Long time asymptotics for forced curvature flow with applications to the motion of a superconducting vortex

K Deckelnick, C M Elliott and G Richardson
Centre for Mathematical Analysis and Its Applications, University of Sussex, Brighton BN1 9QH, UK

Received 31 July 1996
Recommended by S Childress

Abstract. We consider the large time asymptotics for the evolution of a planar curve subject to mean curvature flow and constant forcing. Depending on the sign of the forcing we prove convergence for large times to either a travelling wave or a self-similar profile. The context of our work is the study of the motion of a superconducting vortex.

AMS classification scheme numbers: 35K55, 35Q99, 35R35, 53A04, 78A99

1. Introduction

The response of a bulk superconductor in an applied magnetic field, $H_0$, can be described by the phase diagram drawn in figure 1 which shows a plot of $H_0$ versus $\kappa$. Here $\kappa$ is a material parameter referred to as the Ginzburg–Landau parameter; this, among other things determines whether a superconductor is what is known as a type-I superconductor ($\kappa < 1/\sqrt{2}$) or what is known as a type-II superconductor ($\kappa > 1/\sqrt{2}$). For a type-I superconductor there are two states: a superconducting state for applied magnetic fields below $H_c$ in which the body almost entirely excludes the magnetic field; and a normally conducting (normal) state for applied field, $H_0$, above $H_c$ for which there is no exclusion of the magnetic field. However, for a type-II superconductor a third state exists lying between the normal and superconducting states termed the mixed state. The mixed state allows partial penetration of magnetic field into the superconducting body and has an interesting morphology consisting of many normal filaments, each associated with a quantum of magnetic flux, embedded in a superconducting matrix. These normal filament are frequently called superconducting vortices.

The dynamics of these vortices is the subject of current research. In particular the limit $\kappa \to \infty$ has been much studied both because it is susceptible to the methods of formal asymptotics and because of its practical importance (high-$\kappa$ materials can support high magnetic fields). Using these methods on the time-dependent Ginzburg–Landau equations of superconductivity [4, 9, 12, 13, 19] a vortex velocity law has been derived for a rectilinear vortex [17]. Subsequently this has been generalized in [7] to a three-dimensional curvilinear vortex line where it is shown that the velocity comprises of two terms, the first resulting from an applied current density, $j$, and the second a self-induced term dependent on the

† This effect is known as the Meissner effect.
local curvature of the vortex line. These are such that, where the velocity of the line \( \Gamma(t) \) is denoted by \( v \), the law of motion may be expressed, in certain units, as follows:

\[
v = \mathbf{j} \wedge \mathbf{\tau} + k \mathbf{n}.
\]  

(1.1)

Here \( \mathbf{\tau} \) is the unit tangent, \( k \) the curvature and \( \mathbf{n} \) the unit normal to the vortex line. Another important result on the dynamics of a high-\( \kappa \) vortex has been obtained in [16] where it is shown that a vortex must meet the boundary of the superconductor normally.

The aim of this paper is to study the motion of a curve obeying the law of motion (1.1) in proximity to a boundary which it intersects normally. We shall consider only a simple case in which the current density \( \mathbf{j} \) is constant, parallel to the boundary and normal to a planar curve. In the next section we discuss the reason for studying this particular case which may be briefly expressed here as a desire to understand the boundary conditions on a mean-field model of vortex motion proposed in [5]. Such evolution laws for curves also arise in models for the motion of fronts in excitable media [3]. In section 3 we discuss the long-time asymptotic behaviour of such a curve and formulate some claims about this behaviour, which are proved in section 4. In section 5 a method for the numerical simulation of a curve evolving according to (1.1) is presented together with some results that illustrate the long-time asymptotic behaviour of a curve intersecting a boundary. Finally, in section 6, we present our conclusions.

2. Physical background

The mean-field model proposed by Chapman in [5] provides a method for examining the bulk motion of a large number of superconducting vortices. In a certain geometry, where vortices and the magnetic field lie parallel to the axis of an infinite cylindrical body, the boundary conditions to this model are well understood. In more general geometries, however, no reasonable boundary conditions to this problem have been given. In particular it is not clear what occurs when the vorticity vector, which gives the direction of the vortices, intersects a boundary. In this situation it is known that individual vortices must intersect normally, however imposing the same condition on the vorticity in certain situations overdetermines the model [18]. In order to understand this apparent contradiction we first consider the basis of the mean-field model.

It has been demonstrated in [7] that, given a parametrization of a filamentary vortex in
Long time asymptotics for forced curvature flow

where $\delta$ is the Dirac measure in the tangential direction for the curve $\Gamma$, given by $x = (q_1(s), q_2(s), q_3(s))$, and is defined by

$$\langle \delta, \phi \rangle = \int_0^\infty \phi \tau(s) \, ds$$

for sufficiently smooth test functions $\phi$, where $\tau(s)$ is the unit tangent to the curve $\Gamma$.

Clearly this can be generalized to a system of $n$ vortices, replacing the right-hand side of (2.1) by $(2\pi/\kappa) \sum_{m=1}^n \delta_m$. Furthermore, it has been shown in [7] that the velocity of the $j$th vortex is given approximately by the relations

$$v_j = k n \log \kappa + \kappa \left( \nabla \wedge (\nabla \wedge H_j) \right) \wedge \tau + \cdots$$

$$\nabla \wedge (\nabla \wedge H_j) + H_j = \frac{2\pi}{\kappa} \sum_{m \neq j}^n \delta_m$$

where $\tau$ is the unit tangent, $n$ the unit normal and $k$ the curvature of the vortex line. It is in principle, therefore, possible to calculate the evolutionary behaviour of a collection of vortices using equations (2.3) and (2.4). However, in a practical situation the number of vortices that need to be considered is frequently very large. To circumvent this difficulty Chapman [5] notes that, where the typical separation between vortices $\zeta$ lies in the range $1/\kappa \ll \zeta \ll 1/\log \kappa$ it is possible to make an asymptotic expansion for $H_j$

$$H_j = \frac{h}{\kappa \zeta^2} + \frac{1}{\kappa} H_j^{(1)} + \cdots$$

Here the leading-order term in the expansion is independent of $j$ and satisfies the homogeneous equations

$$\nabla \wedge (\nabla \wedge h) + h = \omega$$

$$\nabla \cdot h = 0$$

where $\omega$ is the averaged vorticity defined by considering a small ball of radius $\eta$, within which there are many vortices so that $\zeta \ll \eta \ll 1$. Then averaging over this ball we get

$$\omega_\eta (x) = \frac{3}{4\pi \eta^3} \sum_m (\delta_m, \tilde{\phi})$$

where

$$\tilde{\phi}(x') = \begin{cases} 
1 & \text{for } |x - x'| < \eta \\
0 & \text{for } |x - x'| > \eta.
\end{cases}$$

Finally, we consider the distinguished limit $\zeta \ll \eta$, $\eta \to 0$ for which

$$\omega = \lim_{\eta \to 0} \zeta^2 \omega_\eta.$$ 

Assuming that locally, on the $\eta$ length scale, all vortices are similarly aligned, a vorticity velocity consistent with (2.3) is defined by†

$$v = (\nabla \wedge h) \wedge \frac{\omega}{|\omega|}.$$  

† The self-induced term in the velocity, due to curvature, is negligible in comparison with that resulting from vortex–vortex interaction.
The model, consisting of equations (2.5)–(2.7) is then closed by considering conservation of vorticity
\[ \omega_t + \nabla \wedge (\omega \wedge v) = 0. \] (2.8)
Appropriate boundary conditions must now be given. Consider a typical configuration in which a superconducting body \( \Omega_1 \) is surrounded by an insulator \( \Omega_1^c \). The magnetic field satisfies Maxwell’s equations with zero current density \( j = 0 \) in \( \Omega_1^c \) together with a far field condition at infinity. Between the two materials the field satisfies jump conditions consistent with Maxwell’s equations, namely
\[ [\mu \mathbf{h} \cdot \hat{n}]_{\Omega_1} = 0 \quad \text{and} \quad [\mathbf{h} \wedge \hat{n}]_{\Omega_1} = 0 \] (2.9)
where \( \mu \) is the magnetic permeability of the material and \( \hat{n} \) is the unit normal to the boundary of the superconductor. Posing boundary conditions on \( \omega \) proves more problematic except in one rather special case, considered in [6], in which the magnetic field is applied parallel to the axis of an infinite cylinder. As a consequence of the geometry both the local magnetic field \( (h = h e_z) \) and the vorticity \( (\omega = \omega e_z) \) lie in the direction of the axis and thus never intersect the boundary of the body. Mathematically well posed boundary conditions on the vorticity take the form of a specified flux on the inflow boundary where \( v \cdot \hat{n} < 0 \) and the characteristics of the system are directed in through the boundary, and no conditions on the outflow boundary where \( v \cdot \hat{n} > 0 \). We shall instead look at another scenario in which the applied magnetic field lies perpendicular to the axis of an infinite cylinder, from which it follows that both vorticity and local magnetic field lie in the plane of the cylinder while the current lies parallel to its axis so that
\[ \omega = (\omega_1(x, y), \omega_2(x, y), 0) \quad \mathbf{h} = (h_1(x, y), h_2(x, y), 0) \quad j = (0, 0, j(x, y)). \]
It is clear then that vorticity may intersect the boundary and, as above, the characteristics of the system would lead one to believe that data for \( \omega \) should only be imposed on sections of the boundary on which \( v \cdot \hat{n} < 0 \). However, it is known that vortices intersect the boundary normally which leads to the naive assumption that the condition \( \omega \wedge \hat{n} = 0 \) should be specified on all sections of the boundary where \( \omega \cdot \hat{n} \neq 0 \). This apparent contradiction leads us to consider a boundary layer analysis.

