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ANALYSIS OF THE TV REGULARIZATION AND H^{-1} FIDELITY MODEL FOR DECOMPOSING AN IMAGE INTO CARTOON PLUS TEXTURE

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Abstract. We study the Osher-Solé-Vese model [11], which is the gradient flow of an energy consisting of the total variation functional plus an H^{-1} fidelity term. A variational inequality weak formulation for this problem is proposed along the lines of that of Feng and Prohl [7] for the Rudin-Osher-Fatemi model [12]. A regularized energy is considered, and the minimization problems corresponding to both the original and regularized energies are shown to be well-posed. The Galerkin method of Lions [9] is used to prove the well-posedness of the weak problem corresponding to the regularized energy. By letting the regularization parameter ϵ tend to 0, we recover the well-posedness of the weak problem corresponding to the original energy. Further, we show that for both energies the solution of the weak problem tends to the minimizer of the energy as $t \to \infty$. Finally, we find the rate of convergence of the weak solution of the regularized problem to that of the original one as $\epsilon \downarrow 0$.

1. Introduction.

Suppose that a gray-scale image f (i.e. a function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ for some bounded open domain Ω, where f measures gray-scale intensity) has been formed by adding Gaussian noise n of known standard deviation σ to a "clean" image g:

$$f = g + n.$$

Clearly, without explicit knowledge of n the recovery of g from f is not possible.

One approach is to apply a "cartoon plus texture" model which splits f into two parts u and v:

$$f = u + v,$$

where u consists of the objects present in g (the "cartoon" part of g) and v consists of the small scale oscillations present in f (n plus the texture in g). The aim is to recover the "cartoon" part.

To this end, Osher, Solé and Vese (OSV) [11] have proposed the minimization problem

$$\inf_{u \in BV(\Omega) \cap \mathcal{F}} J_\lambda (u) , \qquad J_\lambda (u) := \int_\Omega |\nabla u| + \frac{\lambda}{2} \|f - u\|_{-1}^2 ,$$

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where the BV semi-norm $\int_\Omega |\nabla u|$ is a regularising term to remove the texture, $\lambda > 0$ is a weighting parameter and the H^{-1} norm $\|f-u\|_{-1}$ is a fidelity term. Here

$$\int_\Omega |\nabla u| := \sup_{v\in X}\int_\Omega u\nabla\cdot v d\boldsymbol{x},$$

with

$$X := \left\{v=(v_1,\ldots,v_d)\in\left[C_0^1(\Omega)\right]^d : \|v_i\|_{L^\infty(\Omega)}\le 1\ \forall i=1,\ldots,d\right\}.$$

The following function spaces are used:

$$\mathcal{V} := \left\{\eta\in H^1(\Omega) : (\eta,1)=0\right\},\qquad \mathcal{F} := \left\{\eta\in\left(H^1(\Omega)\right)' : \langle\eta,1\rangle=0\right\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\left(H^1(\Omega)\right)'$ and $H^1(\Omega)$ such that

$$\langle\eta,\xi\rangle = \int_\Omega \eta\,\xi\,d\boldsymbol{x}\quad \forall\eta\in L^{\frac{6}{5}}(\Omega),\xi\in H^1(\Omega),\quad d=2,3,$$

the right-hand side being well-defined due to the continuous embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ for $d=2,3$.

By $\mathcal{G}:\mathcal{F}\to\mathcal{V}$ is denoted (minus) the inverse Laplacian operator under Neumann boundary conditions:

$$(\nabla\mathcal{G}\eta,\nabla\xi) = \langle\eta,\xi\rangle\quad\forall\xi\in H^1(\Omega),$$

and $\mathcal{F}$ is equipped with the norm

$$\|\eta\|_{-1} := \|\nabla\mathcal{G}\eta\|\quad\forall\eta\in\mathcal{F}.$$

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the usual norm and inner product on $L^2(\Omega)$.

The Euler-Lagrange equation for formally minimizing $J_\lambda(\cdot)$ is equivalent to

$$0 = -\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)+\lambda\mathcal{G}(u-f)\ \text{in}\ \Omega,\qquad \frac{\partial u}{\partial\nu}=0\quad\text{on}\quad\partial\Omega. \tag{1}$$

Observe that a solution u of equation (1) is a steady state of the evolutionary equations

$$u_t = \nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)-\lambda\mathcal{G}(u-f)\ \text{in}\ \Omega,\qquad \frac{\partial u}{\partial\nu}=0\ \text{on}\ \partial\Omega; \tag{2}$$

$$\mathcal{G}u_t = \nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)-\lambda\mathcal{G}(u-f)\ \text{in}\ \Omega,\quad \frac{\partial u}{\partial\nu}=\frac{\partial}{\partial\nu}\left(\nabla\cdot\left(\frac{\nabla u}{|\nabla u|}\right)\right)=0\ \text{on}\ \partial\Omega. \tag{3}$$

Equations (2) and (3) may be viewed as the L^2 and H^{-1} gradient flows for $J_\lambda(\cdot)$:

$$(2):\frac{d}{dt}J_\lambda(u(t)) = \left(\frac{\nabla u(t)}{|\nabla u(t)|},\nabla u'(t)\right)+\lambda\left(\mathcal{G}[u(t)-f],u'(t)\right) = -\|u'(t)\|^2\le 0,$$

$$(3):\frac{d}{dt}J_\lambda(u(t)) = \left(\frac{\nabla u(t)}{|\nabla u(t)|},\nabla u'(t)\right)+\lambda\left(\mathcal{G}[u(t)-f],u'(t)\right) = -\|u'(t)\|_{-1}^2\le 0.$$

However, note that these are formal calculations because $\frac{\nabla u}{|\nabla u|}$ is not defined when $\nabla u = 0$.

Instead of solving the fourth order equation (3) directly, we introduce a splitting into two coupled second order equations (c.f. [4]):

$$\mathcal{G}u_t = -w - \lambda \mathcal{G}(u-f) \quad \text{in} \quad \Omega, \tag{4}$$

$$w = -\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \quad \text{in} \quad \Omega, \tag{5}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega; \tag{6}$$

Lemma 1.1. *The equations* (4), (5), (6) *are equivalent to the OSV partial differential equation (PDE) (*[11]*):*

$$u_t = \triangle w - \lambda (u-f) \quad \textit{in} \quad \Omega, \tag{7}$$

$$w = -\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) \quad \textit{in} \quad \Omega, \tag{8}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \quad \textit{on} \quad \partial\Omega. \tag{9}$$

Proof. The two problems (4), (5), (6) and (7), (8), (9) have, respectively, the formal variational formulations: for a.e. $t \in (0,T]$,

$$(\mathcal{G}u'(t), \eta) + \left(\frac{\nabla u(t)}{|\nabla u(t)|}, \nabla\eta \right) = -\lambda (\mathcal{G}[u(t)-f], \eta) \qquad \forall \eta \in H^1(\Omega), \tag{10}$$

and

$$\langle u'(t), \eta \rangle + (\nabla w(t), \nabla\eta) = -\lambda (u(t), \eta) + \lambda \langle f, \eta \rangle \qquad \forall \eta \in H^1(\Omega), \tag{11}$$

$$(w(t), \eta) = \left(\frac{\nabla u(t)}{|\nabla u(t)|}, \nabla\eta \right) \qquad \forall \eta \in H^1(\Omega). \tag{12}$$

Defining $w(t) = -\mathcal{G}u'(t) - \lambda\mathcal{G}[u(t)-f]$ in equation (10) gives equation (12) and that

$$\begin{aligned} (\nabla w(t), \nabla\eta) &= -(\nabla\mathcal{G}u'(t), \nabla\eta) - \lambda (\nabla\mathcal{G}[u(t)-f], \nabla\eta) \\ &= -\langle u'(t), \eta \rangle - \lambda (u(t), \eta) + \lambda \langle f, \eta \rangle, \end{aligned}$$

and hence equation (11) holds.

