ contains a subsequence strongly converging in $L^2(\Omega; \mathbb{R}^3)$. This means that $\{\widehat{\mathbf{H}}_{h_n}(t) + \widehat{\mathbf{H}}_{s,h_n}(t)\}_{n=1}^{\infty}$ is relatively compact in $L^2(\Omega; \mathbb{R}^3)$ for any $t \in [0, T]$.

For any $s, t \in [0, T]$ with $s \leqslant t$, we see that by using the inequalities (4.2) and (4.6)

$$\begin{aligned}
&\left\|\widehat{\mathbf{H}}_{h_n}(t) + \widehat{\mathbf{H}}_{s,h_n}(t) - (\widehat{\mathbf{H}}_{h_n}(s) + \widehat{\mathbf{H}}_{s,h_n}(s))\right\|_{L^2(\Omega;\mathbb{R}^3)} \\
&= \left\|\int_s^t \left(\partial_t \widehat{\mathbf{H}}_{h_n}(\tau) + \partial_t \widehat{\mathbf{H}}_{s,h_n}(\tau)\right) \mathrm{d}\tau\right\|_{L^2(\Omega;\mathbb{R}^3)} \\
&\leqslant \left(\left\|\partial_t \widehat{\mathbf{H}}_{h_n}\right\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} + \left\|\partial_t \widehat{\mathbf{H}}_{s,h_n}\right\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\right) |t-s|^{1/2} \\
&\leqslant C(\|\partial_t \mathbf{h}_s\|_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} + \|\mathbf{n} \times \partial_t \mathbf{h}_s\|_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))})|t-s|^{1/2},
\end{aligned}$$

where $C > 0$ is a constant independent of h_n. Therefore, $\{\widehat{\mathbf{H}}_{h_n}(t) + \widehat{\mathbf{H}}_{s,h_n}(t)\}_{n=1}^{\infty}$ is equicontinuous. By applying the Ascoli–Arzelà theorem for $C([0, T]; L^2(\Omega; \mathbb{R}^3))$, we see that there exists $\mathbf{w} \in C([0, T]; L^2(\Omega; \mathbb{R}^3))$ such that, by choosing a subsequence,

$$\widehat{\mathbf{H}}_{h_n} + \widehat{\mathbf{H}}_{s,h_n} \to \mathbf{w} \quad \text{strongly in } C([0, T]; L^2(\Omega; \mathbb{R}^3))$$

as $n \to +\infty$. Moreover, by noting (4.1), (4.3) and (4.6) we can check that $\mathbf{w} = \widehat{\mathbf{H}} + \mathbf{H}_s$, $\widehat{\mathbf{H}}_{h_n}$ strongly converges to $\widehat{\mathbf{H}}$ in $C([0, T]; L^2(\Omega; \mathbb{R}^3))$ and $\overline{\mathbf{H}}_{h_n}$ strongly converges to $\widehat{\mathbf{H}}$ in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ as $n \to +\infty$. The convergences (4.17) and (4.20) are natural consequences of (4.14) and (4.18).

Step 2: We will show that the limit $\widehat{\mathbf{H}}$ is the unique solution of $(\mathbf{P}^B\mathbf{1})$. Take any $\boldsymbol{\phi} \in C([0, T]; L^2(\Omega; \mathbb{R}^3))$ with $\boldsymbol{\phi}(t) \in S$ for all $t \in [0, T]$. Let $\{\boldsymbol{\phi}_l\}_{l=1}^{\infty}$ be the sequence satisfying the properties stated in Lemma 3.4. Define a function $\overline{\boldsymbol{\phi}_{l,h_n}}$ as in Corollary 3.1. Substituting $\boldsymbol{\phi}_{l,h_n}(i\Delta t)$ into (3.9), multiplying by Δt and summing over $i = 1 \to N$, we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \mu \left\langle \partial_t \widehat{\mathbf{H}}_{h_n} + \partial_t \widehat{\mathbf{H}}_{s,h_n}, \overline{\boldsymbol{\phi}_{l,h_n}} - \overline{\mathbf{H}}_{h_n} \right\rangle \mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon} \int_0^T \int_{\Omega_s} g\left(\left|\mathbf{curl}\,\overline{\boldsymbol{\phi}_{l,h_n}}\right|\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t - \frac{1}{\varepsilon} \int_0^T \int_{\Omega_s} g\left(\left|\mathbf{curl}\,\overline{\mathbf{H}}_{h_n}\right|\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t \geqslant 0.
\end{aligned} \tag{4.21}$$

By the properties (2.14) of g, (3.5) and (4.13), we observe that

$$\begin{aligned}
\frac{1}{\varepsilon} \int_0^T \int_{\Omega_s} g\left(\left|\mathbf{curl}\,\overline{\boldsymbol{\phi}_{l,h_n}}\right|\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t &\leqslant \frac{T|\Omega_s|}{\varepsilon} g(Ch_n \|\nabla \mathbf{curl}\,\boldsymbol{\phi}_l\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))} + \mathscr{J}_c) \\
&\leqslant \frac{T|\Omega_s|C_l}{\varepsilon}(A_3 h_n^2 + A_4 h_n) \to 0
\end{aligned} \tag{4.22}$$

as $n \to +\infty$. Thus, by neglecting the last negative term in the left-hand side of (4.21), passing to the limit $n \to +\infty$ and noting the convergence properties (4.5), (4.16), (4.18), (3.6) and (4.22), we obtain

$$\int_0^T \int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \boldsymbol{\phi}_l - \widehat{\mathbf{H}} \rangle \mathrm{d}\mathbf{x}\,\mathrm{d}t \geqslant 0.$$

By sending $l \to +\infty$ we arrive at

$$\int_0^T \int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \boldsymbol{\phi} - \widehat{\mathbf{H}} \rangle \mathrm{d}\mathbf{x}\, \mathrm{d}t \geqslant 0, \tag{4.23}$$

for all $\boldsymbol{\phi} \in C([0,T]; L^2(\Omega; \mathbb{R}^3))$ with $\boldsymbol{\phi}(t) \in S$.