When deriving the mean-field model it has been assumed that the curvature of individual vortices \( k \ll \zeta^2 / \log \kappa \) such that the self-induced vortex velocity is negligible in comparison to that induced by other vortices. However we postulate that, in situations in which the imposition of the boundary condition \( \omega \wedge \hat{n} = 0 \) overdetermines the system, a boundary layer exists in which self-induced velocity is comparable with the mean-field velocity (see figure 2). Consider then the following rescaling of a local coordinate system with the origin in the boundary, and with the \( x \)-axis lying along the normal to the boundary and pointing into the superconductor:
\[ x = \frac{\zeta^2 \log \kappa}{2} x' = \delta x' \quad t = \delta t'. \]
In the boundary layer region, defined by the above rescaling, the mean-field equations (2.5) and (2.6) still hold and, in terms of the rescaled coordinates, may be written as
\[ \frac{\partial h_{i,2}}{\partial x'} - \frac{\partial h_{i,1}}{\partial y'} = \delta j \quad \frac{\partial h_{i,1}}{\partial x'} + \frac{\partial h_{i,2}}{\partial y'} = 0 \]
\[ \frac{\partial j_i}{\partial y'} = \delta (\omega_{i,1} - h_{i,1}) \quad \frac{\partial j_i}{\partial x'} = \delta (h_{i,2} - \omega_{i,2}) \]
where \( i \) denotes the boundary layer solution. Thus, to leading order, \( h_j \) and \( j_i = J \) are constant and equal to the values of \( h \) and \( j \) in the outer region, whilst \( \omega_i \), if it is to match
with the outer solution, must be independent of \( t' \) the short time-scale variable introduced above. Over the small lengthscale of the boundary layer the approximation to the velocity taken in (2.7) breaks down; curvature no longer being negligible. It is therefore appropriate to use the vortex velocity law given in (2.3),

\[ v_i = j_i \wedge \tau + k n. \]  

(2.10)

Since leading-order variations in the inner vorticity only occur in the \( x' \)-direction (a consequence of matching to an outer vorticity that varies only over the much longer outer lengthscale coordinate \( y \)) we may consider one typical vortex, rather than the vorticity, provided we impose boundary conditions on the ends of the vortex line consistent both with normal intersection of the vortex with the boundary and with the direction of the vorticity in the outer region.

Henceforth we consider only the inner problem and so drop the primes on the independent variables and the \( i \)s on the dependent variables.

3. Asymptotic behaviour for large time

We may formulate the vortex velocity law (2.10) for a curve \( \Gamma(t) \) as a system of partial differential equations in several different ways. Perhaps the most intuitive approach is to choose a parametrization \( q = (u(s, t), v(s, t), 0) \) (here \( \sqrt{u_s^2 + v_s^2} \) is the element of arclength rather than \( s \) of \( \Gamma(t) \) that evolves with \( q_t \) normal to the vortex. Then we find the following system for \( u \) and \( v \):

\[
\begin{align*}
u_t - \frac{1}{\sqrt{u_s^2 + v_s^2}} \left( \frac{u_s}{\sqrt{u_s^2 + v_s^2}} \right)_s &= - \frac{J v_s}{\sqrt{u_s^2 + v_s^2}} \\
u_t - \frac{1}{\sqrt{u_s^2 + v_s^2}} \left( \frac{v_s}{\sqrt{u_s^2 + v_s^2}} \right)_s &= \frac{J u_s}{\sqrt{u_s^2 + v_s^2}}.
\end{align*}
\]  

(3.1)

(3.2)

Hereafter we shall assume that \(|J| = 1\) so that \( J \in \{-1, 1\} \). This can be achieved by appropriately rescaling space and time. Imposing normal intersection of the vortex with the planar boundary \( x = 0 \) and enforcing that it lies in the same direction as the far field vorticity at \( x = \infty \), say \((1, m, 0)\), results in the following boundary conditions:

\[
\begin{align*}
u(0, t) &= 0 \\
\frac{v_s}{u_s} &\to m \quad u \to \infty \quad \text{as} \quad s \to \infty.
\end{align*}
\]  

(3.3)
Without loss of generality we may take \( m > 0 \). An initial condition \( u(s, 0) = u_0(s) \), \( v(s, 0) = v_0(s) \) must also be given when studying the evolutionary problem.

An alternative formulation, which results in only one equation, may be used when the vortex lies along some line which is a graph in \( x \) such that its position may be written as \( q = (x, v(x, t), 0) \). We note that adding an arbitrary tangential component to the velocity leaves the motion of a line unchanged and thus we see that, by taking the vector product of equation (2.10) with the tangent vector \( \tau(s) \), the problem may be formulated as follows:

\[
v_t - \frac{v_{xx}}{1 + v_x^2} = J \sqrt{1 + v_x^2}
\]

(3.4)

\[
v_x(0, t) = 0
\]

(3.5)

\[
v_x(x, t) \rightarrow m \quad \text{as} \quad x \rightarrow \infty
\]

(3.6)

\[
v(x, 0) = v_0(x).
\]

(3.7)

where \( J \in \{-1, 1\} \) and \( m > 0 \) is a given constant. When we make the following assumptions about the initial function \( v_0 \):

(A1) \( v_0 \in C^{2+\alpha}_{\geq 0, \leq 0}([0, \infty)) \) for some \( \alpha > 0 \), \( \sup_{x \geq 0} |v_0(x) + v_0(x, t)| < \infty \);

(A2) \( v_0, v_0_x(0) = 0 \) and \( \int_0^\infty |v_0_x| < \infty \),

we can prove the subsequent existence and uniqueness result.

**Theorem 1.** Under assumptions (A1) and (A2) problem (3.4)–(3.7) has a unique global solution \( v \in H^{2+\alpha, 1+\alpha}([0, \infty) \times [0, \infty)) \) such that \( v_x, v_{xx}, v_t \in L^\infty((0, \infty) \times (0, \infty)) \).

Recall that we obtained equations (3.1)–(3.3) (or equivalently (3.4)–(3.7)) in order to regularize a system of partial differential equations with hyperbolic behaviour in proximity to a boundary. These equations are also the regularization of another hyperbolic system, namely

\[
u' = -\frac{\text{sgn}(J) v'_s}{\sqrt{u'^2 + v'^2}} \quad v' = \frac{\text{sgn}(J) u'}{\sqrt{u'^2 + v'^2}}
\]

(3.8)

as can be seen by rescaling distance and time in equations (3.1), (3.2) with \( 1/\gamma \), where \( \gamma \ll 1 \),

\[
u = u'/\gamma \quad v = v'/\gamma \quad t = t'/\gamma
\]

which in the limit \( \gamma \rightarrow 0 \) yields (3.8). Equally one can think of (3.8) as the equation for the intermediate behaviour of the mean-field model (2.5)–(2.8) on a lengthscale \( \delta/\gamma \) where \( \delta \ll \delta/\gamma \ll 1 \). Any initial data to the boundary layer problem (3.1), (3.2) satisfying the far-field conditions (3.3) when rescaled onto the intermediates scale takes the form \( v' = m u' \) for \( u' \geq 0 \) at \( t' = 0 \). Certainly a solution to (3.8) with this initial data is

\[
u' = \frac{s - m t' \text{sgn}(J)}{\sqrt{1 + m^2}} \quad v' = \frac{m s + t' \text{sgn}(J)}{\sqrt{1 + m^2}}
\]

or equivalently

\[
v' = m u' + \sqrt{1 + m^2} t' \text{sgn}(J) \quad u' \geq 0
\]

(3.9)

but it is not clear whether it is a physically reasonable solution. For \( J = 1 \) the solution described by (3.9) lies within the domain of dependence of the initial data and thus appears to be entropy satisfying. It seems likely therefore that there will be a corresponding solution to the curvature problem (3.1), (3.2) which regularizes (3.9) in the boundary layer region. However, for \( J = -1 \) there is a portion of solution (3.9), in the range \( 0 \leq u' \leq -m v' \), which
Long time asymptotics for forced curvature flow

Figure 3. Various solutions to the intermediate problem (3.8) drawn together with characteristics:
(a) an outflow solution (b) an inflow solution with entropy violating shock on the boundary (c) an inflow solution with a rarefaction wave.

does not lie within the domain of dependence of the initial data (see figure 3). Furthermore, if we make the assumption that the correct boundary data should be \( v'_s |_{u=0} = 0 \) (normal intersection of the curve with the boundary) then this particular solution has a shock on the boundary which violates causality.

We now look for a travelling wavesolution to the boundary layer problem (3.4)–(3.6) (equivalently (3.1)–(3.3)) of the form

\[ v = w(x) + ct \]

and find a first-order ODE for \( \chi(x) = w'(x) \)

\[ \chi' = \left(1 + \chi^2\right) \left(c - \text{sgn}(J)\sqrt{1 + \chi^2}\right) \]

(3.10)

\[ \chi(0) = 0 \quad \chi \to m \quad \text{as} \quad x \to \infty. \]  

(3.11)

Clearly this can only satisfy the far-field condition if \( c = \text{sgn}(J)\sqrt{1 + m^2} \).

In the case \( J = 1 \), we have \( \chi' > 0 \) for \( |\chi| < m \) and \( \chi' < 0 \) for \( |\chi| > m \). Thus, a solution to (3.10) satisfying both boundary conditions can be found. This solution corresponds to the V-wave in [3]. The following theorem examines its stability.