Assuming that $f, u(0) \in \mathcal{F}$, letting $\eta = 1$ in equation (11) gives that $u(t), u'(t) \in \mathcal{F}$, and hence equation (11) gives that for a.e. $t \in (0,T]$,

$$(\nabla\mathcal{G}u'(t), \nabla\eta) + (\nabla w(t), \nabla\eta) = -\lambda (\nabla\mathcal{G}[u(t)-f], \nabla\eta) \quad \forall \eta \in H^1(\Omega).$$

It follows that $w(t) = -\mathcal{G}u'(t) - \lambda\mathcal{G}[u(t)-f]$. Substituting this into equation (12) gives equation (10). □

Since $\frac{\nabla u}{|\nabla u|}$ is not defined when $\nabla u = 0$, we introduce standrad (e.g. Nashed-Scherzer [10]) regularized version $J_{\lambda,\epsilon}(\cdot)$ of the energy functional $J_\lambda(\cdot)$:

$$J_{\lambda,\epsilon}(u) := \int_\Omega |\nabla u|_\epsilon \, d\boldsymbol{x} + \frac{\lambda}{2} \|f-u\|_{-1}^2,$$

where $\epsilon > 0$ is a small regularization parameter and

$$|\boldsymbol{p}|_\epsilon := \sqrt{|\boldsymbol{p}|^2 + \epsilon^2} = \sqrt{p_1^2 + \ldots + p_n^2 + \epsilon^2} \quad \text{for} \quad \boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{R}^d.$$

It is convenient to note the following elementary algebraic inequality:

Lemma 1.2. *For $\epsilon > 0$,*

$$- |\boldsymbol{p} - \boldsymbol{q}| \le \frac{(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{q}}{|\boldsymbol{q}|_\epsilon} \le |\boldsymbol{p}|_\epsilon - |\boldsymbol{q}|_\epsilon \le \frac{(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{p}}{|\boldsymbol{p}|_\epsilon} \le |\boldsymbol{p} - \boldsymbol{q}| \quad \forall \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^d. \quad (13)$$

Proof. The first and second inequalities follow from the fourth and third ones on interchanging the rôles of $\boldsymbol{p}$ and $\boldsymbol{q}$. Using

$$\begin{aligned} & |\boldsymbol{p}|^2 - |\boldsymbol{p}| |\boldsymbol{q}| - (|\boldsymbol{p}|_\epsilon - |\boldsymbol{q}|_\epsilon) |\boldsymbol{p}|_\epsilon \\ = \ & \sqrt{|\boldsymbol{p}|^2 |\boldsymbol{q}|^2 + \epsilon^2 \left(|\boldsymbol{p}|^2 + |\boldsymbol{q}|^2\right) + \epsilon^4} - \sqrt{|\boldsymbol{p}|^2 |\boldsymbol{q}|^2 + 2\epsilon^2 |\boldsymbol{p}| |\boldsymbol{q}| + \epsilon^4} \end{aligned}$$

and the inequality $\epsilon^2 \left(|\boldsymbol{p}|^2 + |\boldsymbol{q}|^2\right) \ge 2\epsilon^2 |\boldsymbol{p}| |\boldsymbol{q}|$ gives the third inequality. The fourth inequality follows from using the inequality $|\cdot| \le |\cdot|_\epsilon$. □

The inequalities (13) are a natural extension of the trivial inequalities

$$- |\boldsymbol{p} - \boldsymbol{q}| \le \frac{(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{q}}{|\boldsymbol{q}|} \le |\boldsymbol{p}| - |\boldsymbol{q}| \le \frac{(\boldsymbol{p} - \boldsymbol{q}) \cdot \boldsymbol{p}}{|\boldsymbol{p}|} \le |\boldsymbol{p} - \boldsymbol{q}| \quad \forall \boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^d. \quad (14)$$

We will make use of the Poincaré inequality

$$\|\eta\| \le C_P \left(|(\eta, 1)| + \|\nabla \eta\|\right) \quad \forall \eta \in H^1(\Omega). \quad (15)$$

It is easy to show that $\|\cdot\|_{-1}$ and $\|\cdot\|_{(H^1(\Omega))'}$ are equivalent norms on $\mathcal{F}$ with

$$\frac{1}{C_P + 1} \|\eta\|_{-1} \le \|\eta\|_{(H^1(\Omega))'} \le \|\eta\|_{-1} \quad \forall \eta \in \mathcal{F}, \quad (16)$$

and

$$\|\eta\|_{-1} \le C_P \|\eta\| \quad \forall \eta \in \mathcal{F} \cap L^2(\Omega), \quad (17)$$

and that for $d = 2, 3$, there exists $C \equiv C(\Omega)$ such that

$$\|\eta\|_{-1} \le C \|\eta\|_{\frac{6}{5}} \quad \forall \eta \in \mathcal{F} \cap L^{\frac{6}{5}}(\Omega). \quad (18)$$

The Poincaré-Wirtinger inequality ([2], p. 148)

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u d\boldsymbol{x} \right\|_1 \le C \int_\Omega |\nabla u| \quad \forall u \in BV(\Omega),$$

with $C \equiv C(\Omega)$, gives that

$$\|u\|_{BV(\Omega)} \le (C + 1) \int_\Omega |\nabla u| \quad \forall u \in BV(\Omega) \cap \mathcal{F}. \quad (19)$$

It is easy to see that

Lemma 1.3. *For $v \in L^2(0, T; \mathcal{F})$, the map $F_v : L^2(0, T; \mathcal{F}) \to \mathbb{R}$ defined by*

$$F_v(\eta) = \int_0^T \langle v(t), \mathcal{G}\eta(t) \rangle \, dt \quad \forall \eta \in L^2(0, T; \mathcal{F})$$

satisfies $F_v \in \left(L^2(0, T; \mathcal{F})\right)'$.

This paper is organized as follows. In Section 2, the initial boundary value problems for the H^{-1} gradient flows for $J_\lambda(\cdot)$ and $J_{\lambda,\epsilon}(\cdot)$ are formulated. A notion of weak solution is introduced for each problem. In Section 3, the minimization problems for the two energies are shown to be well-posed. Well-posedness of the weak formulations is established in Section 4. The convergence of the weak solution of each problem to the minimizer of the corresponding energy as $t \to \infty$ is established in Section 5. In Section 6, a rate of convergence of the sequence of weak solutions of the H^{-1} gradient flow for $J_{\lambda,\epsilon}(\cdot)$ to the weak solution of the H^{-1} gradient flow for $J_\lambda(\cdot)$ as the regularization parameter $\epsilon \downarrow 0$ is established.