Note that by the weak lower semicontinuity of the functional $\int_{\Omega_s} g(|\mathbf{curl} \cdot |)\mathrm{d}\mathbf{x}$ in $H(\mathbf{curl}; \Omega)$ and sending $n \to \infty$ in (4.7), we obtain that

$$\int_{\Omega_s} g(|\mathbf{curl}\, \widehat{\mathbf{H}}(t)|)\mathrm{d}\mathbf{x} = 0 \quad \text{for a.e. } t \in [0,T],$$

which implies $\widehat{\mathbf{H}}(t) \in S$ for all $t \in [0,T]$. By taking $\upsilon \in C^\infty([0,T])$ with $0 \leqslant \upsilon \leqslant 1$ and replacing $\boldsymbol{\phi}$ by $\upsilon\boldsymbol{\phi} + (1-\upsilon)\widehat{\mathbf{H}}$ in (4.23), we deduce that

$$\int_0^T \upsilon \int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \boldsymbol{\phi} - \widehat{\mathbf{H}} \rangle \mathrm{d}\mathbf{x}\, \mathrm{d}t \geqslant 0,$$

which implies that

$$\int_\Omega \mu \langle \partial_t \widehat{\mathbf{H}} + \partial_t \mathbf{H}_s, \boldsymbol{\phi} - \widehat{\mathbf{H}} \rangle \mathrm{d}\mathbf{x} \geqslant 0$$

for a.e. $t \in (0,T)$ and any $\boldsymbol{\phi} \in S$. Therefore, $\widehat{\mathbf{H}}$ is the solution of ($\mathbf{P}^B\mathbf{1}$) and the unique solvability of ($\mathbf{P}^B\mathbf{1}$) ensures the convergence properties (4.14)–(4.20) without extracting a subsequence of Λ. We have thus completed the proof. □

Let us define the discrete functions $\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,\varepsilon} \in C([0,T]; H(\mathbf{curl}; \Omega))$ and $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,\varepsilon} \in L^\infty(0,T; H(\mathbf{curl}; \Omega))$ consisting of the minimizers of the hybrid optimization problem ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$) by

$$\begin{aligned}\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,\varepsilon}(t) := {} & \frac{t - \Delta t(n-1)}{\Delta t}(\boldsymbol{\psi}_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon}) \\ & + \frac{\Delta t n - t}{\Delta t}(\boldsymbol{\psi}_{h,n-1,\varepsilon}|\nabla u_{h,n-1,\varepsilon}) \quad \text{in } [\Delta t(n-1), \Delta t n],\end{aligned}$$

$$\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,\varepsilon}(t) := \begin{cases}(\boldsymbol{\psi}_{h,n,\varepsilon}|\nabla u_{h,n,\varepsilon}) & \text{in } (\Delta t(n-1), \Delta t n], \\ (\boldsymbol{\psi}_{h,0,\varepsilon}|\nabla u_{h,0,\varepsilon}) & \text{on } \{t=0\},\end{cases}$$

for $n = 1, \dots, N$, where $(\boldsymbol{\psi}_{h,0,\varepsilon}|\nabla u_{h,0,\varepsilon}) = \mathbf{0}$. We see that $\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,\varepsilon}(t) + \widehat{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ and $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,\varepsilon}(t) + \overline{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ for all $t \in [0,T]$.

Under Assumption (4.13), Proposition 3.1 and Theorem 4.1 immediately yield the following result.

COROLLARY 4.1 The discrete approximations $\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t(h),\varepsilon(h)}$, $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t(h),\varepsilon(h)}$ converge to the unique solution $(\boldsymbol{\psi}|\nabla u)$ of ($\mathbf{P}^B\mathbf{2}$) in the same sense as (4.14)–(4.20) for $\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} = \widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t(h),\varepsilon(h)}$, $\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon(h)} = \overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t(h),\varepsilon(h)}$ and $\widehat{\mathbf{H}} = (\boldsymbol{\psi}|\nabla u)$ as $h \searrow 0$, $h \in \Lambda$.

REMARK 4.1 In the case that the penalty coefficient $\varepsilon > 0$ is fixed and it is assumed that Δt depends on h, satisfying $\sup_{h\in\Lambda} \Delta t(h) < 1$ and $\lim_{h\searrow 0,h\in\Lambda} \Delta t(h) = 0$, by using (3.7) we can similarly prove the convergence of the discrete solutions $\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon}$, $\partial_t\widehat{\mathbf{H}}_{h,\Delta t(h),\varepsilon}$ and $\overline{\mathbf{H}}_{h,\Delta t(h),\varepsilon}$ to the solution $\widehat{\mathbf{H}}_\varepsilon$ of ($\mathbf{P}^{mB}_\varepsilon\mathbf{1}$) in the same sense as (4.14)–(4.20) for $\widehat{\mathbf{H}} = \widehat{\mathbf{H}}_\varepsilon$.

4.2 *Convergence of the discrete solutions solving ($\mathbf{P}^P_{h,\Delta t,p}\mathbf{1}$) and ($\mathbf{P}^P_{h,\Delta t,p}\mathbf{2}$)*

We will prove that the discrete solutions consisting of the minimizers of ($\mathbf{P}^P_{h,\Delta t,p}\mathbf{1}$) and ($\mathbf{P}^P_{h,\Delta t,p}\mathbf{2}$) converge to the solution of ($\mathbf{P}^B\mathbf{1}$) and ($\mathbf{P}^B\mathbf{2}$), respectively.

We define the piecewise linear in-time functions $\widehat{\mathbf{H}}_{h,\Delta t,p} \in C([0,T]; H(\mathbf{curl};\Omega))$ and the piecewise constant in-time function $\overline{\mathbf{H}}_{h,\Delta t,p} \in L^\infty(0,T; H(\mathbf{curl};\Omega))$ in the same way as $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}$ and $\overline{\mathbf{H}}_{h,\Delta t,\varepsilon}$ by using the minimizer $\widehat{\mathbf{H}}_{h,n,p}$ of ($\mathbf{P}^P_{h,\Delta t,p}\mathbf{1}$). Note that $\widehat{\mathbf{H}}_{h,\Delta t,p}(t), \overline{\mathbf{H}}_{h,\Delta t,p}(t) \in V_h(\Omega)$, $\widehat{\mathbf{H}}_{h,\Delta t,p}(t) + \widehat{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ and $\overline{\mathbf{H}}_{h,\Delta t,p}(t) + \overline{\mathbf{H}}_{s,h,\Delta t}(t) \in X_h^{(\mu)}(\Omega)$ for all $t \in [0,T]$. By the same calculation as in Proposition 4.1 we can prove the following bounds.

PROPOSITION 4.2 Take any $\tau \in (0,1)$. The following inequalities hold. For any $h \in \Lambda$, $p \geqslant 2$, $\Delta t \in (0,\tau]$,

$$\begin{aligned}&\|\partial_t\widehat{\mathbf{H}}_{h,\Delta t,p}\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))}\\&\quad\leqslant C\frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}}\big(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} + \|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}\big),\end{aligned} \tag{4.24}$$

$$\begin{aligned}&\operatorname*{ess\,sup}_{t\in[0,T]}\left\{\int_{\Omega_s}\frac{\mathscr{J}_c}{p}|\mathbf{curl}\,\overline{\mathbf{H}}_{h,\Delta t,p}(t)/\mathscr{J}_c|^p\,\mathrm{d}\mathbf{x}\right\}\\&\quad\leqslant C\max\{\mu_d,\mu_s\}\big(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} + \|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}\big),\end{aligned} \tag{4.25}$$

$$\begin{aligned}&\|\overline{\mathbf{H}}_{h,\Delta t,p}\|^2_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))}\\&\quad\leqslant \frac{C}{1-\tau}\mathrm{e}^{T/(1-\tau)}\frac{\max\{\mu_d,\mu_s\}}{\min\{\mu_d,\mu_s\}}\big(h^2\|\partial_t\mathbf{h}_s\|^2_{L^2(0,T;H^{1/2}(\partial\Omega;\mathbb{R}^3))} + \|\mathbf{n}\times\partial_t\mathbf{h}_s\|^2_{L^2(0,T;L^2(\partial\Omega;\mathbb{R}^3))}\big),\end{aligned} \tag{4.26}$$

where $C > 0$ is a constant independent of h, p, Δt and μ.