**Theorem 2.** Let \( J = 1 \) and assume that

(A3) \( v_{0,\perp} > -m \) for all \( x \geq 0 \).

(A4) There is a \( R_0 > 0 \) such that \( v_{0,\perp}(x) = m \) for \( x \geq R_0 \).

Then the solution to (3.4)–(3.7) converges to a travelling wave as \( t \to \infty \) in the sense that

\[ \sup_{x \geq 0} \left| v(x, t) - \left(\sqrt{1 + m^2}t + w(x)\right)\right| \to 0 \quad t \to \infty \]

where \( w(x) \) is the solution of (3.10), (3.11) with \( c = \sqrt{1 + m^2} \).

In the case \( J = -1 \) we have \( \chi' < 0 \) for \( |\chi| < m \) and \( \chi' > 0 \) for \( |\chi| > m \). Thus, the solution to (3.10) with initial condition \( \chi(0) = 0 \) tends to \( -m \) as \( x \to \infty \) and there is no travelling wave solution. In an attempt to find the correct behaviour we look for a similarity solution to the direct analogue of equation (3.8) for the graph \( q = (x, v(x, t), 0) \), namely

\[ v_t = \text{sgn}(J)\sqrt{1 + v_x^2} = -\sqrt{1 + v_x^2}. \]

(3.12)

Looking for a solution of the form \( Q = Q(\theta) \), where \( Q = v/t \) and \( \theta = x/t \) are the similarity variables, we find the following ODE:

\[ Q(\theta) - \theta Q'(\theta) = -\sqrt{1 + Q'(\theta)^2} \]
which has solutions of the form
\[ Q(\theta) = k\theta - \sqrt{1 + k^2} \quad \text{and} \quad Q(\theta)^2 + \theta^2 = 1 \quad (\text{for } Q < 0). \]

We note that it is possible to make a hybrid solution with a continuous first derivative that satisfies the boundary conditions
\[ Q'(0) = 0 \quad \text{and} \quad Q'(\theta) \to m \quad \text{as} \quad \theta \to \infty; \]
this is
\[ Q(\theta) = \begin{cases} 
-\sqrt{1 - \theta^2} & \theta \leq m \\
m\theta - \sqrt{1 + m^2} & \theta > m 
\end{cases} \quad (3.13) \]

Not only does this solution satisfy both boundary conditions, and thus give a vortex which intersects the boundary normally and which lies in the same direction as the far field vorticity at infinity, it also lies within the domain of dependence of the initial conditions. In fact it describes a rarefaction fan (see figure 3). The following theorem examines the convergence to the similarity solution.

**Theorem 3.** Let \( J = -1 \). Then the solution, \( v \), to \((3.4)–(3.7)\) converges to a self-similar solution in the sense that
\[ \frac{1}{t} \left| v(x, t) - tQ \left( \frac{x}{t} \right) \right| \to 0 \quad t \to \infty \]
uniformly on the sets \( \{(x, t) \in [0, \infty)^2 | x \leq Rx \} \) for all \( 0 < R < \infty \) where \( Q \) is given by (3.13).

Equations \((3.4)–(3.7)\) have been studied in [1] on a bounded interval (see also [14, 2] for results in higher dimensions). The presence of the term \( \sqrt{1 + v^2} \) on the right-hand side of \((3.4)\) creates difficulties in the proof of theorem 2 since the energy techniques employed in [1] do not work any more. In addition, the unboundedness of the underlying domain requires some knowledge of the behaviour of \( v \) at infinity.

### 4. Proofs of the theorems

#### 4.1. Proof of theorem 1

Let us start by establishing the existence and uniqueness of the problem \((3.4)–(3.7)\) which we first study on bounded domains. For \( R > 0 \) and \( 0 < T < \infty \) let \( \Omega_{R,T} := (0, R) \times (0, T) \) and consider for \( J \in \{-1, 1\} \) the initial-boundary value problem
\[ \begin{aligned}
&v_t - \frac{v_{xx}}{1 + v^2} = J\sqrt{1 + v^2} \quad \text{in} \ \Omega_{R,T} \\
v_x(0, t) = 0 \quad 0 < t < T \\
v_x(R, t) = m \quad 0 < t < T \\
v(., 0) = v_0^R \quad \text{in} \ (0, R)
\end{aligned} \quad (4.1) \]

where \( v_0^R \) is a suitable approximation of \( v_0 \) satisfying \( v_{0,x}^R(0) = 0, v_{0,x}^R(R) = m, v_0^R \to v_0, R \to \infty \) in \( C^2(\bar{I}) \) for all \( I \subset \subset \mathbb{R} \) as well as
\[ \sup_{(0,R)} (|v_{0,x}^R| + |v_{0,x}^R|) \leq M \quad \int_0^R |v_{0,x}^R - m| \leq \tilde{M} \quad \text{uniformly in} \ R. \quad (4.2) \]

The existence of a solution to \((4.1)\) can be obtained with the help of the Leray–Schauder principle (see e.g. [15]). Rather than giving all the details we shall concentrate on the crucial step in the application of the Leray–Schauder theorem which essentially consists of deriving \textit{a priori} estimates for all possible solutions of \((4.1)\).
Let us start by estimating the maximum norm of $v_x$. For this purpose we differentiate (4.1) with respect to $x$ to obtain an evolution equation for $y = v_x$:

$$\frac{y_{t}}{1+y^2} + \frac{y_{xx}}{2(1+y^2)} - J \frac{y_{y}}{\sqrt{1+y^2}} = 0 \quad \text{in } \Omega_{R,T}$$

$$y(0,t) = 0 \quad y(R,t) = m \quad t > 0$$

$$y(.,0) = v_{0,x} \quad \text{in } (0,R). \quad (4.3)$$

The maximum principle and (4.2) imply that

$$\sup_{\Omega_{R,T}} |v_x| \leq \max \left( \sup_{(0,R)} |v_{0,x}|, m \right) \leq M. \quad (4.4)$$

Using the fact that $v_{xx}(0,t) = v_{xx}(R,t) = 0$ for all $t > 0$ we obtain in a similar way a bound on $v_t$, namely

$$\sup_{\Omega_{R,T}} |v_t| \leq \sup_{(0,R)} |v_{(.,0)}| \leq \sup_{(0,R)} \left( |v_{0,R,x}| + \sqrt{1 + (v_{0,R,x})^2} \right) =: M_1. \quad (4.5)$$

If we combine (4.4) and (4.5) with the first equation of (4.1) we also obtain

$$\sup_{\Omega_{R,T}} |v_{xx}| \leq \left( 1 + \sup_{\Omega_{R,T}} |v_t|^2 \right) \sup_{\Omega_{R,T}} |v_t| + \sup_{\Omega_{R,T}} \sqrt{1 + v_t^2} \leq M_2. \quad (4.6)$$

Note that $M, M_1$ and $M_2$ are independent of $R$ and $T$. Finally, we may use (4.5) in order to derive a bound on the maximum norm of $v$, namely

$$\sup_{\Omega_{R,T}} |v| \leq \sup_{(0,R)} |v_{0}| + M_1T = C(R,T).$$

As already pointed out the above estimates are the main step in the application of the Leray–Schauder theory from which we derive the existence of a solution $v^R$ of (4.1). This solution can be extended to the domain $(-R, R) \times (0, T)$ via

$$v^R(-x,t) := v^R(x,t) \quad 0 \leq x \leq R, t \geq 0.$$ 

Inequalities (4.4)–(4.6) imply that the functions $v^R$ are bounded uniformly in $C^{2,1}(I \times [0,T])$ for large $R$ where $I$ is a bounded open subset of $\mathbb{R}$. Using the Arzela–Ascoli theorem and a diagonal argument we conclude that there is a sequence $(R_j)_{j \in \mathbb{N}}$, $R_j \to \infty$, $j \to \infty$ and a function $v$ such that

$$v^R_j \to v \quad \text{in } H^{1+\gamma, \frac{1+\gamma}{\gamma}}(I \times [0,T]) \quad \text{for all } \gamma < 1, I \subset \subset \mathbb{R}$$

$$v_{xx}^R \to v_{xx} \quad \text{in } L^\infty(I \times (0,T)) \quad \text{for all } I \subset \subset \mathbb{R} \quad (4.7)$$

$$v^R \to v \quad \text{in } L^\infty(I \times (0,T)) \quad \text{for all } I \subset \subset \mathbb{R}.$$ 

In particular, (4.7) implies that $v(.,0) = v_0$ and $v_t(.,0) = 0$ for $0 \leq t \leq T$. Next, let $\phi \in C_0^\infty(\mathbb{R} \times (0,T))$ be arbitrary; we derive from (4.1) and (4.7) after sending $j \to \infty$:

$$\int_0^T \int_{-\infty}^\infty v_t \phi = \int_0^T \int_{-\infty}^\infty \frac{v_{xx} \phi}{1+v^2} + J \int_0^T \int_{-\infty}^\infty \sqrt{1+v^2} \phi.$$ 