Our approach is similar to that of Feng, van Oehsen and Prohl in [6] and Feng and Prohl [7] for the second order Rudin-Osher-Fatemi model [12]. However, we use the Faedo-Galerkin method of Lions [9] to prove the existence of a weak solution for $\epsilon > 0$ in place of the maximal monotone operator approach of Feng and Prohl. See also [5] for a related Cahn-Hilliard model.

2. Mathematical formulations of initial boundary value problems, definitions of weak solutions.

The problems considered are the OSV initial boundary value problem and the analogous problem for $J_{\lambda,\epsilon}(\cdot)$ ($\epsilon > 0$), denoted by **($\mathbf{P}^\epsilon$)**:

(P) given $T > 0$, find $u(\boldsymbol{x},t), w(\boldsymbol{x},t) : \Omega_T := \Omega \times (0,T] \to \mathbb{R}$ such that

$$u_t(\boldsymbol{x},t) - \triangle w(\boldsymbol{x},t) = -\lambda(u(\boldsymbol{x},t) - f(\boldsymbol{x})) \qquad \forall(\boldsymbol{x},t) \in \Omega_T, \tag{20}$$

$$w(\boldsymbol{x},t) = -\nabla\cdot\left(\frac{\nabla u(\boldsymbol{x},t)}{|\nabla u(\boldsymbol{x},t)|}\right) \qquad \forall(\boldsymbol{x},t) \in \Omega_T, \tag{21}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega, \tag{22}$$

$$\frac{\partial u}{\partial \nu}(\boldsymbol{x},t) = \frac{\partial w}{\partial \nu}(\boldsymbol{x},t) = 0 \qquad \forall(\boldsymbol{x},t) \in \partial\Omega_T; \tag{23}$$

($\mathbf{P}^\epsilon$) given $T > 0$, find $u_\epsilon(\boldsymbol{x},t), w_\epsilon(\boldsymbol{x},t) : \Omega_T \to \mathbb{R}$ such that

$$u_{\epsilon,t}(\boldsymbol{x},t) - \triangle w_\epsilon(\boldsymbol{x},t) = -\lambda(u_\epsilon(\boldsymbol{x},t) - f(\boldsymbol{x})) \qquad \forall(\boldsymbol{x},t) \in \Omega_T, \tag{24}$$

$$w_\epsilon(\boldsymbol{x},t) = -\nabla\cdot\left(\frac{\nabla u_\epsilon(\boldsymbol{x},t)}{|\nabla u_\epsilon(\boldsymbol{x},t)|_\epsilon}\right) \qquad \forall(\boldsymbol{x},t) \in \Omega_T, \tag{25}$$

$$u_\epsilon(\boldsymbol{x},0) = u_{0,\epsilon}(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega, \tag{26}$$

$$\frac{\partial u_\epsilon}{\partial \nu}(\boldsymbol{x},t) = \frac{\partial w_\epsilon}{\partial \nu}(\boldsymbol{x},t) = 0 \qquad \forall(\boldsymbol{x},t) \in \partial\Omega_T; \tag{27}$$

where $\partial\Omega_T := \partial\Omega \times (0,T]$.

Since the expression $\frac{\nabla u}{|\nabla u|}$ is not defined when $\nabla u = \mathbf{0}$, the PDEs (20) and (21) are only formal statements. In order to give a rigorous definition of solution, convex analysis and variational inequalities are used.

Remark 1. The natural image processing assumptions, that $u_0 = f$ and $u_{0,\epsilon} = f$, are not made here. This allows for a more general analysis of **(P)** and **($\mathbf{P}^\epsilon$)**, in particular under different regularity assumptions on u_0, $u_{0,\epsilon}$ and f.

It follows from equation (3), $(a-b)(c-a) \leq \frac{1}{2}[(c-b)^2 - (a-b)^2]$ and equation (14) that

$$(\mathcal{G}u_t, v-u) + J_\lambda(v) - J_\lambda(u) \geq 0,$$

for all suitably smooth test functions v, and similarly Lemma 1.2 gives that

$$(\mathcal{G}u_{\epsilon,t}, v-u_\epsilon) + J_{\lambda,\epsilon}(v) - J_{\lambda,\epsilon}(u_\epsilon) \geq 0.$$

These last inequalities motivate the following definitions of weak solutions of **(P)**, **($\mathbf{P}^\epsilon$)**:

Definition 2.1. Let $\Omega \subset \mathbb{R}^d$ ($2 \leq d \leq 3$) be a bounded open domain with Lipschitz boundary $\partial\Omega$ and suppose that u_0, $u_{0,\epsilon} \in BV(\Omega) \cap \mathcal{F}$ and $f \in \mathcal{F}$. Then

- u is said to be a weak solution of the initial boundary value problem **(P)** if $u \in C(0,T;\mathcal{F}) \cap L^\infty(0,T;BV(\Omega)) \cap H^1(0,T;\mathcal{F})$, $u(0) = u_0$ a.e. and u satisfies for any $s \in [0,T]$,

$$\int_0^s \langle u'(t), \mathcal{G}[v(t) - u(t)] \rangle \, dt + \int_0^s [J_\lambda(v(t)) - J_\lambda(u(t))] \, dt \geq 0$$
$$\forall\, v \in L^1(0,T;BV(\Omega)) \cap L^2(0,T;\mathcal{F}); \quad (28)$$

- u_ϵ is said to be a weak solution of the initial boundary value problem **($\mathbf{P}^\epsilon$)**if $u_\epsilon \in C(0,T;\mathcal{F}) \cap L^\infty(0,T;BV(\Omega)) \cap H^1(0,T;\mathcal{F})$, $u_\epsilon(0) = u_{0,\epsilon}$ a.e. and u_ϵ satisfies for any $s \in [0,T]$,

$$\int_0^s \langle u_\epsilon'(t), \mathcal{G}[v(t) - u_\epsilon(t)] \rangle \, dt + \int_0^s [J_{\lambda,\epsilon}(v(t)) - J_{\lambda,\epsilon}(u_\epsilon(t))] \, dt \geq 0$$
$$\forall\, v \in L^1(0,T;BV(\Omega)) \cap L^2(0,T;\mathcal{F}). \quad (29)$$

Note that, since

$$\langle v'(t) - u'(t), \mathcal{G}[v(t) - u(t)] \rangle = \frac{1}{2}\frac{d}{dt}\|v(t) - u(t)\|_{-1}^2,$$

and similarly for $v(t) - u_\epsilon(t)$, the inequalities (28), (29) are equivalent to

$$\begin{aligned} &\int_0^s \langle v'(t), \mathcal{G}[v(t) - u(t)] \rangle \, dt + \int_0^s [J_\lambda(v(t)) - J_\lambda(u(t))] \, dt \\ \geq\ & \frac{1}{2}\left[\|v(s) - u(s)\|_{-1}^2 - \|v(0) - u_0\|_{-1}^2\right] \qquad (30) \\ & \forall\, v \in L^1(0,T;BV(\Omega)) \cap C(0,T;\mathcal{F}) \ :\ v' \in L^2(0,T;\mathcal{F}), \end{aligned}$$

$$\begin{aligned} &\int_0^s \langle v'(t), \mathcal{G}[v(t) - u_\epsilon(t)] \rangle \, dt + \int_0^s [J_{\lambda,\epsilon}(v(t)) - J_{\lambda,\epsilon}(u_\epsilon(t))] \, dt \\ \geq\ & \frac{1}{2}\left[\|v(s) - u_\epsilon(s)\|_{-1}^2 - \|v(0) - u_{0,\epsilon}\|_{-1}^2\right] \qquad (31) \\ & \forall\, v \in L^1(0,T;BV(\Omega)) \cap C(0,T;\mathcal{F}) \ :\ v' \in L^2(0,T;\mathcal{F}) \qquad (32) \end{aligned}$$

respectively.