Let us assume that Δt and p are positive functions of h satisfying

$$\begin{aligned}&\sup_{h\in\Lambda}\Delta t(h) < 1, \quad \lim_{h\searrow 0,h\in\Lambda}\Delta t(h) = \lim_{h\searrow 0,h\in\Lambda} 1/p(h) = 0,\\&\inf_{h\in\Lambda} p(h) \geqslant 2, \quad \sup_{h\in\Lambda} hp(h) < +\infty.\end{aligned} \tag{4.27}$$

THEOREM 4.2 The piecewise linear in-time approximation $\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)}$ and the piecewise constant in-time approximation $\overline{\mathbf{H}}_{h,\Delta t(h),p(h)}$ converge to the unique solution $\widehat{\mathbf{H}}$ of ($\mathbf{P}^B\mathbf{1}$) in the following sense.

$$\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)} \to \widehat{\mathbf{H}} \quad \text{strongly in } C([0,T]; L^2(\Omega;\mathbb{R}^3)), \tag{4.28}$$

$$\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)} \rightharpoonup \widehat{\mathbf{H}} \quad \text{weak}\,*\ \text{ in } L^\infty(0,T; H(\mathbf{curl};\Omega)), \tag{4.29}$$

$$\partial_t \widehat{\mathbf{H}}_{h,\Delta t(h),p(h)} \rightharpoonup \partial_t \widehat{\mathbf{H}} \quad \text{weakly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.30}$$

$$\mathbf{curl}\,\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)} \to \mathbf{curl}\,\widehat{\mathbf{H}} \quad \text{strongly in } C([0,T];(H(\mathbf{curl};\Omega))^*), \tag{4.31}$$

$$\overline{\mathbf{H}}_{h,\Delta t(h),p(h)} \to \widehat{\mathbf{H}} \quad \text{strongly in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{4.32}$$

$$\overline{\mathbf{H}}_{h,\Delta t(h),p(h)} \rightharpoonup \widehat{\mathbf{H}} \quad \text{weak} * \text{ in } L^\infty(0,T;H(\mathbf{curl};\Omega)), \tag{4.33}$$

$$\mathbf{curl}\,\overline{\mathbf{H}}_{h,\Delta t(h),p(h)} \to \mathbf{curl}\,\widehat{\mathbf{H}} \quad \text{strongly in } L^\infty(0,T;(H(\mathbf{curl};\Omega))^*), \tag{4.34}$$

as $h \searrow 0$, $h \in \Lambda$.

Proof. To simplify the notation let $\widehat{\mathbf{H}}_h, \overline{\mathbf{H}}_h, \widehat{\mathbf{H}}_{s,h}$ denote $\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)}, \overline{\mathbf{H}}_{h,\Delta t(h),p(h)}, \widehat{\mathbf{H}}_{s,h,\Delta t(h)}$, respectively.

By Corollary 2.1 (1) and the bound (4.25), we see that $\{\mathbf{curl}\,\overline{\mathbf{H}}_h\}_{h\in\Lambda}$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ and by (4.24) $\{\partial_t\widehat{\mathbf{H}}_h\}_{h\in\Lambda}$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$. Thus, by taking a subsequence $\{h_n\}_{n=1}^\infty \subset \Lambda$, the weak $*$ convergences (4.29), (4.30), (4.33) hold true for some $\widehat{\mathbf{H}} \in L^\infty(0,T;H(\mathbf{curl};\Omega)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^3))$ satisfying $\widehat{\mathbf{H}}(t) \in V(\Omega)$ a.e. $t \in [0,T]$. Moreover, using Lemma 3.1 (1) and the same argument as Theorem 4.1, we can apply the Ascoli–Arzelà theorem to prove the strong convergences (4.28) and (4.32), which also yield the convergences (4.31) and (4.34).

We show that the limit $\widehat{\mathbf{H}}$ is the solution of $(\mathbf{P}^B\mathbf{1})$. By substituting $\boldsymbol{\phi}_{h_n} = \boldsymbol{\phi}_{l,h_n}(\Delta t i)$ into the inequality corresponding to (3.9), multiplying by Δt and summing over $i = 1 \to N$, we have

$$\begin{aligned}&\int_0^T\int_\Omega \mu\left\langle \partial_t\widehat{\mathbf{H}}_{h_n} + \partial_t\widehat{\mathbf{H}}_{s,h_n}, \overline{\boldsymbol{\phi}_{l,h_n}} - \overline{\mathbf{H}}_{h_n}\right\rangle \mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad + \frac{\mathscr{J}_c}{p}\int_0^T\int_{\Omega_s}\left|\mathbf{curl}\,\overline{\boldsymbol{\phi}_{l,h_n}}/\mathscr{J}_c\right|^p \mathrm{d}\mathbf{x}\,\mathrm{d}t - \frac{\mathscr{J}_c}{p}\int_0^T\int_{\Omega_s}\left|\mathbf{curl}\,\overline{\mathbf{H}}_{h_n}/\mathscr{J}_c\right|^p \mathrm{d}\mathbf{x}\,\mathrm{d}t \geqslant 0.\end{aligned} \tag{4.35}$$

Noting the fact that there is a constant $C > 0$ such that $h_n \leqslant C/p$ by Conditions (4.27) and (3.5), we see that

$$\begin{aligned}\frac{\mathscr{J}_c}{p}\int_0^T\int_{\Omega_s}\left|\mathbf{curl}\,\overline{\boldsymbol{\phi}_{l,h_n}}/\mathscr{J}_c\right|^p \mathrm{d}\mathbf{x}\,\mathrm{d}t &\leqslant \frac{\mathscr{J}_c T|\Omega_s|}{p}(Ch_n\|\nabla\mathbf{curl}\,\boldsymbol{\phi}_l\|_{L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^9))}+1)^p \\ &\leqslant \frac{\mathscr{J}_c T|\Omega_s|}{p}(C_l/p+1)^p \to 0,\end{aligned} \tag{4.36}$$

as $n \to +\infty$. Moreover, the bound (4.25) and Corollary 2.1 (2) show that $\widehat{\mathbf{H}}(t) \in S$ for all $t \in [0,T]$.