As we already know that $v \in H^{\gamma + \frac{2}{\gamma}}(\mathbb{R} \times [0,T])$ for all $\gamma < 1$, a standard regularity result for linear parabolic equations (see e.g. theorem 12.2, chapter III in [15]) implies that $v \in H^{2+\alpha, 1+\frac{2}{\alpha}}(\mathbb{R} \times [0,T])$ (as in (A1)), so that (4.1) is satisfied in the classical sense. It remains to clarify in which way the condition at infinity is attained. To this purpose we
return to the approximate solutions \( v^R \). Multiplying (4.3) by \( v^R_x - m \) and integrating by parts gives
\[
\frac{1}{2} \frac{d}{dt} \int_0^R \left| v^R_x - m \right|^2 + \int_0^R \frac{|v^R_x|^2}{1 + (v^R)^2} = \int_0^R v^R_x (v^R_x - m) \left. \right|_{x=0}^{x=R} + \int_0^R \frac{v^R_x v^R (v^R_x - m)}{\sqrt{1 + (v^R)^2}}
\]
\[
\leq m |v^R_x(0, t)| + \int_0^R |v^R_x|^2 |v^R_x - m|
\]
\[
\leq m M_2 + \frac{1}{2} \int_0^R \frac{|v^R_x|^2}{1 + (v^R)^2} + \int_0^R \frac{1 + M^2}{2} |v^R - m|^2
\]
by (4.6) and (4.4). Gronwall’s lemma together with (4.4) and (4.2) yields
\[
\sup_{0 \leq t \leq T} \int_0^R |v(\cdot, t)|^2 + \int_0^T \int_0^R |v_x|^2 \leq C(v_0, T) \tag{4.8}
\]
uniformly in \( R \). From (4.7) and the lower semicontinuity of the \( L^2 \)-norm with respect to weak convergence we infer
\[
\sup_{0 \leq t \leq T} \int_0^N |v(\cdot, t)|^2 + \int_0^T \int_0^N |v_x|^2 \leq C(v_0, T)
\]
for all \( N \in \mathbb{N} \) and sending \( N \to \infty \) finally gives
\[
\sup_{0 \leq t \leq T} \int_0^\infty |v(\cdot, t)|^2 + \int_0^T \int_0^\infty |v_x|^2 \leq C(v_0, T) \tag{4.9}
\]
in particular \( v_1(x, t) \to m, x \to \infty \) for a.a. \( t \in (0, T) \). Since \( 0 < T < \infty \) is arbitrary, the existence part of theorem 1 is proved. Note that in view of (4.4)–(4.6) \( v_x, v_{xx} \) and \( v_t \) are bounded on \( (0, \infty) \times (0, \infty) \).

In order to show uniqueness suppose that \( v_1, v_2 \in H^{2,a,1+\frac{a}{2}}([0, \infty) \times [0, T]) \) are two solutions of (3.4)–(3.7). Extending them in the same way as above to \( \mathbb{R} \times [0, T] \) we obtain for \( v := v_1 - v_2 \) an equation of the form
\[
v_t - av_{xx} + bv_x = 0 \quad \text{in} \ \mathbb{R} \times (0, T)
\]
\[
v(\cdot, 0) = 0 \quad \text{in} \ \mathbb{R}
\]
where the coefficients
\[
a = \frac{1}{1 + v^2_{1,x}} \quad \text{and} \quad b = \frac{v_{2,xx}(v_{1,x} + v_{2,x})}{(1 + v^2_{1,x})(1 + v^2_{2,x})} - \frac{J}{\sqrt{1 + v^2_{1,x} + 1 + v^2_{2,x}}}
\]
are bounded on \( \mathbb{R} \times [0, T] \). Note further that since \( v(\cdot, 0) = 0 \) we have
\[
|v(x, t)| \leq \int_0^t |v_t(x, s)| ds \leq \|v_t\|_{L^\infty} T \quad x \in \mathbb{R}, \ t \in [0, T].
\]
Thus, theorem 2.5 in [15] implies that \( v \equiv 0 \) in \( \mathbb{R} \times (0, T) \).

4.2. Proof of theorem 2

Let \( J = 1 \). We start by establishing the existence of problem (3.10), (3.11).

**Lemma 1.** Problem (3.10), (3.11) with \( c = \sqrt{1 + m^2} \) has a unique solution \( \chi \in C^\infty(\mathbb{R}) \) and it satisfies \( \chi(-x) = -\chi(x) \) for all \( x \in \mathbb{R} \). Furthermore,
\[
\int_0^\infty |m - \chi| dx < \infty.
\]
**Proof.** The theory of the ODEs implies that
\[
\chi' = (1 + \chi^2)(\sqrt{1 + m^2} - \sqrt{1 + \chi^2}) \quad \chi(0) = 0
\]  
(4.10)

has a unique local solution \( \chi \). Furthermore, it is not hard to see that \( \chi \) can be extended to \( \mathbb{R} \) and that \( \chi(x) \to m \) as \( x \to \infty \). A uniqueness argument shows that \( \chi(-x) = -\chi(x) \) for all \( x \in \mathbb{R} \). Finally, let us write (4.10) as
\[
\chi' = a(x)(m - \chi) \quad \text{where} \quad a(x) = \frac{(1 + \chi^2)(m + \chi)}{\sqrt{1 + m^2} + \sqrt{1 + \chi^2}} \geq \frac{m}{2\sqrt{1 + m^2}} \quad x \geq 0.
\]

Integration with respect to \( x \in (0, R) \) gives
\[
\frac{m}{2\sqrt{1 + m^2}} \int_0^R |m - \chi| \leq \int_0^R \chi_s(x) \, dx = \chi(R) \leq m \quad x \geq 0
\]
so that \( \int_0^\infty |m - \chi| < \infty \). \( \square \)

Note that the function \( w(x) = \int_0^x \chi(s) \, ds \) satisfies
\[
\frac{w_{xx}}{1 + w_x^2} + \sqrt{1 + w_x^2} = \sqrt{1 + m^2} \quad x \in \mathbb{R}
\]
\[
w_x(0) = 0 \quad w_x(x) \to m \quad x \to \infty.
\]  
(4.11)

In the following we shall study the asymptotic behaviour of the solution \( v \) of (3.4)–(3.7) in the case \( J = 1 \). To do so, we need precise information about \( v_x \) as \( x \to \infty \). We infer from (A1)–(A4) that there is a \( \delta \in (0, m) \) with
\[
v_{0,x}(x) \geq -m + \delta \quad x \geq 0.
\]  
(4.12)

We shall again work with the approximate solutions \( v^R \) from the proof of theorem 1. In view of the additional assumption (A4) we may choose \( v^R_0 = v_0 \) for \( R \geq R_0 \) in order to satisfy (4.2).

**Lemma 2.** Positive constants \( \rho, \tilde{R} \) and \( \tilde{M} \) exist such that
\[
w_x(x - \tilde{R}) \leq v_x(x, t) \leq m + \tilde{M}e^{-\rho x} \quad x \geq 0, \ t \geq 0.
\]

**Proof.** We start with the estimate from below. Since \( w_x(x) \to -m \), \( x \to -\infty \) there is \( R_1 > 0 \) such that
\[
w_x(x) \leq -m + \delta \quad \text{for} \quad x \leq -R_1.
\]  
(4.13)

Let us define \( \phi(x, t) := w_x(x - R_0 - R_1) \). An easy calculation based on (4.11) shows that \( \phi \) satisfies the equation
\[
\phi_t - \frac{\phi_{xx}}{1 + \phi^2} + 2\phi\phi_x^2 \frac{\phi_x^2}{(1 + \phi^2)^2} - \frac{\phi\phi_x}{\sqrt{1 + \phi^2}} = 0 \quad \text{in} \quad \Omega_{R,T}, \ R \geq R_0.
\]

Furthermore, by using (4.13) we get \( \phi(0, t) = w_x(-R_0 - R_1) \leq -m + \delta \leq 0 = v^R_x(0, t) \) and \( \phi(R, t) = w_x(R - R_0 - R_1) \leq m = v^R_x(R, t) \) for \( 0 \leq t \leq T \). It remains to compare the initial data. We have by (4.13), (4.12) and (A3)
\[
\phi(x, 0) = w_x(x - R_0 - R_1) \leq -m + \delta \leq v_{0,x}(x) \quad 0 \leq x \leq R_0
\]
\[
\phi(x, 0) = w_x(x - R_0 - R_1) \leq m = v_{0,x}(x) \quad R_0 \leq x \leq R.
\]

The comparison principle (see e.g. [11]) then implies
\[
w_x(x - R_0 - R_1) \leq v^R_x(x, t) \quad \text{in} \quad \Omega_{R,T}.
\]
In order to derive an upper bound on $v_x$ we introduce

$$\psi(x, t) := m + \tilde{M}e^{\rho(R_0 - x)}$$

where $\tilde{M} = \sup_{[x_0, 1]} |v_0| - m$ and $\rho > 0$ will be chosen later. It is clear that $\psi(x, t) \geq v_x^R(x, t)$ for $x = 0, R$ and $0 \leq t \leq T$. Furthermore,

$$\psi(x, 0) \geq m + \tilde{M} \geq v_0(x) \quad 0 \leq x \leq R_0$$

$$\psi(x, 0) = \psi(0, x) \quad R_0 \leq x \leq R$$

by (A4). Finally we compute

$$\psi_t - \frac{\psi_{xx}}{1 + \psi^2} + 2 \frac{\psi_x \psi_{x}^2}{(1 + \psi^2)^2} - \frac{\psi \psi_x}{\sqrt{1 + \psi^2}} \geq -\rho^2 \tilde{M}e^{\rho(R_0 - x)} + \frac{(m + \tilde{M}e^{\rho(R_0 - x)})\rho \tilde{M}e^{\rho(R_0 - x)}}{\sqrt{1 + (m + \tilde{M}e^{\rho(R_0 - x)})^2}}$$

$$\geq \rho \tilde{M}e^{\rho(R_0 - x)} \left( -\rho + \frac{m}{\sqrt{1 + (m + \tilde{M}e^{\rho(R_0)})^2}} \right) \geq 0$$

provided $\rho$ is chosen sufficiently small. Thus, again by the comparison principle

$$v_x^R(x, t) \leq m + \tilde{M}e^{\rho(R_0 - x)} \quad \text{in } \Omega_{R,T}.$$  

Since both lower and upper bounds do not depend on $R$ the result follows from (4.7) upon sending $j \to \infty$.