3. Well-posedness of the energy minimization problems.

Theorem 3.1. *[$d = 2,3$] Suppose that $\lambda > 0$ and $f \in \mathcal{F}$. Then*

- *the minimization problem*

$$\inf_{u \in BV(\Omega) \cap \mathcal{F}} J_\lambda(u) \quad (33)$$

  has a unique solution;
- *for each $\epsilon \geq 0$, the minimization problem*

$$\inf_{u_\epsilon \in BV(\Omega) \cap \mathcal{F}} J_{\lambda,\epsilon}(u_\epsilon) \quad (34)$$

  has a unique solution.

Proof. The result for $J_{\lambda,\epsilon}(\cdot)$ is proved (the result for $J_\lambda(\cdot)$ can be proved analogously). The argument used is a standard one; see for example [1], [3], [11].

Let $\{u_{\epsilon,n}\}_{n\in\mathbb{N}}$ be a minimizing sequence. There exists a constant $M > 0$ such that

$$\int_{\Omega} |\nabla u_{\epsilon,n}| \leq \int_{\Omega} |\nabla u_{\epsilon,n}|_{\epsilon} \leq J_{\lambda,\epsilon}(u_{\epsilon,n}) \leq M \quad \forall n \in \mathbb{N}.$$

It follows from inequality (19) that

$$\|u_{\epsilon,n}\|_{BV(\Omega)} \leq (C+1)\, M.$$

Also,

$$\|u_{\epsilon,n}\|_{-1}^2 \leq 2\,\|u_{\epsilon,n} - f\|_{-1}^2 + 2\,\|f\|_{-1}^2 \leq \frac{4}{\lambda} J_{\lambda,\epsilon}(u_{\epsilon,n}) + 2\,\|f\|_{-1}^2 \leq \frac{4M}{\lambda} + 2\,\|f\|_{-1}^2 .$$

It follows (see [2], p. 125) that there exists $u_\epsilon \in BV(\Omega) \cap \mathcal{F}$ and a subsequence of $\{u_{\epsilon,n}\}_{n\in\mathbb{N}}$ (still denoted by $\{u_{\epsilon,n}\}_{n\in\mathbb{N}}$) such that

$$u_{\epsilon,n} \underset{BV-w^*}{\overset{*}{\rightharpoonup}} u_\epsilon, \qquad u_{\epsilon,n} \underset{\mathcal{F}}{\rightharpoonup} u_\epsilon, \qquad u_{\epsilon,n} \underset{L^1(\Omega)}{\rightarrow} u_\epsilon \qquad \text{as} \qquad n \rightarrow \infty.$$

Moreover,

$$0 \leq \left| \int_{\Omega} u_\epsilon d\boldsymbol{x} \right| = \left| \int_{\Omega} (u_{\epsilon,n} - u_\epsilon)\, d\boldsymbol{x} \right| \leq \|u_{\epsilon,n} - u_\epsilon\|_1 \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

$$\Rightarrow \quad \int_{\Omega} u_\epsilon d\boldsymbol{x} = 0.$$

Recall that a functional J is said to be "convex" if

$$J(\gamma u_1 + (1-\gamma)\, u_2) \leq \gamma J(u_1) + (1-\gamma)\, J(u_2)$$

whenever $u_1 \neq u_2$ and $\gamma \in (0,1)$, and "strictly convex" if the inequality is strict.

The strict convexity of $J_{\lambda,\epsilon}(\cdot)$ follows from the convexity of $J_{0,\epsilon}(\cdot)$ ([1], Theorem 2.4) and the strict convexity of $\|(\cdot) - f\|_{-1}^2$:

$$\gamma\,\|u_1 - f\|_{-1}^2 + (1-\gamma)\,\|u_2 - f\|_{-1}^2 - \|\gamma u_1 + (1-\gamma)\, u_2 - f\|_{-1}^2 = \gamma\,(1-\gamma)\,\|u_1 - u_2\|_{-1}^2 .$$

Since $J_{\lambda,\epsilon}(\cdot)$ is convex, it is lower semi-continuous in $BV(\Omega)$ with respect to convergence in $L^1(\Omega)$ (see [7]). It follows that

$$J_{\lambda,\epsilon}(u_\epsilon) \leq \liminf_{n\rightarrow\infty} J_{\lambda,\epsilon}(u_{\epsilon,n}),$$

and hence that u_ϵ is a solution of the minimization problem (34).

Suppose that $\tilde{u}_\epsilon \neq u_\epsilon$ is another solution. The strict convexity of $J_{\lambda,\epsilon}(\cdot)$, gives that

$$J_{\lambda,\epsilon}\left(\frac{u_\epsilon + \tilde{u}_\epsilon}{2}\right) < \frac{J_{\lambda,\epsilon}(u_\epsilon) + J_{\lambda,\epsilon}(\tilde{u}_\epsilon)}{2} = \inf_{u_\epsilon \in BV(\Omega)\cap\mathcal{F}} J_{\lambda,\epsilon}(u_\epsilon),$$

a contradiction. Hence the solution of the minimization problem (34) is unique. □

4. **Well-posedness of the weak formulations.**

4.1. **Statement of result.**

Theorem 4.1 (c.f. [7], Theorems 1.1 to 1.4). *Let $\Omega \subset \mathbb{R}^d$ $(2 \leq d \leq 3)$ be a bounded open domain with Lipschitz boundary $\partial\Omega$ and suppose that u_0, $u_{0,\epsilon} \in BV(\Omega) \cap \mathcal{F}$ and $f \in \mathcal{F}$. Then*

- *there exists a unique weak solution u of **(P)**,*
- *there exists a unique weak solution u_ϵ of **($\mathbf{P}^\epsilon$)**.*

Further,

- *if u_i $(i = 1, 2)$ are weak solutions of **(P)** for data $u_{0,i} \in BV(\Omega) \cap \mathcal{F}$ and $f_i \in \mathcal{F}$ then*

$$\|u_2(s) - u_1(s)\|_{-1} \leq \|u_{0,2} - u_{0,1}\|_{-1} + \sqrt{\lambda T}\,\|f_2 - f_1\|_{-1} \quad \forall s \in [0,T]\,; \tag{35}$$

- *if $u_{\epsilon,i}$ $(i = 1, 2)$ are weak solutions of **($\mathbf{P}^\epsilon$)** for data $u_{0,\epsilon,i} \in BV(\Omega) \cap \mathcal{F}$ and $f_i \in \mathcal{F}$ then*

$$\|u_{\epsilon,2}(s) - u_{\epsilon,1}(s)\|_{-1} \leq \|u_{0,\epsilon,2} - u_{0,\epsilon,1}\|_{-1} + \sqrt{\lambda T}\,\|f_2 - f_1\|_{-1} \quad \forall s \in [0,T]\,. \tag{36}$$