Now by neglecting the last term on the left-hand side of (4.35), noting (4.36) and letting $n \to +\infty$ and $l \to +\infty$ in (4.35), we obtain

$$\int_0^T\int_\Omega \mu\langle\partial_t\widehat{\mathbf{H}} + \partial_t\mathbf{H}_s, \boldsymbol{\phi} - \widehat{\mathbf{H}}\rangle \mathrm{d}\mathbf{x}\,\mathrm{d}t \geqslant 0,$$

which is equivalent to $(\mathbf{P}^B\mathbf{1})$. Therefore, $\widehat{\mathbf{H}}$ is the solution of $(\mathbf{P}^B\mathbf{1})$. The uniqueness of $(\mathbf{P}^B\mathbf{1})$ assures the convergences as $h \searrow 0$ without extracting a subsequence. □

Let us define the discrete functions $\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,p} \in C([0,T];H(\mathbf{curl};\Omega))$ and $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,p} \in L^\infty(0,T;H(\mathbf{curl};\Omega))$ consisting of the minimizers of the hybrid optimization problem $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ in the same way as $\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,\varepsilon}$ and $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,\varepsilon}$. We see that $\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,p}(t)+\widehat{\mathbf{H}}_{s,h,\Delta t}(t)\in X_h^{(\mu)}(\Omega)$ and $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t,p}(t)+\overline{\mathbf{H}}_{s,h,\Delta t}(t)\in X_h^{(\mu)}(\Omega)$ for all $t\in[0,T]$.

Under Assumption (4.27), Proposition 3.1 and Theorem 4.2 yield

COROLLARY 4.2 The discrete approximations $\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t(h),p(h)}$, $\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t(h),p(h)}$ converge to the unique solution $(\boldsymbol{\psi}|\nabla u)$ of $(\mathbf{P}^B\mathbf{2})$ as $h\searrow 0$, $h\in\Lambda$ in the same sense as (4.28)–(4.34) for $\widehat{\mathbf{H}}_{h,\Delta t(h),p(h)}=\widehat{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t(h),p(h)}$, $\overline{\mathbf{H}}_{h,\Delta t(h),p(h)}=\overline{(\boldsymbol{\psi}|\nabla u)}_{h,\Delta t(h),p(h)}$ and $\widehat{\mathbf{H}}=(\boldsymbol{\psi}|\nabla u)$ as $h\searrow 0$, $h\in\Lambda$.

REMARK 4.2 If we fix $p\geqslant 2$ and assume the relations $\sup_{h\in\Lambda}\Delta t(h)<1$ and $\lim_{h\searrow 0}\Delta t(h)=0$, by using (3.7) we can similarly prove that the discrete solutions $\widehat{\mathbf{H}}_{h,\Delta t(h),p}$, $\partial_t\widehat{\mathbf{H}}_{h,\Delta t(h),p}$ and $\overline{\mathbf{H}}_{h,\Delta t(h),p}$ converge to the solution of $(\mathbf{P}_p^P\mathbf{1})$ in the same sense as (4.28)–(4.34) and

$$\mathbf{curl}\,\widehat{\mathbf{H}}_{h,\Delta t(h),p}\rightharpoonup\mathbf{curl}\,\widehat{\mathbf{H}}_p \text{ weak} * \text{ in } L^\infty(0,T;L^p(\Omega;\mathbb{R}^3)), \tag{4.37}$$

$$\mathbf{curl}\,\overline{\mathbf{H}}_{h,\Delta t(h),p}\rightharpoonup\mathbf{curl}\,\widehat{\mathbf{H}}_p \text{ weak} * \text{ in } L^\infty(0,T;L^p(\Omega;\mathbb{R}^3)). \tag{4.38}$$

The weak convergence properties (4.37) and (4.38) are consequences of the bound (4.25).

5. Numerical results

In this section, we present numerical results by computing the unconstrained optimization problems $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ and $(\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2})$. All the examples in this section are computed in the situation where Ω and Ω_s are parallelepipeds whose faces are either parallel or perpendicular to the x–y, y–z, z–x planes in the (x,y,z) coordinate system.

We apply the external magnetic field $\mathbf{H}_s$ to be uniform in space and perpendicular to the x–y plane, so the boundary value $\mathbf{h}_s$ is given as $\mathbf{h}_s(t)=(0,0,\eta(t))$, where $\eta\in C^{1,1}([0,T])$ and $\eta(0)=0$. In this case, Conditions (2.36) and (2.37) are satisfied and the unique solution $\mathbf{H}_s$ to System (2.7)–(2.10) is naturally given as $\mathbf{H}_s(t)=(0,0,\eta(t))$.

Let us note another equivalent characterization of the space $W_h(\Omega)$:

$$W_h(\Omega)=\{(\boldsymbol{\phi}_h|\nabla u_h)\in L^2(\Omega;\mathbb{R}^3)|(\boldsymbol{\phi}_h,u_h)\in U_h(\Omega_s)\times Z_h(\Omega_d),$$
$$\mathbf{n}\times\boldsymbol{\phi}_h=\mathbf{n}\times\nabla u_h \text{ on } \partial\Omega_s,\ u_h|_{\partial\Omega}=0\},$$

where $\mathbf{n}$ is the unit normal to $\partial\Omega_s$. Lemma 3.1 implies that the equality $\mathbf{n}\times\boldsymbol{\phi}_h=\mathbf{n}\times\nabla u_h$ on $\partial\Omega_s$ holds if, and only if,

$$M_e(\boldsymbol{\phi}_h-\nabla u_h)=0, \tag{5.1}$$

for all edges e on $\partial\Omega_s$. Condition (5.1) is equivalent to the equality

$$M_e(\boldsymbol{\phi}_h)=m_{v_2}(u_h)-m_{v_1}(u_h), \tag{5.2}$$

where v_1 and v_2 are the initial vertex and the terminal vertex of the edge e, respectively. The relation (5.2) has to be always satisfied in the implementation of $W_h(\Omega)$ to fulfill the tangential continuity constraint $\mathbf{n}\times\boldsymbol{\phi}_h=\mathbf{n}\times\nabla u_h$ on $\partial\Omega_s$.

Problems $(\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2})$ and $(\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2})$ are computed by Newton's method coupled with the conjugate gradient method. The code with which we obtained the results was based on ALBERTA (Schmidt & Siebert, 2005).