The next step consists of obtaining lower bounds for $v_{xx}$. To this purpose we define

$$h^R(x, t) := \frac{v_{xx}^R(x, t)}{1 + (v_x^R(x, t))^2} \quad (x, t) \in \Omega_{R,T}.$$  

A long but straightforward calculation shows that $h^R$ satisfies the following evolution equation

$$h^R_t - \frac{h^R_{xx}}{1 + (v_x^R)^2} - \frac{v_x^R}{\sqrt{1 + (v_x^R)^2}} h^R_x + \frac{2v_x^R}{1 + (v_x^R)^2} h^R h^R_x - \frac{1}{\sqrt{1 + (v_x^R)^2}} (h^R)^2 = 0. \quad (4.14)$$

Differentiating the identity $v_t^R = h^R + \sqrt{1 + (v_x^R)^2}$ with respect to $x$ and observing that $v_{xt}^R(x, t) = 0$ for $x = 0, R$ and $0 < t \leq T$ we get

$$0 = h^R_x(0, t) + \frac{v_x^R(0, t)v_{xx}^R(0, t)}{\sqrt{1 + v_x^R(0, t)^2}} = h^R_x(0, t)$$

$$0 = h^R_x(R, t) + \frac{v_x^R(R, t)v_{xx}^R(R, t)}{\sqrt{1 + v_x^R(R, t)^2}} = h^R_x(R, t) + m \sqrt{1 + m^2 h^R(R, t)}.$$

Now we are in a position to prove this.

**Lemma 3.** There exist positive constants $\gamma, K$ such that

$$v_{xx}(x, t) \geq -\frac{K}{\gamma t + 1} \quad x \geq 0, \ t \geq 0$$

$$v_{xx}(x, t) \geq -\frac{K}{(\gamma x + 1)^2} \quad x \geq 0, \ t \geq 0.$$  

(4.16)

(4.17)
To prove (4.17) we first observe that
\[ \eta(x,t) = \phi_x(R,t), \]
by (4.4) provided \( \gamma \) is small enough. Thus, by comparison
\[ \phi_t - \phi_{xx} = \frac{v^R}{1 + (v^R)^2} \phi_x + \frac{2v^R}{1 + (v^R)^2} \phi = \frac{1}{\sqrt{1 + (v^R)^2}} \phi \]
Clearly, \( \phi(x,0) = -K \leq \frac{\nu_{0,x}}{1 + \nu_{0,t}} = h^R(x,0), \ 0 \leq x \leq R \) and \( \phi_t(0,t) = 0 \).
\[ \phi_x(R,t) + m\sqrt{1 + m^2} \phi(R,t) \leq 0, \ 0 \leq t \leq T. \]
Furthermore,
\[ \phi - \phi_{xx} = \frac{v^R}{1 + (v^R)^2} \phi_x + \frac{2v^R}{1 + (v^R)^2} \phi \phi_x - \frac{1}{\sqrt{1 + (v^R)^2}} \phi \]
by (4.4) provided \( \gamma \) is small enough. Thus, by comparison
\[ v^R_{xx} = (1 + (v^R)^2)h^R \geq -(1 + (v^R)^2)(\frac{K}{\gamma^2 + 1}) \]
in \( \Omega_{R,T}. \) (4.18)
To prove (4.17) we first observe that
\[ v^R_x(x,t) \geq w_x(x - \tilde{R}) \geq 0 \quad \text{for } \tilde{R} \leq x \leq R, \ t \geq 0 \] (4.19)
by lemma 2. We introduce the comparison function
\[ \eta(x,t) := -\frac{\tilde{K}}{(\tilde{\gamma}x + 1)^2} \quad (x,t) \in (\tilde{R}, R) \times (0, T) \]
where \( \tilde{K} \geq 4M_3 \) and \( 0 < \tilde{\gamma} \leq \min(\tilde{R}^{-1}, \frac{1}{4}m\sqrt{1 + m^2}) \) a priori. We then have
\[ \eta(x,0) \leq h^R(x,0) \]
since \( \nu_{0,xx} = 0 \) for \( x \geq \tilde{R} > R_0 \) by (A4). On the other hand,
\[ \eta(\tilde{R},t) = -\frac{\tilde{K}}{(\tilde{\gamma}R + 1)^2} \leq -\frac{\tilde{K}}{4} \leq -M_2 \leq h^R(\tilde{R},t) \quad 0 \leq t \leq T \]
by (4.6) and the choices of \( \tilde{K} \) and \( \tilde{\gamma} \). Next,
\[ \eta_t - \eta_{xx} = \frac{\nu^R}{(\tilde{\gamma}x + 1)^3} \tilde{K} \tilde{\gamma} - m\sqrt{1 + m^2} \frac{\tilde{K}}{(\tilde{\gamma}R + 1)^2} \]
\[ \leq \frac{\tilde{K}}{(\tilde{\gamma}R + 1)^2} (2\tilde{\gamma} - m\sqrt{1 + m^2}) \leq 0 \]
again by the choice of \( \tilde{\gamma} \). Finally, in view of (4.19) and (4.4)
\[ \eta_t - \eta_{xx} = \frac{\nu^R}{(\tilde{\gamma}x + 1)^3} \eta_t + \frac{2v^R}{1 + (v^R)^2} \eta_{xx} - \frac{1}{\sqrt{1 + (v^R)^2}} \eta^2 \]
\[ = \frac{6\tilde{K}\tilde{\gamma}^2}{(\tilde{\gamma}x + 1)^3} - \frac{v^R}{(\tilde{\gamma}x + 1)^3} \frac{2\tilde{K}\tilde{\gamma}}{(\tilde{\gamma}x + 1)^5} - \frac{2v^R}{1 + (v^R)^2} \frac{\tilde{K}^2\tilde{\gamma}}{(\tilde{\gamma}x + 1)^5} \]
\[ \leq \frac{\tilde{K}}{(\tilde{\gamma}x + 1)^2} \left( 6\tilde{\gamma}^2 - \frac{\tilde{K}}{\sqrt{1 + M_2^2}} \right) \leq 0 \]
if $\tilde{\gamma}$ is small enough. Similarly as above we get by comparison
\[ v_{tt}^R(x, t) \geq -(1 + M_1^2) \frac{\tilde{K}}{(\tilde{\gamma} x + 1)^2} \quad \tilde{R} \leq x \leq R, \ t \geq 0. \tag{4.20} \]

As $\gamma, \tilde{\gamma}$ and $K, \tilde{K}$ do not depend on $R$ the result follows from (4.18) and (4.20) by sending $R \to \infty$. \hfill $\square$

The following lemma provides a first step towards the asymptotic behaviour of the solution.

**Lemma 4.** There exists a sequence $(t_j)_{j \in \mathbb{N}}, t_j \to \infty, \ j \to \infty$ such that
\[ \int_0^\infty |v_{i,j}(., t_j)|^2 \, dx \to 0 \quad j \to \infty. \]

**Proof.** Let $\zeta \in C^\infty(\mathbb{R})$ be a cut-off function with the properties $0 \leq \zeta \leq 1, \ \zeta \equiv 1$ for $x \leq 1, \ \zeta \equiv 0$ for $x \geq 2$ and set $\zeta_R(x) := \zeta(\frac{x}{R})$. Differentiating (3.4) with respect to $t$ gives
\[ v_{tt} - \frac{v_{txx}}{1 + v_x^2} + 2 \frac{v_{tx} v_{xx}}{(1 + v_x^2)^2} - \frac{v_{x} v_{tx}}{\sqrt{1 + v_x^2}} = 0. \]

Let us multiply this identity by $\zeta_R^2(v_t - \sqrt{1 + m^2})$ and integrate with respect to $x$. Keeping in mind that $v_1(0, t) = v_{i,x}(0, t) = 0$ we obtain after integration by parts
\[ \frac{1}{2} \frac{d}{dt} \int_0^\infty \zeta_R^2 \left| v_t - \sqrt{1 + m^2} \right|^2 + \int_0^\infty \zeta_R^2 \frac{|v_{tx}|^2}{1 + v_x^2} + \frac{1}{2} \frac{\partial}{\partial t} \int_0^\infty \zeta_R^2 \frac{v_{xx}}{\sqrt{1 + v_x^2}} (v_t - \sqrt{1 + m^2})^2 \]
\[ = -2 \int_0^\infty \zeta_R^2 \frac{v_{tx}}{1 + v_x^2} (v_t - \sqrt{1 + m^2}) \]
\[ - \int_0^\infty \zeta_R^2 \frac{v_x}{\sqrt{1 + v_x^2}} (v_t - \sqrt{1 + m^2})^2 \]
\[ \leq \frac{1}{2} \frac{\partial}{\partial t} \int_0^\infty \zeta_R^2 \frac{|v_{tx}|^2}{1 + v_x^2} + C \int_0^\infty \zeta_R^2 \left| v_t - \sqrt{1 + m^2} \right|^2 \]
\[ \leq \frac{1}{2} \frac{\partial}{\partial t} \int_0^\infty \zeta_R^2 \frac{|v_{xx}|^2}{1 + v_x^2} + C \int_0^\infty \left( |v_{xx}|^2 + |v_t - m|^2 \right) \]