4.2. **Overview of proof.** Firstly, the existence of a weak solution of **($\mathbf{P}^\epsilon$)** is established by using the Faedo-Galerkin method of Lions ([9]). This consists of three parts:

> **Proof of local existence and uniqueness** (Sections 4.3, 4.4) A countable orthogonal basis of $\{\eta^i\}_{i\in\mathbb{N}}$ of $\mathcal{V}$ is constructed using eigenfunctions of the Neumann Laplacian. For $k \in \mathbb{N}$, we seek $u_{\epsilon,k}$ in the span of $\{\eta^i\}_{i=1}^k$ which solves the variational PDE with test space spanned by $\{\eta^i\}_{i=1}^k$. We deduce local existence and uniqueness of $u_{\epsilon,k}$.
>
> **Proof of global existence** (Section 4.5) Specific choice(s) of test function(s) yield bounds on relevant norms of $u_{\epsilon,k}$ which are independent of k and remain finite as $\epsilon \downarrow 0$.
>
> **Passage to the limit** (Section 4.6) The bounds on the seequence $\{u_{\epsilon,k}\}_{k\in\mathbb{N}}$ give various convergence results as $k \to \infty$ (the limit is denoted by u_ϵ). These convergence results are used to pass to the limit of each term in a weaker reformulation of the finite dimensional problem (which is analogous to the weaker reformulation (31) of the equations (24) and (25)), yielding that u_ϵ satisfies inequality (31). An argument of Lichnewsky and Temam [8] is used to show that $u_\epsilon(0) = u_{0,\epsilon}$ and $u_\epsilon \in C(0,T;\mathcal{F})$.

Uniqueness of the weak solution of **($\mathbf{P}^\epsilon$)** is an immediate consequence of inequality (36), which is proved via an argument of Lichnewsky and Temam [8] in Section 4.7.

In Section 4.8, the existence of a weak solution of **(P)** is established. This is achieved by using the bounds found in Section 4.5 to give various convergence results for the sequence $\{u_\epsilon\}_{\epsilon>0}$ of weak solutions of **($\mathbf{P}^\epsilon$)** for initial data $u_{0,\epsilon} = u_0$ as $\epsilon \downarrow 0$ (the limit is denoted by u). These convergence results are used to pass to the limit of each term in inequality (31), giving that u satisfies inequality (30).

The proofs that $u(0) = u_0$, that $u \in C(0,T;\mathcal{F})$ and of inequality (35) are analogous to those of the corresponding results for **($\mathbf{P}^\epsilon$)**.

4.3. **Definition of Galerkin problems.** The orthogonal basis $\left\{\eta^i\right\}_{i\in\mathbb{N}} \subset \mathcal{V}$ of $\mathcal{V}$ consisting of the (L^2-normalized) eigenfunctions of the Laplacian operator with zero Neumann boundary conditions together with the corresponding non-decreasing sequence of eigenvalues $\left\{\lambda_i\right\}_{i\in\mathbb{N}}$ is considered:

$$\begin{aligned} &-\triangle\eta^i\left(\boldsymbol{x}\right) = \lambda_i\eta^i\left(\boldsymbol{x}\right)\ \forall\boldsymbol{x}\in\Omega, && \frac{\partial\eta^i}{\partial\nu}\left(\boldsymbol{x}\right) = 0\ \forall\boldsymbol{x}\in\partial\Omega,\\ &\left(\eta^i,\eta^i\right) = 1, && 0<\lambda_1\le\lambda_2\le\dots\ . \end{aligned}$$

It follows that

$$\left(\nabla\eta^i,\nabla\eta\right) = \lambda_i\left(\eta^i,\eta\right)\quad\forall\eta\in\mathcal{V}, i\in\mathbb{N},\qquad \mathcal{G}\eta^i = \frac{1}{\lambda_i}\eta^i\quad\forall i\in\mathbb{N}. \tag{37}$$

For $k\in\mathbb{N}$, V^k is defined to be the finite dimensional subspace of $\mathcal{V}$ spanned by $\left\{\eta^i\right\}_{i=1}^k$.

The kth Galerkin problem is to find $u_{\epsilon,k}\left(t\right), w_{\epsilon,k}\left(t\right)\in V^k$ such that for a.e. $t\in(0,T]$,

$$\left(u_{\epsilon,k}'\left(t\right),\eta_k\right) + \left(\nabla w_{\epsilon,k}\left(t\right),\nabla\eta_k\right) = -\lambda\left(u_{\epsilon,k}\left(t\right),\eta_k\right) + \lambda\left\langle f,\eta_k\right\rangle \qquad \forall\eta_k\in V^k, \tag{38}$$

$$\left(w_{\epsilon,k}\left(t\right),\eta_k\right) = \left(\frac{\nabla u_{\epsilon,k}\left(t\right)}{\left|\nabla u_{\epsilon,k}\left(t\right)\right|_\epsilon},\nabla\eta_k\right) \qquad \forall\eta_k\in V^k, \tag{39}$$

$$u_{\epsilon,k}\left(0\right) = P^k u_{0,\epsilon,k}, \tag{40}$$

where $\left\{u_{0,\epsilon,k}\right\}_{k\in\mathbb{N}}\subset C^\infty\left(\Omega\right)\cap\mathcal{F}$ is chosen such that for each $p\in\left[1,\frac{d}{d-1}\right)$ (see [13], p. 225)

$$\left\|u_{0,\epsilon,k}-u_{0,\epsilon}\right\|_p\to 0\quad\text{and}\quad\int_\Omega\left|\nabla u_{0,\epsilon,k}\right|\to\int_\Omega\left|\nabla u_{0,\epsilon}\right|\quad\text{as}\quad k\to\infty, \tag{41}$$

and $P^k:\mathcal{V}\to V^k$ is taken to be the (Faedo-)Galerkin projection operator:

$$P^k\eta := \sum_{i=1}^k\left(\eta,\eta^i\right)\eta^i\quad\forall\eta\in\mathcal{V}.$$

The operator P^k satisfies

$$\forall\eta\in\mathcal{V},\ \eta_k\in V^k,\qquad \left(P^k\eta,\eta_k\right) = \left(\eta,\eta_k\right),\ \left(\nabla P^k\eta,\nabla\eta_k\right) = \left(\nabla\eta,\nabla\eta_k\right); \tag{42}$$

$$\forall\eta\in\mathcal{V},\qquad P^k\eta\to\eta\quad\text{in}\quad H^1\left(\Omega\right)\quad\text{as}\quad k\to\infty; \tag{43}$$

$$\forall\eta\in\mathcal{V},\qquad \left\|P^k\eta\right\|_{-1}\le\left\|\eta\right\|_{-1}; \tag{44}$$

$$\forall\eta\in C\left(0,T;\mathcal{V}\right),\qquad P^k\eta\to\eta\quad\text{in}\quad C\left(0,T;H^1\left(\Omega\right)\right)\quad\text{as}\quad k\to\infty; \tag{45}$$

$$\forall\eta\in C^1\left(0,T;\mathcal{V}\right),\qquad \frac{\partial}{\partial t}\left(P^k\eta\right) = P^k\left(\frac{\partial\eta}{\partial t}\right). \tag{46}$$