5.1 *Definition of the penalized energy*

In order to search for the minimizer of ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$) by means of Newton's method, we use C^2 class energy density so that we can calculate the Hessian of the energy functional. In our numerical simulation, we employ the following regularized energy density g. For $0 < \alpha_1 < \alpha_2 < \alpha_3$, let $f_{\alpha_1,\alpha_2,\alpha_3} \in C^2(\mathbb{R})$ be a function satisfying that $f_{\alpha_1,\alpha_2,\alpha_3}(x) = 0$ for all $x \leqslant 0$,

$$f''_{\alpha_1,\alpha_2,\alpha_3}(x) = \begin{cases} x/\alpha_1 & \text{in } [0, \alpha_1], \\ 1 & \text{in } [\alpha_1, \alpha_2], \\ (-x+\alpha_3)/(\alpha_3-\alpha_2) & \text{in } [\alpha_2, \alpha_3], \\ 0 & \text{in } [\alpha_3, \infty). \end{cases}$$

Now $f_{\alpha_1,\alpha_2,\alpha_3}$ is a polynomial of degree 3 in $[0, \alpha_1]$, of degree 2 in $[\alpha_1, \alpha_2]$, of degree 3 in $[\alpha_2, \alpha_3]$ and of degree 1 in $[\alpha_3, \infty)$. Define $g(x) := f_{\alpha_1,\alpha_2,\alpha_3}(x^2 - \mathscr{J}_c^2)$, which is found to satisfy the required properties (2.14). This energy density $g(|\mathbf{v}|)/\varepsilon$ with $\varepsilon > 0$, $\mathbf{v} \in \mathbb{R}^3$ is a regularized version of the energy density $\gamma^{mB}_{\varepsilon}$ of the modified Bean's model defined in (2.13) which is not continuously differentiable.

5.2 *Estimated errors for discrete solutions solving ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$) and ($\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2}$)*

Let us compute the errors between two discrete solutions $\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}$ and $\widehat{\mathbf{H}}_{\hat{h},\hat{\Delta t},\varepsilon}$ solving ($\mathbf{P}^{mB}_{h,\Delta t,\varepsilon}\mathbf{2}$) and the errors between $\widehat{\mathbf{H}}_{h,\Delta t,p}$ and $\widehat{\mathbf{H}}_{\hat{h},\hat{\Delta t},p}$ solving ($\mathbf{P}^{P}_{h,\Delta t,p}\mathbf{2}$), respectively. We consider the situation where $\Omega = (-2,2)^3$, $\Omega_s = (-1,1)^3$, $\mu_s = \mu_d = 1$, $\mathscr{J}_c = 1$, $\varepsilon = 0.1$, $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\alpha_3 = 1$, $p = 10$, and the uniform external magnetic field $\mathbf{H}_s(t) = (0, 0, 0.1t)$ is applied. The domain Ω is meshed uniformly by tetrahedra of the same size.

The parameters for $\widehat{\mathbf{H}}_{\hat{h},\hat{\Delta t},\varepsilon}$ and $\widehat{\mathbf{H}}_{\hat{h},\hat{\Delta t},p}$ are relatively small and fixed as $\hat{h} \approx 1/32$, $\hat{\Delta t} = 1/3200$, for which 440 512 degrees of freedom are involved. For different parameters h and Δt, we compute the errors $\|\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) - \widehat{\mathbf{H}}_{\hat{h},\hat{\Delta t},\varepsilon}(t)\|_{L^2(\Omega;\mathbb{R}^3)} \times 100$ and $\|\widehat{\mathbf{H}}_{h,\Delta t,p}(t) - \widehat{\mathbf{H}}_{\hat{h},\hat{\Delta t},p}(t)\|_{L^2(\Omega;\mathbb{R}^3)} \times 100$. The results are shown in Tables 1 and 2, where DOF stands for the degrees of freedom.

These tables suggest that the rate of convergence is consistent with the order $\mathrm{O}(h^{1/2})$ in both cases.

5.3 *The current density and the magnetic field*

We display some numerical results showing the behaviour of the electric current and the magnetic field, where $\mu_d = \mu_s = 1$, $\mathscr{J}_c = 1$, $\alpha_1 = 0.1$, $\alpha_2 = 0.5$, $\alpha_3 = 1$, and $\mathbf{H}_s(t) = (0, 0, 0.1t)$ is applied in $\Omega = (-2,2)^3$. The time step size is fixed as $\Delta t = 0.001$.

TABLE 1 *The error* $\|\widehat{\mathbf{H}}_{h,\Delta t,\varepsilon}(t) - \widehat{\mathbf{H}}_{\hat{h},\hat{\Delta t},\varepsilon}(t)\|_{L^2(\Omega;\mathbb{R}^3)} \times 100$

h	Δt	DOF	$t = 0.05$	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.25$	$t = 0.3$
1/2	1/200	52	1.1793	2.3675	3.5558	4.7441	5.9323	7.1109
1/4	1/400	632	1.0224	2.0219	3.0031	3.9763	4.9460	5.9040
1/8	1/800	6064	0.7646	1.4621	2.1645	2.8724	3.5831	4.2828
1/16	1/1600	52 832	0.4911	0.9873	1.4870	1.9875	2.4882	2.9771

TABLE 2 *The error* $\|\widehat{\mathbf{H}}_{h,\Delta t,p}(t) - \widehat{\mathbf{H}}_{\hat{h},\hat{\Delta t},p}(t)\|_{L^2(\Omega;\mathbb{R}^3)} \times 100$

h	Δt	DOF	$t = 0.05$	$t = 0.1$	$t = 0.15$	$t = 0.2$	$t = 0.25$	$t = 0.3$
$1/2$	$1/200$	52	1.1729	2.3598	3.5321	4.6658	5.7896	6.9061
$1/4$	$1/400$	632	1.0222	2.0429	3.0277	3.9562	4.8643	5.7593
$1/8$	$1/800$	6064	0.8005	1.5166	2.1920	2.8217	3.4343	4.0495
$1/16$	$1/1600$	52 832	0.4954	0.9817	1.4543	1.8802	2.3009	2.7081

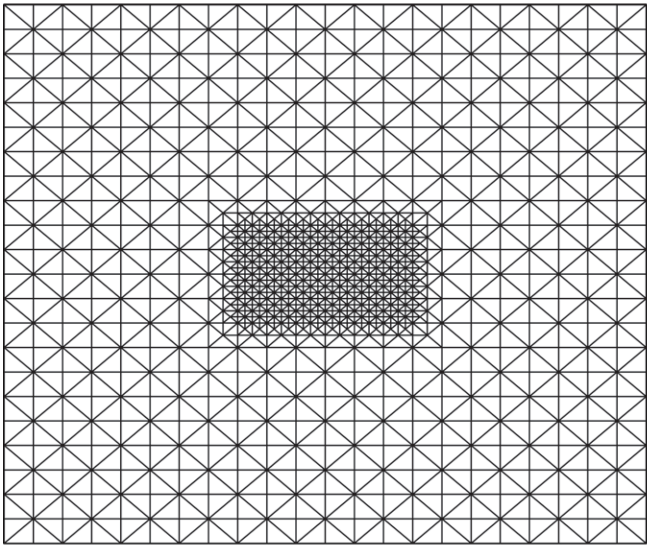

FIG. 3. A locally refined mesh.

5.3.1 *The current density for each* **E**–**J** *relation.* We assume that $\Omega_s = (-3/8, 3/8) \times (-2/8, 2/8) \times (-1/8, 1/8)$. The computation involves 205 996 degrees of freedom. The mesh is set to be relatively fine in Ω_s and coarse in Ω_d. Figure 3 shows an example of such mesh on the cross-section of the domain cut by the plane $z = 0$.