where we used Young's inequality and (3.4). Integration with respect to $t \in (0, T)$ yields
\[ \int_0^\infty \zeta_R^2 \left| v_t - \sqrt{1 + m^2} \right|^2 \, dx + \int_t^T \int_0^\infty \zeta_R^2 \frac{|v_{tx}|^2}{1 + v_x^2} \]
\[ + \int_0^T \int_0^\infty \zeta_R^2 \frac{v_{xx}}{\sqrt{1 + v_x^2}} |v_t - \sqrt{1 + m^2}| \]
\[ \leq \int_0^\infty \zeta_R^2 \left| v_t - \sqrt{1 + m^2} \right|^2 \, dx + \frac{C}{R} \int_0^T \int_0^\infty (|v_{xx}|^2 + |v_t - m|^2) \]
\[ \leq C + \frac{1}{R} C(v_0, T) \]

by (4.9). Using (4.4) and taking the limit $R \to \infty$ we end up with
\[ \frac{1}{1 + M_1^2} \int_0^T \| v_{i,x} \|_{L^2(0, \infty)}^2 \leq - \int_0^T \int_0^\infty \frac{v_{xx}}{\sqrt{1 + v_x^2}} |v_t - \sqrt{1 + m^2}|^2 + C. \]
Lemma 3 implies
\[ v_{xx}(x, t) \geq -\frac{K}{(\gamma t + 1)(\gamma x + 1)^{3/4}} \quad x \geq 0, \ t \geq 0 \]
so that
\[
\int_0^t \| v_{xx} \|^2_{L^2(0, \infty)} \leq C + K \int_0^t (\gamma s + 1)^{-1/4} \int_0^\infty (\gamma x + 1)^{-1/4} \left| v - \sqrt{1 + m^2} \right|^2 \, dx \, ds
\]
\[ \leq C + C \int_0^t (\gamma s + 1)^{-1/4} \, ds \]
\[ \leq C(1 + t^{3/4}) \]
since \(|v_t| \leq M_1\) by (4.5). The last inequality implies the assertion of the lemma. \qed

In addition to the uniform bounds (4.4)–(4.6) on \(v_x, v_t\) and \(v_{xx}\) respectively we shall now derive an estimate involving the solution \(v\) itself. To this purpose we introduce \(\tilde{w}(x, t) := \sqrt{1 + m^2 t} + w(x)\) where \(w\) appeared in (4.11). Thus \(\tilde{w}\) satisfies
\[ \tilde{w}_t - \frac{\tilde{w}_{xx}}{1 + \tilde{w}_x^2} = 1 + \tilde{w}_x^2 \quad \text{in} \ \mathbb{R} \times (0, T) \]
so that we obtain for \(z := v - \tilde{w}\) an equation of the form
\[ z_t - az_{xx} + bz_x = 0 \quad \text{in} \ \mathbb{R} \times (0, T) \]
where the coefficients
\[ a = \frac{1}{1 + v_x^2} \quad \text{and} \quad b = \frac{v_{xx}(1 + v_x^2) - v_x + \tilde{w}_x}{(1 + v_x^2)(1 + \tilde{w}_x^2)} \]
are bounded on \(\mathbb{R} \times (0, T)\). For the initial values we have according to (A4) and lemma 1
\[
|z(x, 0)| = |v_0(x) - w(0)| \leq |v_0(0) - w(0)| + \int_0^x |v_{0,x}(y) - w_x(y)| \, dy
\]
\[ \leq |v_0(0)| + R_0 \sup_{0 \leq x \leq R_0} |v_{0,x}(x) - w_x(x)| + \int_{R_0}^{\max(x, R_0)} |m - w_x(y)| \, dy
\]
\[ \leq C(v_0, m). \]
Since \(|z(x, t)| \leq M(T)(1 + |x|)\) for \((x, t) \in \mathbb{R} \times (0, T)\) the maximum principle (see theorem 2.5 and the following remark 2.2 in chapter I of [15]) implies
\[ \sup_{\mathbb{R} \times (0,T)} |v - \tilde{w}| \leq \sup_{\mathbb{R}} |v_0 - w| \leq C(v_0, m) \quad \text{for all} \ T \geq 0. \quad (4.21) \]

Using the Arzela–Ascoli theorem and a diagonal argument we may assume in view of (4.4)–(4.6), (4.21) and lemma 4 that there is a sequence \((t_j)_{j \in \mathbb{N}}, t_j \to \infty, j \to \infty\) such that
\[
v(\cdot, t_j) - \tilde{w}(\cdot, t_j) \to f \quad \text{in} \ C^0(\bar{I}) \quad \text{for all} \ I \subset \subset [0, \infty)
\]
v_{x}(\cdot, t_j) \to f_x \quad \text{in} \ C^0(\bar{I}) \quad \text{for all} \ I \subset \subset [0, \infty)
\]
v_{xx}(\cdot, t_j) \to f_{xx} \quad \text{in} \ L^\infty(0, \infty)
\]
v(\cdot, t_j) \to g \quad \text{in} \ C^0(\bar{I}) \quad \text{for all} \ I \subset \subset [0, \infty)
\]
\[ \|v_{tt}(\cdot, t_j)\|_{L^2(0, \infty)} \to 0. \]

\[ \quad (4.22) \]
Let $\zeta \in C_0^\infty(0, \infty)$ be arbitrary. Then we have for all $j \in \mathbb{N}$
\[
\int_0^\infty v_t(., t_j)\zeta = \int_0^\infty \frac{v_{xx}(., t_j)\zeta}{1 + v_x(., t_j)^2} + \int_0^\infty \sqrt{1 + v_x(., t_j)^2}\zeta.
\]
Passing to the limit $j \to \infty$ we obtain from (4.22)
\[
\int_0^\infty g\zeta = \int_0^\infty \frac{h_x\zeta}{1 + h^2} + \int_0^\infty \sqrt{1 + h^2}\zeta
\]
with $h := w_x + f_x$. Thus,
\[
h_x = (1 + h^2) \left( g - \sqrt{1 + h^2} \right) \quad \text{in } [0, \infty)
\]
h(0) = 0.

In addition, (4.22)$_5$ yields for all $\zeta \in C_0^\infty(0, \infty)$
\[
\int_0^\infty g_x \zeta \to \int_0^\infty v_t(., t_j)\zeta = -\int_0^\infty v_t, x(., t_j)\zeta \to 0
\]
so that $g$ must be a constant, say $g \equiv g_0$. Next, lemmas 1 and 2 imply for $j \in \mathbb{N}$ and $x \geq \hat{R}$
\[
v_t(x, t_j) = \frac{v_{xx}(x, t_j)}{1 + v_x(x, t_j)^2} + \sqrt{1 + v_x(x, t_j)^2} \geq -\frac{K}{\gamma t_j + 1} + \sqrt{1 + w_x(x - \hat{R})^2}.
\]
Sending $j \to \infty$ gives
\[
g_0 \geq \sqrt{1 + w_x(x - \hat{R})^2} \quad \text{for all } x \geq \hat{R}
\]
and taking the limit $x \to \infty$ finally yields $g_0 \geq \sqrt{1 + m^2}$. Let us assume that $g_0 > \sqrt{1 + m^2}$, say $g_0 = \sqrt{1 + \tilde{m}^2}$ for some $\tilde{m} > m$. Since $h$ is a solution of
\[
h_x = (1 + h^2) \left( \sqrt{1 + \tilde{m}^2} - \sqrt{1 + h^2} \right) \quad h(0) = 0
\]
we conclude in a similar way to in the proof of lemma 1 that $\hat{R}$ exists such that
\[
h(x) \geq \frac{1}{2}(m + \tilde{m}) \quad x \geq \hat{R}.
\]
On the other hand, lemma 2 permits us to find $\tilde{R}$ with
\[
|v_t(x, t) - m| \leq \frac{1}{2}(\tilde{m} - m) \quad x \geq \tilde{R}, t \geq 0.
\]
Let $R := \max(\hat{R}, \tilde{R})$. Then
\[
|v_t(R, t_j) - h(R)| \geq |h(R) - m| - |v_t(R, t_j) - m| \\
\geq \frac{1}{2}(\tilde{m} - m) - \frac{1}{2}(\tilde{m} - m) = \frac{1}{2}(\tilde{m} - m)
\]
for all $j \in \mathbb{N}$. This contradicts (4.22)$_2$ and therefore we must have $g_0 = \sqrt{1 + m^2}$. From lemma 1 we then immediately obtain $h = w_x$ or in other words $f_x = 0$. Thus, (4.22)$_1$ implies
\[
v(., t_j) - (\bar{w}(., t_j) + c) \to 0 \quad \text{in } C^0(I) \text{ for all } I \subset \subset \mathbb{R}
\]
for some constant $c$. Redefining $w$ if necessary we may assume $c = 0$. Let us finally prove that
\[
\sup_{\mathbb{R}} |v(., t) - \bar{w}(., t)| \to 0 \quad t \to \infty.
\]
Let $\epsilon > 0$ be given and choose $x_0 \geq 0$ so large that
\[
\int_{x_0 - \bar{R}}^{\infty} |m - w_x(y)| \, dy + \frac{M}{\rho} e^{-\rho x_0} \leq \frac{\epsilon}{2}
\] (4.24)
where $\bar{R}, \tilde{M}$ and $\rho$ appear in lemma 2. Afterwards choose $j_0 \in \mathbb{N}$ so large that
\[
\sup_{0 \leq x \leq x_0} |v(x, t_{j_0}) - \tilde{w}(x, t_{j_0})| \leq \frac{\epsilon}{2}.
\] (4.25)
For $x \geq x_0$ we then have
\[
|v(x, t_{j_0}) - \tilde{w}(x, t_{j_0})| \leq \frac{\epsilon}{2} + \int_{x_0}^{x} |v_x(y, t_{j_0}) - w_x(y)| \, dy.
\]
From lemma 2 we infer that
\[
|v_x(x, t) - w_x(x)| \leq |m - w_x(x - \bar{R})| + \tilde{M} e^{-\rho x} \quad x \geq 0, \ t \geq 0
\]
so that
\[
|v(x, t_{j_0}) - \tilde{w}(x, t_{j_0})| \leq \frac{\epsilon}{2} + \int_{x_0}^{x} |m - w_x(y - \bar{R})| \, dy + \tilde{M} \int_{x_0}^{x} e^{-\rho y} \, dy
\]
\[
\leq \frac{\epsilon}{2} + \int_{x_0 - \bar{R}}^{\infty} |m - w_x(y)| \, dy + \frac{\tilde{M}}{\rho} e^{-\rho x_0} \leq \epsilon
\]
by (4.24). Thus, $\sup_{\mathbb{R}} |v(., t_{j_0}) - \tilde{w}(., t_{j_0})| \leq \epsilon$ and the maximum principle just as above gives
\[
\sup_{\mathbb{R}} |v(., t) - \tilde{w}(., t)| \leq \epsilon \quad t \geq t_{j_0}
\]
which is (4.23).