Equations (42), (43), (45) and (46) are standard results, and equation (44) follows from equations (37) and (42):

$$\begin{aligned}\left\|P^k\eta\right\|_{-1}^2 &= \left\|\nabla\mathcal{G}\left[P^k\eta\right]\right\|^2 = \left(P^k\eta,\mathcal{G}\left[P^k\eta\right]\right) = \left(\eta,\mathcal{G}\left[P^k\eta\right]\right) = \left(\nabla\mathcal{G}\eta,\nabla\mathcal{G}\left[P^k\eta\right]\right)\\ &\le\left\|\nabla\mathcal{G}\eta\right\|\left\|\nabla\mathcal{G}\left[P^k\eta\right]\right\| = \left\|\eta\right\|_{-1}\left\|P^k\eta\right\|_{-1}.\end{aligned}$$

Since $H^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^{\frac{d}{d-1}}(\Omega)$ for $d = 2, 3$, equations (41) and (43) give that for $p \in \left[1, \frac{d}{d-1}\right)$,

$$\|u_{\epsilon,k}(0) - u_{0,\epsilon}\|_p \to 0 \quad \text{and} \quad (|\nabla u_{\epsilon,k}(0)|, 1) \to \int_\Omega |\nabla u_{0,\epsilon}| \quad \text{as} \quad k \to \infty. \tag{47}$$

Lemma 4.2. *The equations* (38)*,* (39) *are equivalent to*

$$\left(\mathcal{G}u'_{\epsilon,k}(t), \eta_k\right) = -\left(\frac{\nabla u_{\epsilon,k}(t)}{|\nabla u_{\epsilon,k}(t)|_\epsilon}, \nabla \eta_k\right) - \lambda\left(\mathcal{G}\left[u_{\epsilon,k}(t) - f\right], \eta_k\right)$$
$$\forall \eta_k \in V^k,\ a.e.\ t \in (0, T]. \tag{48}$$

Proof. Defining $w_{\epsilon,k}(t) = -\mathcal{G}u'_{\epsilon,k}(t) - \lambda\mathcal{G}u_{\epsilon,k}(t) + \lambda P^k[\mathcal{G}f]$ in equation (48) and using equation (42) gives equation (39), and a further usage of equation (42) gives

$$\begin{aligned}(\nabla w_{\epsilon,k}(t), \nabla\eta_k) &= -\left(\nabla\mathcal{G}u'_{\epsilon,k}(t), \nabla\eta_k\right) - \lambda\left(\nabla\mathcal{G}\left[u_{\epsilon,k}(t) - f\right], \nabla\eta_k\right)\\ &= -\left(u'_{\epsilon,k}(t), \eta_k\right) - \lambda\left(u_{\epsilon,k}(t), \eta_k\right) + \lambda\langle f, \eta_k\rangle,\end{aligned}$$

and hence equation (38) holds.

Since $u_{\epsilon,k}(t), u'_{\epsilon,k}(t) \in \mathcal{V}$, equation (38) gives that for a.e. $t \in (0, T]$,

$$\left(\nabla\mathcal{G}u'_{\epsilon,k}(t), \nabla\eta_k\right) + (\nabla w_{\epsilon,k}(t), \nabla\eta_k) = -\lambda\left(\nabla\mathcal{G}\left[u_{\epsilon,k}(t) - f\right], \nabla\eta_k\right) \quad \forall\eta_k \in V^k.$$

It follows by equation (42) that $w_{\epsilon,k}(t) = -\mathcal{G}u'_{\epsilon,k}(t) - \lambda\mathcal{G}u_{\epsilon,k}(t) + \lambda P^k[\mathcal{G}f]$. Substituting this into equation (39) and using equation (42) gives equation (48). □

4.4. **Local existence and uniqueness for Galerkin problems.**

Take

$$u_{\epsilon,k}(t) = \sum_{i=1}^k c_{\epsilon,k,i}(t)\,\eta^i \quad \forall k \in \mathbb{N},$$

and define the vectors $\boldsymbol{c}_{\epsilon,k}(t)$, $\boldsymbol{c}_{0,\epsilon,k}$, $\boldsymbol{f}_k$ and $\boldsymbol{\eta}_k$ of length k by

$$[\boldsymbol{c}_{\epsilon,k}(t)]_i = c_{\epsilon,k,i}(t),\ [\boldsymbol{c}_{0,\epsilon,k}]_i = \left(u_{\epsilon,k}(0), \eta^i\right),\ [\boldsymbol{f}_k]_i = \left\langle f, \eta^i\right\rangle,\ [\boldsymbol{\eta}_k]_i = \eta^i \quad \forall i \le k.$$

The non-linear operator $\boldsymbol{A}_{\epsilon,k} : \mathbb{R}^k \to \mathbb{R}^k$ is defined by

$$[\boldsymbol{A}_{\epsilon,k}(\boldsymbol{c})]_i := \lambda_i\left(\frac{\nabla[\boldsymbol{c}\cdot\boldsymbol{\eta}_k]}{|\nabla[\boldsymbol{c}\cdot\boldsymbol{\eta}_k]|_\epsilon}, \nabla\eta^i\right) \quad \forall i \le k.$$

The kth Galerkin problem (48) is equivalent to

$$\boldsymbol{c}'_{\epsilon,k}(t) + \boldsymbol{A}_{\epsilon,k}(\boldsymbol{c}_{\epsilon,k}(t)) = -\lambda\boldsymbol{c}_{\epsilon,k}(t) + \lambda\boldsymbol{f}_k, \qquad \boldsymbol{c}_{\epsilon,k}(0) = \boldsymbol{c}_{0,\epsilon,k}.$$

In order to invoke the standard Picard Theorem and obtain the existence of a unique solution of this problem on some time interval $[0, T_k]$, it is sufficient to show that $\boldsymbol{A}_{\epsilon,k}$ is globally Lipschitz. For a vector $\boldsymbol{c}$ of length k and $i \le k$,

$$[\boldsymbol{A}_{\epsilon,k}(\boldsymbol{c})]_i = \lambda_i \sum_{j=1}^d \left(G_d\left(\frac{\partial[\boldsymbol{c}\cdot\boldsymbol{\eta}_k]}{\partial x_1}, \dots, \frac{\partial[\boldsymbol{c}\cdot\boldsymbol{\eta}_k]}{\partial x_{j-1}}, \frac{\partial[\boldsymbol{c}\cdot\boldsymbol{\eta}_k]}{\partial x_{j+1}}, \dots, \frac{\partial[\boldsymbol{c}\cdot\boldsymbol{\eta}_k]}{\partial x_d}, \frac{\partial[\boldsymbol{c}\cdot\boldsymbol{\eta}_k]}{\partial x_j}\right), \frac{\partial\eta^i}{\partial x_j}\right),$$

where the nonlinear operator $G_d : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$G_d(\boldsymbol{c}) := \frac{c_d}{|\boldsymbol{c}|_\epsilon} \qquad \forall \boldsymbol{c} = (c_1, \dots, c_d) \in \mathbb{R}^d.$$