In Figs 4–6 the current density $|\mathbf{J}|$ on the surface and the cross-section of the superconductor Ω_s cut by the plane $z = 0$ are displayed.

5.3.2 *Motion of the subcritical region.* In the same situation as Section 5.3.1, we show the motion of the subcritical region where there is no current or the current $\mathbf{J}$ with $|\mathbf{J}| \leqslant 1/10$ is flowing in Fig. 7 by solving $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ with $p = 100$.

5.3.3 *The magnetic field.* We consider the case that $\Omega_s = (-5/8, 5/8)^3$ and a uniform mesh is used. The computation involves 35 192 degrees of freedom. We show the penetration of the magnetic flux $\mathbf{B} = \mu\widehat{\mathbf{H}} + \mu\mathbf{H}_s$ into the superconductor by solving $(\mathbf{P}^P_{h,\Delta t,p}\mathbf{2})$ with $p = 100$. In Fig. 8, the cross-section of Ω cut by the plane $y = 0$ is displayed.

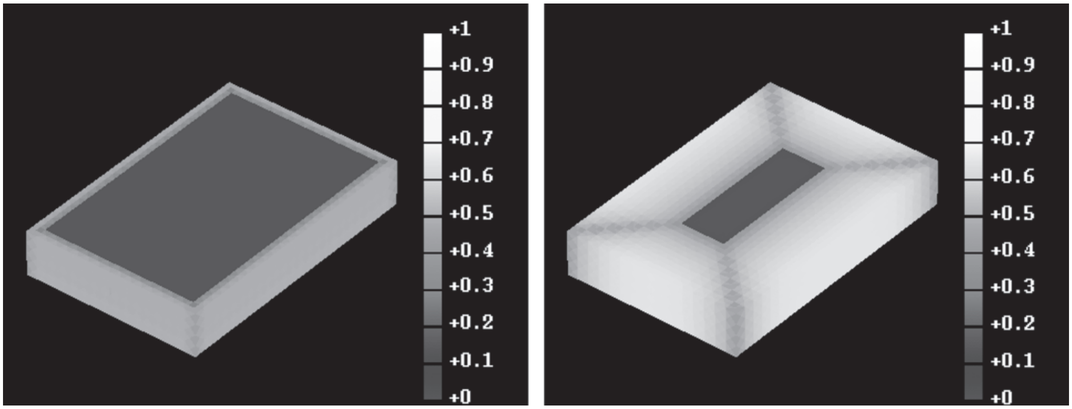

FIG. 4. The current density $|\mathbf{J}|$ of the power law with $p = 10$ at $t = 0.1$ left, $t = 1$ right.

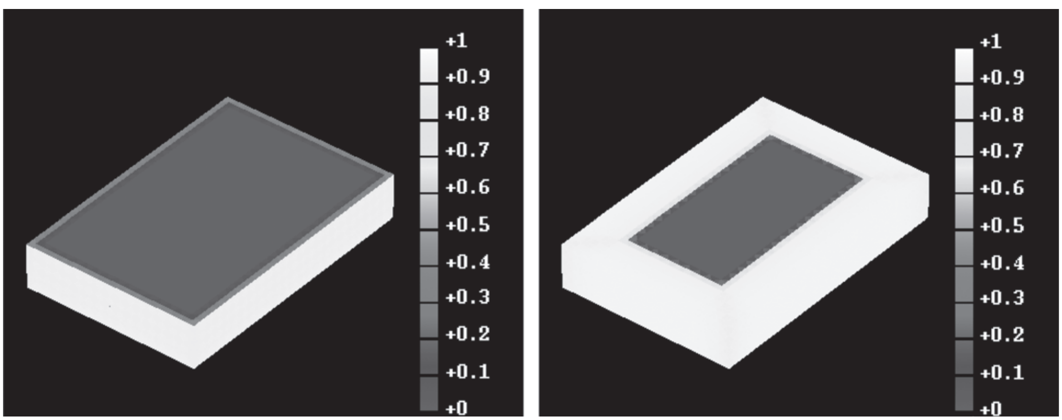

FIG. 5. The current density $|\mathbf{J}|$ of the power law with $p = 100$ at $t = 0.1$ left, $t = 1$ right.

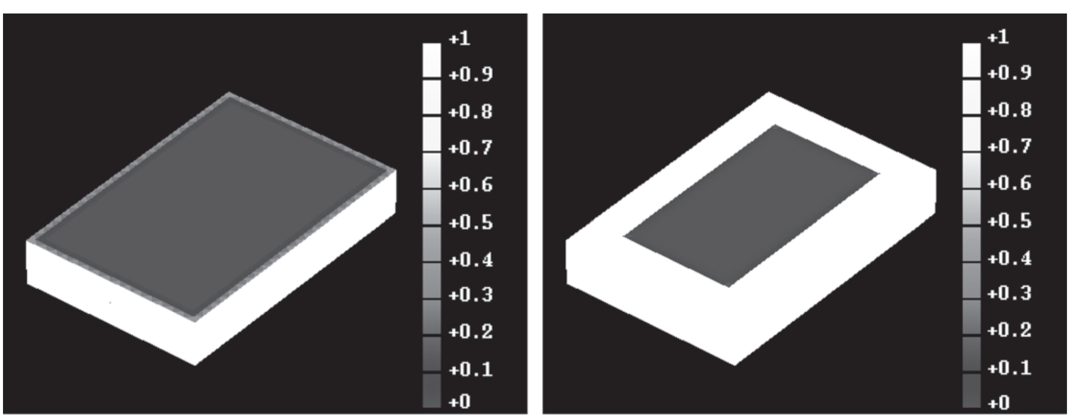

FIG. 6. The current density $|\mathbf{J}|$ of the modified Bean model with $\varepsilon = 0.01$ at $t = 0.1$ left, $t = 1$ right.