4.3. Proof of theorem 3

Let $v$ be the solution of (3.4)–(3.7) with $J = -1$. As above we can extend $v$ to $\mathbb{R} \times (0, \infty)$ by setting
\[
v(-x, t) = v(x, t) \quad x \geq 0, \ t \geq 0.
\]
Suppose $(t_k)_{k \in \mathbb{N}}$ is an arbitrary sequence, $t_k \to \infty, k \to \infty$. We define
\[
v_k(x, t) := \frac{1}{t_k} v(t_k x, t_k t) \quad (x, t) \in \mathbb{R} \times (0, \infty).
\]
Then $v_k$ is a solution of the problem
\[
v_k, t - \frac{1}{t_k^2} \frac{v_k, xx}{1 + v_k^2} = -\sqrt{1 + v_k^2} \quad \text{in} \ \mathbb{R} \times (0, \infty).
\]
In view of (4.4) and (4.5) we have the following estimates
\[
\sup_{\mathbb{R} \times (0, \infty)} |v_{k, xx}| \leq M \quad \sup_{\mathbb{R} \times (0, \infty)} |v_{k, t}| \leq M_1.
\] (4.26)
The function \( v_k \) itself satisfies for \((x, t) \in (-R, R) \times (0, T)\)

\[
|v_k(x, t)| \leq |v_k(x, 0)| + \int_0^t |v_{k,r}(x, s)| \, ds
\]

\[
\leq \frac{1}{t_k} |v_0(0)| + \frac{1}{t_k} |v_0(t_k x) - v_0(0)| + M_1 T
\]

\[
\leq \frac{1}{t_k} |v_0(0)| + \int_0^{t_k} |v_{0,x}(y)| \, dy + M_1 T
\]

\[
\leq |v_0(0)| + R \sup_{\mathbb{R}} |v_{0,x}| + M_1 T \leq C(R, T)
\]

uniformly in \( k \in \mathbb{N} \). Combining this estimate with the Arzela–Ascoli theorem and a diagonal argument we conclude that there is a subsequence \((t_{k_j})_{j \in \mathbb{N}}\) and a function \( \tilde{v} \in C^0(\mathbb{R} \times [0, \infty)) \) such that

\[
v_{k_j} \rightarrow \tilde{v}, \quad j \rightarrow \infty \quad \text{uniformly on } [-R, R] \times [0, T] \text{ for all } R, T.
\]

Furthermore, a standard argument on the theory of viscosity solutions for Hamilton–Jacobi equations (cf [8]) shows that \( \tilde{v} \) is a viscosity solution of

\[
\tilde{v}_t + \sqrt{1 + \tilde{v}^2} = 0 \quad \text{in } \mathbb{R} \times (0, \infty)
\]

\[
\tilde{v}(x, 0) = m|x|.
\]

To see that the initial values of \( \tilde{v} \) are in fact given by \( m|x| \) we estimate for \( x \geq 0 \)

\[
|v_{k_j}(x, 0) - mx| \leq \frac{1}{t_{k_j}} |v_0(t_{k_j} x) - v_0(0) - mx| + \frac{1}{t_{k_j}} |v_0(0)|
\]

\[
\leq \frac{1}{t_{k_j}} \int_0^{t_{k_j}} |v_{0,x}(y) - m| \, dy + \frac{1}{t_{k_j}} |v_0(0)|
\]

\[
\leq C \rightarrow 0, \quad j \rightarrow \infty
\]

by (A2), so that \( \tilde{v}(x, 0) = mx, x \geq 0 \).

Our next aim is to prove that \( v(x, t) = t Q(\frac{x}{t}) \) in \( \mathbb{R} \times (0, \infty) \) where \( Q \) is given by (3.13) and extended to \( \mathbb{R} \) by setting \( Q(\theta) = \bar{Q}(-\theta) \) for \( \theta < 0 \). Note that \( Q \in C^1(\mathbb{R}) \) and \( Q(\theta) - \theta \bar{Q}'(\theta) = -\sqrt{1 + Q'(\theta)^2} \) in \( \mathbb{R} \). Therefore \( \tilde{w}(x, t) := t \bar{Q}(\frac{x}{t}) \) satisfies \( \tilde{w} \in C^1(\mathbb{R} \times (0, \infty)) \) and

\[
\tilde{w}_t + \sqrt{1 + \tilde{w}^2} = 0 \quad \text{in } \mathbb{R} \times (0, \infty)
\]

\[
\tilde{w}(x, 0) = m|x| \quad x \in \mathbb{R}
\]

(4.29)

where the second equality is an immediate consequence of the explicit form of \( Q \). If we knew that the viscosity solution of (4.28) were unique we could infer \( \tilde{v} = \tilde{w} \). The corresponding uniqueness result in [8] however requires the solutions to be bounded which is obviously not the case in our situation. To overcome this difficulty we first prove that \( \tilde{z} := v - \tilde{w} \) is bounded in \( \mathbb{R} \times [0, T] \) for \( T < \infty \).

Let us define \( \epsilon_{k_j} := \frac{1}{t_{k_j}} > 0 \) and

\[
w_{k_j}(x, t) := \int_{-\infty}^{\infty} \phi_{\epsilon_{k_j}}(x - y)\bar{w}(y, t) \, dy
\]
Long time asymptotics for forced curvature flow

where \( \phi(t) = \frac{1}{t} \phi \left( \frac{t}{x} \right) \) and \( \phi \) is the usual mollifier. Clearly \( w_k \to \tilde{w}, j \to \infty \) uniformly on compact subsets of \( \mathbb{R} \times [0, T] \) and

\[
\sup_{x \in \mathbb{R} \times [0, T]} \left( |w_{k,x}| + |w_{k,t}| + \frac{1}{t_k} |w_{k,xx}| \right) \leq C \quad \text{uniformly in } j. \tag{4.30}
\]

Thus, \( z_k := v_k - w_k \) satisfies

\[
z_{k,t} - \frac{1}{t_k} z_{k,xx} = f_k := -\sqrt{1 + (v_{k,x})^2 - w_{k,t} + \frac{1}{t_k} (v_{k,xx})^2}.
\]

In view of (4.26) and (4.30) \( f_k \) can be estimated by

\[
\sup_{x \in \mathbb{R} \times [0, T]} |f_k| \leq \sqrt{1 + M^2 + C}.
\]

Furthermore, a similar computation as above shows that \( z_k(x,0) \leq C \) uniformly in \( j \in \mathbb{N} \). As \( z_k(x,t) \leq M(T)(1 + |x|) \) theorem 2.5, chapter II in [15] implies

\[
\sup_{x \in \mathbb{R} \times [0, T]} |z_k| \leq \sup_{x \in \mathbb{R}} |z_k(.,0)| + T \sup_{x \in \mathbb{R} \times (0,T)} |f_k| \leq C(1 + T).
\]

Sending \( j \to \infty \) we obtain

\[
|\tilde{v}(x,t) - \tilde{w}(x,t)| \leq C(T) \quad x \in \mathbb{R}, \ 0 \leq t \leq T. \tag{4.31}
\]

Next, let us fix a smooth function \( z \in C^2(\mathbb{R}) \) which satisfies \( z(x) = m|x| \) for \( |x| \geq 1 \). Using (4.31) and the explicit form of \( \tilde{w} \) it is not hard to see that

\[
|\tilde{v}(x,t) - z(x)| + |\tilde{w}(x,t) - z(x)| \leq C(T) \quad x \in \mathbb{R}, \ 0 \leq t \leq T.
\]