Since

$$\frac{\partial G_d(\boldsymbol{c})}{\partial c_d} = \frac{|\boldsymbol{c}|_\epsilon^2 - c_d^2}{|\boldsymbol{c}|_\epsilon^3} \qquad \text{and} \qquad \frac{\partial G_d(\boldsymbol{c})}{\partial c_i} = -\frac{c_i c_d}{|\boldsymbol{c}|_\epsilon^3} \quad \forall i = 1, \ldots, d-1,$$

it follows that

$$\left|\frac{\partial G_d(\boldsymbol{c})}{\partial c_i}\right| \le \frac{1}{|\boldsymbol{c}|_\epsilon} \le \frac{1}{\epsilon} \qquad \forall i = 1, \ldots, d.$$

Hence, by Taylor's Theorem, $G_d(\cdot)$ is globally Lipschitz:

$$|G_d(\boldsymbol{c}) - G_d(\tilde{\boldsymbol{c}})| \le \frac{1}{\epsilon}\|\boldsymbol{c} - \tilde{\boldsymbol{c}}\|_1 \quad \forall \boldsymbol{c}, \tilde{\boldsymbol{c}} \in \mathbb{R}^d,$$

where $\|\cdot\|_1$ is the discrete L^1 norm: $\|\boldsymbol{c}\|_1 := |c_1| + \ldots + |c_d|$. It follows that $\boldsymbol{A}_{\epsilon,k}(\cdot)$ is globally Lipschitz:

$$\|\boldsymbol{A}_{\epsilon,k}(\boldsymbol{c}) - \boldsymbol{A}_{\epsilon,k}(\tilde{\boldsymbol{c}})\|_1 \le \frac{kd^2\lambda_k^2}{\epsilon}\|\boldsymbol{c} - \tilde{\boldsymbol{c}}\|_1 \quad \forall \boldsymbol{c}, \tilde{\boldsymbol{c}} \in \mathbb{R}^d.$$

4.5. **Global existence for Galerkin problems.** Since $\frac{d}{d-1} > \frac{6}{5}$ for $d = 2, 3$, inequality (18) and the limit (47) give that

$$\|u_{\epsilon,k}(0) - u_{0,\epsilon}\|_{-1} \to 0 \quad \text{as} \quad k \to \infty. \tag{49}$$

Using that $|\boldsymbol{p}|_\epsilon \le |\boldsymbol{p}| + \epsilon$ and $\frac{1}{2}(a+b)^2 \le a^2 + b^2$ gives

$$\begin{aligned} J_{\lambda,\epsilon}(u_{\epsilon,k}(0)) &= \left(|\nabla u_{\epsilon,k}(0)|_\epsilon, 1\right) + \frac{\lambda}{2}\|f - u_{\epsilon,k}(0)\|_{-1}^2 \\ &\le (|\nabla u_{\epsilon,k}(0)|, 1) + \epsilon|\Omega| + \lambda\|u_{\epsilon,k}(0)\|_{-1}^2 + \lambda\|f\|_{-1}^2. \end{aligned} \tag{50}$$

It follows from the limits (47) and (49) that $\{J_{\lambda,\epsilon}(u_{\epsilon,k}(0))\}_{k\in\mathbb{N}}$ is bounded.

Lemma 4.3. *For $s \in [0, T]$,*

$$\begin{aligned} (i)\quad & \|u_{\epsilon,k}(s)\|_{-1}^2 + 2\int_0^s \left(\frac{\nabla u_{\epsilon,k}(t)}{|\nabla u_{\epsilon,k}(t)|_\epsilon}, \nabla u_{\epsilon,k}(t)\right) dt + \lambda\int_0^s \|u_{\epsilon,k}(t)\|_{-1}^2\, dt \\ & \le \|u_{\epsilon,k}(0)\|_{-1}^2 + \lambda T\|f\|_{-1}^2, \end{aligned} \tag{51}$$

$$(ii)\quad \int_0^s \left\|u'_{\epsilon,k}(t)\right\|_{-1}^2 dt = J_{\lambda,\epsilon}(u_{\epsilon,k}(0)) - J_{\lambda,\epsilon}(u_{\epsilon,k}(s)). \tag{52}$$

Proof. **(i)** Letting $\eta = u_{\epsilon,k}(t)$ in equation (48) gives that

$$\frac{d}{dt}\|u_{\epsilon,k}(t)\|_{-1}^2 + 2\left(\frac{\nabla u_{\epsilon,k}(t)}{|\nabla u_{\epsilon,k}(t)|_\epsilon}, \nabla u_{\epsilon,k}(t)\right) + \lambda\|u_{\epsilon,k}(t)\|_{-1}^2 \le \lambda\|f\|_{-1}^2,$$

and integrating with respect to t from 0 to s gives inequality (51).

(ii) Taking $\eta = u'_{\epsilon,k}(t)$ in equation (48) gives that

$$\left\|u'_{\epsilon,k}(t)\right\|_{-1}^2 = -\frac{d}{dt}\left[J_{\lambda,\epsilon}(u_{\epsilon,k}(t))\right],$$

and integrating with respect to t from 0 to s gives inequality (52). □

It follows from inequalities (50) and (51), limit (49), equation (52) and the non-negativity of $J_{\lambda,\epsilon}(\cdot)$ that there exists $C \equiv C(u_{0,\epsilon}, f, \lambda, T, \epsilon, \Omega)$ such that

$$\|u_{\epsilon,k}\|_{L^\infty(0,T;\mathcal{F})}, \ \left\|u'_{\epsilon,k}\right\|_{L^2(0,T;\mathcal{F})} \le C. \tag{53}$$

It follows from

$$V^k \subset \mathcal{V} \subset H^1(\Omega) \hookrightarrow W^{1,1}(\Omega) \hookrightarrow BV(\Omega) \quad \forall k \in \mathbb{N}$$

that

$$u_{\epsilon,k}(s) \in BV(\Omega) \cap \mathcal{V} \quad \forall k \in \mathbb{N}.$$

Inequality (19) gives that

$$\|u_{\epsilon,k}(s)\|_{BV(\Omega)} \leq (C+1)(|\nabla u_{\epsilon,k}(s)|, 1).$$

Further, inequality (50) and equation (52) give that there exists $C \equiv C(u_{0,\epsilon}, f, \lambda, \epsilon, \Omega)$ such that

$$(|\nabla u_{\epsilon,k}(s)|, 1) \leq J_{\lambda,\epsilon}(u_{\epsilon,k}(s)) \leq J_{\lambda,\epsilon}(u_{\epsilon,k}(0)) \leq C.$$

Hence there exists $C \equiv C(u_{0,\epsilon}, f, \lambda, \epsilon, \Omega)$ such that

$$\|u_{\epsilon,k}\|_{L^\infty(0,T;BV(\Omega))} \leq C. \tag{54}$$

4.6. **Passage to the limit.** Since $(H^1(\Omega))'$ is a Hilbert space and $\mathcal{F}$ is a closed subspace of $(H^1(\Omega))'$, $\mathcal{F}$ is a Hilbert space. It follows that $\mathcal{F}$ is a reflexive Banach space, and hence so too is $L^2(0,s;\mathcal{F})$ for $s \in (0,T]$. Further, $BV(\Omega)$ is the dual of a separable space and hence so too is $L^\infty(0,T;BV(\Omega))$.