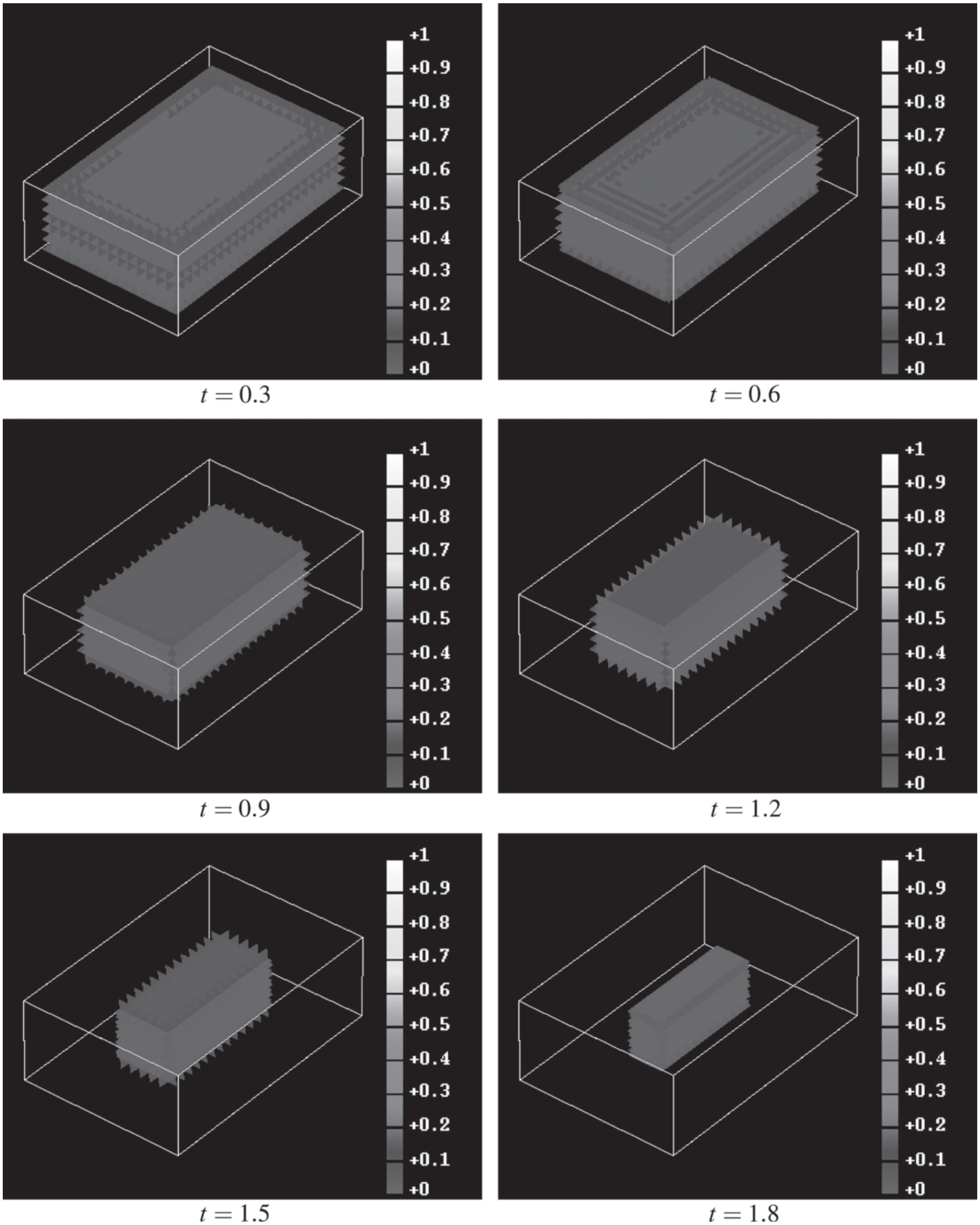

FIG. 7. The subcritical region where $|\mathbf{J}| \leqslant 1/10$.

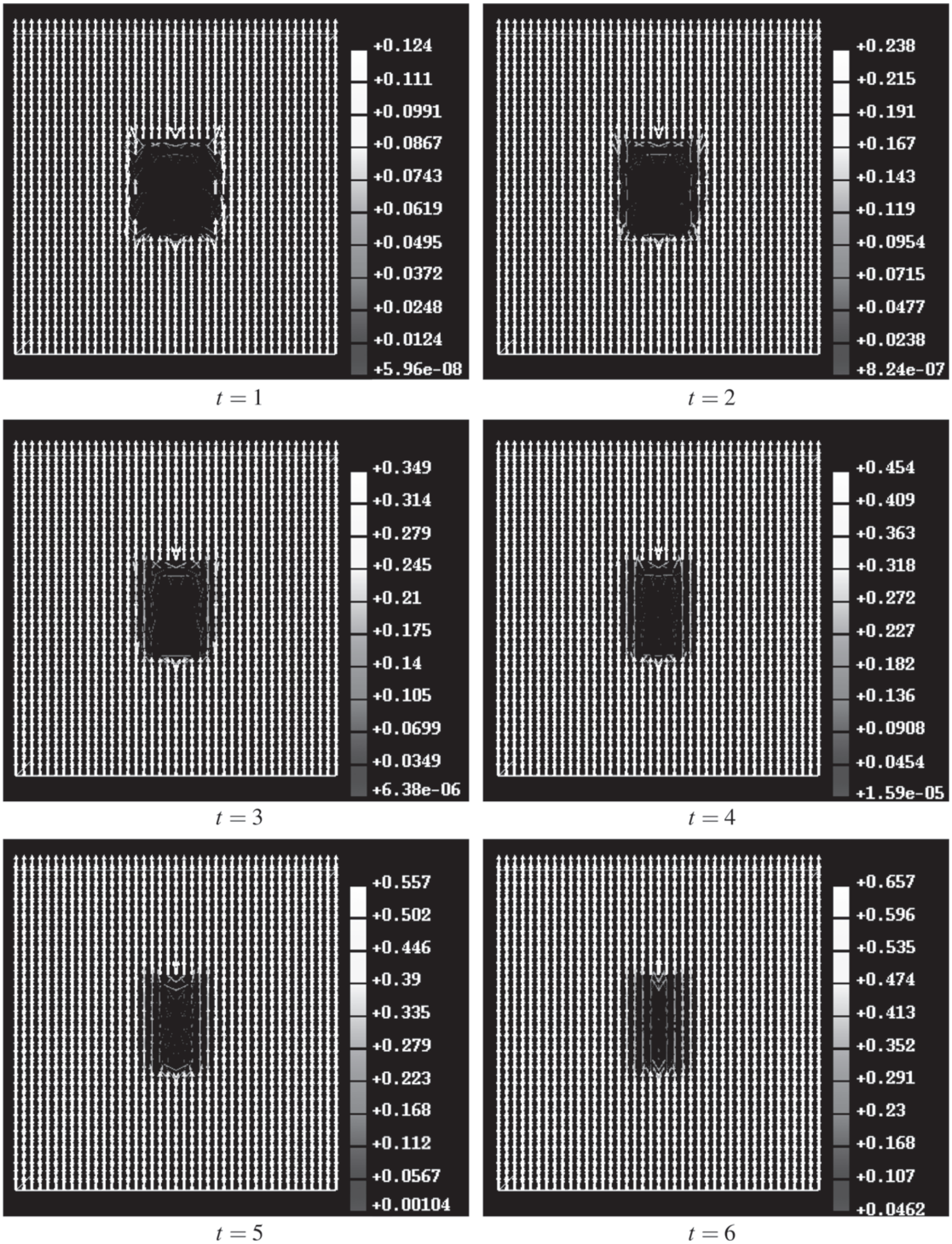

FIG. 8. The penetration of the magnetic flux density $\mathbf{B} = \mu\widehat{\mathbf{H}} + \mu\mathbf{H}_s$.