The functions \( \tilde{v}(x,t) := \tilde{v}(x,t) - z(x) \) and \( \tilde{w}(x,t) := \tilde{w}(x,t) - z(x) \) are then viscosity solutions of

\[
\phi_x + H(x, \phi_x) = 0 \quad \text{in } \mathbb{R} \times (0, T),
\]

\[
\phi(0,0) = m|x| - z(x) \quad \text{in } \mathbb{R}
\]

where \( H(x, p) := \sqrt{1 + (p + z_x(x))^2} \), \( x \in \mathbb{R}, p \in \mathbb{R} \). In addition \( \tilde{v}, \tilde{w} \in BUC(\mathbb{R} \times [0, T]) \), the space of bounded, uniformly continuous functions on \( \mathbb{R} \times [0, T] \). Note that the uniform continuity of \( \tilde{v} \) is a consequence of the bounds on the derivatives (4.26). A careful inspection of the proof of theorem 4.1 in [8] (which implies uniqueness for the case \( \phi_x + H(\phi_x) = 0 \)) shows that the assertion remains valid for functions \( H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) which are uniformly continuous on \( \mathbb{R}^2 \). Thus, we can conclude \( \tilde{v} = \tilde{w} \) and therefore \( \tilde{v} = \tilde{w} \) in \( \mathbb{R} \times (0, T) \). Since the limit is unique we must have for the whole sequence

\[
v_k \to \tilde{w}, k \to \infty \quad \text{locally uniformly in } \mathbb{R} \times (0, \infty).
\]

Specifying \( t = 1 \) we obtain in particular

\[
\frac{1}{t_k} v(t_k x, t_k) \to \tilde{w}(x, 1) \quad k \to \infty \quad \text{on compact subsets of } \mathbb{R}.
\]

Since the sequence \( (t_k)_{k \in \mathbb{N}} \) was arbitrary we have shown that

\[
\frac{1}{t} v(t x, t) \to \tilde{w}(x, 1) = Q(x) \quad t \to \infty \quad \text{on compact subsets of } \mathbb{R}.
\]

Replacing \( x \) by \( \frac{x}{t} \) this can be rewritten as

\[
\frac{1}{t} \left| v(x, t) - tQ \left( \frac{x}{t} \right) \right| \to 0 \quad t \to \infty
\]

uniformly on \( |x| \leq Rt \) which completes the proof of theorem 3.
5. Numerical solution

In attempting to derive a numerical algorithm to simulate the evolution of a vortex line with the law of motion given by (2.10) we are faced with a choice between two methods of solution. For example, we can choose to start with the formulation for the graph (3.4)–(3.6) or the parametric formulation given in (3.1)–(3.3). We shall adopt the latter approach simply because it allows us to study a greater class of initial data.

We follow Dziuk [10], who studied the curve shortening flow $v = kn$, and write down a finite element formulation for the parametric formulation (3.1)–(3.3). However, before doing so some thought must be given to the approximation of the far-field boundary conditions on a finite domain. Noting that the velocity given by this parametrization is normal to the
Figure 5. Evolution of vortex curves with various initial conditions and with forcing $J = -1$.  
(a) The short-time evolution of a curve with graph-like initial data for times $t = 0, 1, 2, \ldots, 10$.  
(b) The evolution of the same initial data for longer times $t = 10, 20, 30, \ldots, 60$.  
(c) The short-time evolution of a curve with non-graph-like initial data into a graph; plotted at times $0, 0.3, 0.6, 0.9, 1.2$.  
(d) The short-time evolution of a curve with non-graph-like initial data for which intersection with the boundary occurs rather than evolution to a travelling wave; plotted at times $t = 0, 0.3, 0.6, 0.9, 1.2$.

A weak formulation can then be written down for the finite domain approximation in terms of test functions $\phi$ and $\eta$,

$$\int_{0}^{s_f} \phi u_s \sqrt{u_s^2 + v_s^2} + \frac{\phi u_s}{\sqrt{u_s^2 + v_s^2}} \, ds = -J \int_{0}^{s_f} \phi v_s \, ds$$
\[
\int_0^{s_f} \left( \eta v_1 \sqrt{u_x^2 + v_x^2} + \frac{\eta v_1}{\sqrt{u_x^2 + v_x^2}} \right) ds - \left( \frac{\eta v_1}{\sqrt{u_x^2 + v_x^2}} \right) \bigg|_{x=s_f} = J \int_0^{s_f} \eta u_x ds
\]

where \( \phi(0) = 0 \) and \( \phi(s_f) = 0 \). Consider approximating \( u \) and \( v \) by the piecewise linear functions \( u_h \) and \( v_h \), such that

\[
u_h = \sum_{j=1}^{n-1} U_j(t) \eta_j(s) \quad \alpha_h = \sum_{j=1}^n V_j(t) \eta_j(s)
\]

and the functions \( \eta_j(s) \) are piecewise linear basis functions defined on a uniform grid \( s_j = jh \) \( (j = 0, \ldots, n) \). Substituting \( u_h \) and \( v_h \) into the weak formulation results in the following finite difference scheme, where \( g_i = \sqrt{(U_i - U_{i-1})^2 + (V_i - V_{i-1})^2} \) is the discrete element of arclength:

\[
\begin{align*}
\frac{U_{i+1}g_{i+1}}{6} + U_i \left( \frac{g_i + g_{i+1}}{3} \right) + \frac{U_{i-1}g_i}{6} + \frac{U_i - U_{i-1}}{g_i} + \frac{U_j - U_{j+1}}{g_{j+1}} &= -J \left( \frac{U_{i+1}}{2} - \frac{U_{i-1}}{2} \right) \\
\frac{V_{i+1}g_{i+1}}{6} + V_i \left( \frac{g_i + g_{i+1}}{3} \right) + \frac{V_{i-1}g_i}{6} + \frac{V_i - V_{i-1}}{g_i} + \frac{V_j - V_{j+1}}{g_{j+1}} &= J \left( \frac{U_{i+1}}{2} - \frac{U_{i-1}}{2} \right)
\end{align*}
\]

for interior points \( i = 1, \ldots, n - 1 \) while for the boundary points

\[
\begin{align*}
\frac{V_0g_1}{3} + \frac{V_1g_1}{6} + \frac{V_0 - V_1}{g_1} &= \frac{J}{2} (U_1 - U_0) \\
U_0 &= 0 \\
\frac{V_ng_n}{3} + \frac{V_{n-1}g_n}{6} + \frac{V_n - V_{n-1}}{g_n} - \frac{m}{\sqrt{m^2 + 1}} &= \frac{J}{2} (U_n - U_{n-1}) \\
U_n &= c - mV_n.
\end{align*}
\]

We use a time discretization based on the backward Euler scheme, evaluating \( U_i \) and \( V_i \), where they appear on the left-hand side of the equations above, at the new time level and evaluating all other terms including \( g_i \) at the old time level. This results in a simple semi-implicit scheme. The results of our computations are shown in figures 4 and 5.

**Practical issues.** Unlike the curvature flow considered in [10] there is no guarantee that nodes do not cluster around some points on the curve and become sparse around others such that \( g_i \) becomes very small and very large respectively. To make an accurate calculation using this method one must therefore reparametrize the curve at appropriate times during the computation. This can be achieved by measuring the length of the piecewise linear curve given by the old parametrization and then placing new nodes on the curve an equal distance apart. Alternatively these new nodes may be placed according to some criterion based on the local curvature along the length of the curve (more nodes are required where the curvature is high). We adopted both approaches, choosing an inter-node distance inversely proportional to \( \sqrt{K} \) for the latter, and found, provided the number of nodes was sufficiently large, that the results of the computations were extremely close.

Another problem that arises when calculating solutions to the problem with \( J > 0 \) and for large time is that the curve moves into part of the domain where the artificial surface, on which far-field data is posed, lies close to the boundary. This is overcome by extending the curve linearly back out to a new artificial surface \( F_c \) lying further from the boundary (i.e. \( c \) is increased). This is done before parts of the curve close to the artificial surface are affected by the presence of the boundary (i.e. the curvature becomes non-negligible).
6. Conclusion

We have examined the evolution of a semi-infinite curve lying in the $x$–$y$ plane evolving with the following velocity law:

$$v = (0, 0, J) \wedge \tau + kn,$$

where $J$ is a constant. Furthermore, the curve we have considered ends on the planar boundary $x = 0$, which it intersects normally, and has a specified gradient as $x \to \infty$. We found that where the initial configuration was a graph in $x$ the curve either evolves towards a growing self-similar solution or a travelling wave depending on whether the velocity of the curve at infinity has a component in the positive $x$-direction or the negative $x$-direction. Where the initial configuration is not a graph the numerical solutions we obtained indicate a greater range of possibilities: either the curves can evolve as for the graph, reconnect with the boundary, or self-intersect.

We can apply the results for the graph to the two-dimensional mean-field model in which

$$\omega = (\omega_1(x, y), \omega_2(x, y), 0), \quad h = (h_1(x, y), h_2(x, y), 0), \quad j = (0, 0, j(x, y)).$$

Doing so we find that the appropriate boundary conditions on the vorticity $\omega$, on sections of the boundary where $\omega \cdot \hat{n}|_{\partial \Omega} \neq 0$, may be expressed in the form

$$v \cdot \hat{n}|_{\partial \Omega} \geq 0 \quad \left( v = (\nabla \wedge h) \wedge \frac{\omega}{|\omega|} \right).$$

Thus, either $v \cdot \hat{n}|_{\partial \Omega} > 0$ and no condition need be applied on the vorticity or $v \cdot \hat{n}|_{\partial \Omega} = 0$ in which case either $v = 0$ or $\omega \wedge \hat{n}|_{\partial \Omega} = 0$. We note that where the sign of the current is reversed on the boundary that a rarefaction in $\omega$ occurs whose short-time behaviour is given by the self-similar solution (3.13) obtained to the intermediate problem (3.12).

Acknowledgments

The authors would like to thank Dr B Stoth and Dr P Bates for their helpful comments during the preparation of this manuscript. KD was supported by a research grant of the German Science Foundation (DFG). GR was supported by a CMAIA Research Fellowship.

References

[16] Pearl J 1966 Structure of superconductive vortices near a metal air interface *J. Appl. Phys.* **37** 4139