Acknowledgements

The authors are grateful to Dr David Kay for valuable comments on the computations. The work of Y. Kashima was supported by the ORS scheme no. 2003041013 from Universities UK and the GTA scheme from the University of Sussex.

REFERENCES

AMROUCHE, C., BERNARDI, C., DAUGE, M. & GIRAULT, V. (1998) Vector potentials in three dimensional nonsmooth domains. *Math. Methods Appl. Sci.*, **21**, 823–864.

ATTOUCH, H. (1978) *Convergence de functionnelles convexes, Journes d'Analyse Non Linaire, Proceedings, Besancon, France, on (1977)*. Lecture Notes in Mathematics, vol. 665. Berlin: Springer, pp. 1–40.

ATTOUCH, H. (1984) *Variational Convergence for Functions and Operators.* Boston, MA: Pitman Advanced Publishing Program.

AUCHMUTY, G. & ALEXANDER, J. C. (2005) L^2-well-posedness of 3D div-**curl** boundary value problems. *Q. Appl. Math.*, **63**, 479–508.

BARBU, V. & PRECUPANU, TH. (1986) *Convexity and Optimization in Banach Spaces.* Dordrecht, The Netherlands: D. Reidel Publishing Company.

BARNES, G., MCCULLOCH, M. & DEW-HUGHES, D. (1999) Computer modelling of type II superconductors in applications. *Supercond. Sci. Technol.*, **12**, 518–522.

BARRETT, J. W. & PRIGOZHIN, L. (2000) Bean's critical-state model as the $p \to \infty$ limit of an evolutionary $p-$Laplacian equation. *Nonlinear Anal.*, **42**, 977–993.

BARRETT, J. W. & PRIGOZHIN, L. (2006) Dual formulations in critical state problems. *Interfaces and Free Boundaries*, **8**, (to appear).

BEAN, C. P. (1964) Magnetization of high-field superconductors. *Rev. Mod. Phys.*, **36**, 31–39.

BERMÚDEZ, A., RODRÍGUEZ, R. & SALGADO, P. (2002) A finite element method with Lagrange multipliers for low-frequency harmonic Maxwell equations. *SIAM J. Numer. Anal.*, **40**, 1823–1849.

BOSSAVIT, A. (1994) Numerical modelling of superconductors in three dimensions: a model and a finite element method. *IEEE Trans. Magn.*, **30**, 3363–3366.

BRANDT, E. H. (1996) Superconductors of finite thickness in a perpendicular magnetic field: strips and slabs. *Phys. Rev. B*, **54**, 4246–4264.

BREZIS, H. (1971) Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations. *Contributions to Nonlinear Functional Analysis* (E. Zarantonello ed.), Madison, WI: Academic Press, pp. 101–156.

CESSENAT, M. (1996) *Mathematical Methods in Electromagnetism.* Singapore: World Scientific.

CHAPMAN, S. J. (2000) A hierarchy of models for type-II superconductors. *SIAM Rev.*, **42**, 555–598.

DU, Q., GUNZBURGER, M. & LEE, H. K. (1999) Analysis and computation of a mean-field model for superconductivity. *Numer. Math.*, **81**, 539–560.

ELLIOTT, C. M., KAY, D. & STYLES, V. (2004) A finite element approximation of a variational formulation of Bean's model for superconductivity. *SIAM J. Numer. Anal.*, **42**, 1324–1341.

ELLIOTT, C. M., KAY, D. & STYLES, V. (2005) Finite element analysis of a current density—electric field formulation of Bean's model for superconductivity. *IMA J. Numer. Anal.*, **25**, 182–204.

GIRAULT, V. & RAVIART, P. A. (1986) *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms.* Berlin: Springer.

GRILLI, F., STABREV, S., FLOCH, Y. L., COSTA-BOUZO, M., VINOT, E., KLUTSCH, I., MEUNIER, G., TIXADOR, P. & DUTOIT, B. (2005) Finite-element method modeling of superconductors: from 2-D to 3-D. *IEEE Trans. Appl. Supercond.*, **15**, 17–25.

KIKUCHI, F. (1989) On a discrete compactness property for the Nédélec finite elements. *J. Fac. Sci. Univ. Tokyo Sect. 1A Math.*, **36**, 479–490.

MONK, P. (2003) *Finite Element Methods for Maxwell's Equations.* Oxford: Oxford University Press.

NÉDÉLEC, J. C. (1980) Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, **35**, 315–341.

PECHER, R. (2003) NablaX—Software for Solving Highly Nonlinear Maxwell Equations in 3D. University of Oxford, Oxford. http://kron1.eng.ox.ac.uk/~nablax/index.htm.

PECHER, R., MCCULLOCH, M. D., CHAPMAN, S. J., PRIGOZHIN, L. & ELLIOTT, C. M. (2003) 3D-modelling of bulk type-II superconductors using unconstrained H-formulation. *Proceedings of the 6th European Conference on Applied Superconductivity (EUCAS)*. Sorrento, Italy.

PRIGOZHIN, L. (1996a) On the Bean critical-state model in superconductivity. *Eur. J. Appl. Math.*, **7**, 237–247.

PRIGOZHIN, L. (1996b) The Bean model in superconductivity: variational formulation and numerical solution. *J. Comput. Phys.*, **129**, 190–200.

PRIGOZHIN, L. (1997) Analysis of critical-state problems in type-II super conductivity. *IEEE Trans. Appl. Supercond.*, **7**, 3866–3873.

PRIGOZHIN, L. (1998) Solution of thin film magnetization problems in type-II superconductivity. *J. Comput. Phys.*, **144**, 180–193.

PRIGOZHIN, L. (2004) Variational inequalities in critical-state problems. *Physica D*, **197**, 197–210.

RHYNER, J. (1993) Magnetic properties and AC-losses of superconductors with power law current-voltage characteristics. *Physica C*, **212**, 292–300.

RUBINACCI, G., TAMBURRINO, A., VENTRE, S. & VILLONE, F. (2000) Macroscopic electrodynamic modelling of superconductors. *Cryogenics*, **40**, 671–676.

RUBINACCI, G., TAMBURRINO, A. & VILLONE, F. (2002) Three dimensional finite elements modeling of superconductors. *IEEE Trans. Magn.*, **36**, 1276–1279.

SCHMIDT, A. & SIEBERT, K. G. (2005) *Design of Adaptive Finite Element Software, the Finite Element Toolbox ALBERTA*. Lecture Notes in Computational Science and Engineering, vol. 42. Berlin: Springer.

THOMÉE, V. (1997) *Galerkin Finite Element Methods for Parabolic Problems.* Berlin: Springer.

YIN, H.-M. (2001) On a p-Laplacian type of evolution system and applications to the Bean model in the type-II superconductivity theory. *Q. Appl. Math.*, **59**, 47–66.

YIN, H.-M., LI, B. Q. & ZOU, J. (2002) A degenerate evolution system modeling Bean's critical-state type-II superconductors. *Discrete Conti. Dyn. Syst.*, **8**, 781–794.