It follows from the bounds (53) and (54) that there exist a subsequence of $\{u_{\epsilon,k}\}_{k\in\mathbb{N}}$, still denoted by $\{u_{\epsilon,k}\}_{k\in\mathbb{N}}$, and $u_\epsilon \in L^\infty(0,T;BV(\Omega)) \cap L^\infty(0,T;\mathcal{F})$ such that $u_\epsilon' \in L^2(0,T;\mathcal{F})$ and as $k \to \infty$,

$$\begin{aligned} &u_{\epsilon,k} \rightharpoonup u_\epsilon \text{ in } L^2(0,s;\mathcal{F}) \ \forall s \in (0,T], \\ &\left.\begin{aligned} &u_{\epsilon,k}(s) \rightharpoonup u_\epsilon(s) \text{ in } \mathcal{F} \\ &u_{\epsilon,k}(s) \to u_\epsilon(s) \in BV(\Omega) \text{ in } L^1(\Omega) \end{aligned}\right\} \text{for a.e. } s \in [0,T]. \end{aligned} \tag{55}$$

Suppose that $v \in C^1(0,T;C^1(\Omega)) \cap C^1(0,T;\mathcal{V})$ (a density argument is given further on which shows that it is sufficient to consider such functions), and take $v_k(t) = P^k v(t)$.

Inequality (17) and limit (43) give that

$$\|v(t) - v_k(t)\|_{-1} \leq C_P \|v(t) - v_k(t)\| \to 0 \quad \text{as } k \to \infty. \tag{56}$$

Inequality (44) and equation (46) give that for all $k \in \mathbb{N}$,

$$\|v_k(t)\|_{-1} \leq \|v(t)\|_{-1}, \qquad \|v_k'(t)\|_{-1} \leq \|v'(t)\|_{-1}. \tag{57}$$

Hence for all $k \in \mathbb{N}$,

$$\|v_k\|_{L^2(0,T;\mathcal{F})} \leq \|v\|_{L^2(0,T;\mathcal{F})}, \qquad \|v_k'\|_{L^2(0,T;\mathcal{F})} \leq \|v'\|_{L^2(0,T;\mathcal{F})}. \tag{58}$$

Also, it follows from inequality (17), limit (45) and equation (46) that

$$\|v - v_k\|_{L^2(0,T;\mathcal{F})}, \ \|v' - v_k'\|_{L^2(0,T;\mathcal{F})}, \ \|\nabla v - \nabla v_k\|_{L^2(\Omega_T)} \to 0 \quad \text{as} \quad k \to \infty. \tag{59}$$

Since the limits (55) are insufficient to identify the limit of each term in the Galerkin problem (48) as $k \to \infty$, a resulting variational inequality for which passage to the limit is possible is found. The process of deducing this variational inequality from equation (48) is the finite dimensional analogue of the deduction of the variational inequality (31) from equation (3): taking $\eta_k = v_k(t) - u_{\epsilon,k}(t)$ in equation (48), using that

$(a-b)(c-a) \le \frac{1}{2}\left[(c-b)^2-(a-b)^2\right]$ and Lemma 1.2, and integrating with respect to t from 0 to s gives

$$\begin{aligned}&\int_0^s \left(v_k'(t),\mathcal{G}\left[v_k(t)-u_{\epsilon,k}(t)\right]\right)dt+\int_0^s \left[J_{\lambda,\epsilon}(v_k(t))-J_{\lambda,\epsilon}(u_{\epsilon,k}(t))\right]dt\\ \ge\ &\frac{1}{2}\left[\|v_k(s)-u_{\epsilon,k}(s)\|_{-1}^2-\|v_k(0)-u_{\epsilon,k}(0)\|_{-1}^2\right]. \qquad (60)\end{aligned}$$

Lemma 4.4 gives the $k\to\infty$ limit of each term in the variational inequality (60).

Lemma 4.4. *For $v\in C^1\left(0,T;C^1(\Omega)\right)\cap C^1(0,T;\mathcal{V})$ and $v_k(t)=P^k v(t)$ $(k\in\mathbb{N})$,*

$$\begin{aligned}&(i) && \lim_{k\to\infty}\int_0^s (v_k'(t),\mathcal{G}v_k(t))\,dt=\int_0^s (v'(t),\mathcal{G}v(t))\,dt,\\ &(ii) && \lim_{k\to\infty}\int_0^s (v_k'(t),\mathcal{G}u_{\epsilon,k}(t))\,dt=\int_0^s (v'(t),\mathcal{G}u_\epsilon(t))\,dt,\\ &(iii) && \lim_{k\to\infty}\int_0^s J_{\lambda,\epsilon}(v_k(t))\,dt=\int_0^s J_{\lambda,\epsilon}(v(t))\,dt,\\ &(iv) && \liminf_{k\to\infty}\int_0^s J_{\lambda,\epsilon}(u_{\epsilon,k}(t))\,dt\ge\int_0^s J_{\lambda,\epsilon}(u_\epsilon(t))\,dt,\\ &(v) && \liminf_{k\to\infty}\|v_k(s)-u_{\epsilon,k}(s)\|_{-1}^2\ge\|v(s)-u_\epsilon(s)\|_{-1}^2,\\ &(vi) && \lim_{k\to\infty}\|v_k(0)-u_{\epsilon,k}(0)\|_{-1}^2=\|v(0)-u_{0,\epsilon}\|_{-1}^2.\end{aligned}$$

Proof. **(i)** The bounds (58) and limits (59) give

$$\begin{aligned}&\left|\int_0^s (v_k'(t),\mathcal{G}v_k(t))\,dt-\int_0^s (v'(t),\mathcal{G}v(t))\,dt\right|\\ \le&\left|\int_0^s \left(v_k'(t),\mathcal{G}\left[v_k(t)-v(t)\right]\right)dt\right|+\left|\int_0^s (v_k'(t)-v'(t),\mathcal{G}v(t))\,dt\right|\\ \le&\left[\|v_k'\|_{L^2(0,T;\mathcal{F})}\|v_k-v\|_{L^2(0,T;\mathcal{F})}+\|v_k'-v'\|_{L^2(0,T;\mathcal{F})}\|v\|_{L^2(0,T;\mathcal{F})}\right]\\ \to&0 \quad\text{as}\quad k\to\infty.\end{aligned}$$

(ii) Define T_1 and T_2 by

$$\begin{aligned}&\left|\int_0^s (v_k'(t),\mathcal{G}u_{\epsilon,k}(t))\,dt-\int_0^s (v'(t),\mathcal{G}u_\epsilon(t))\,dt\right|\\ \le&\left|\int_0^s (v_k'(t)-v'(t),\mathcal{G}u_{\epsilon,k}(t))\,dt\right|+\left|\int_0^s \left(v'(t),\mathcal{G}\left[u_{\epsilon,k}(t)-u_\epsilon(t)\right]\right)dt\right|\\ =:&\,T_1+T_2.\end{aligned}$$

The bounds (53) and limits (59) give that

$$T_1\le\|v_k'-v'\|_{L^2(0,T;\mathcal{F})}\|u_{\epsilon,k}\|_{L^2(0,T;\mathcal{F})}\to 0 \quad\text{as}\quad k\to\infty.$$

The limits (55) and Lemma 1.3 give that

$$\int_0^s \left(v'(t),\mathcal{G}\left[u_{\epsilon,k}(t)-u_\epsilon(t)\right]\right)dt\to 0 \quad\text{as}\quad k\to\infty,$$

and hence $T_2\to 0$ as $k\to\infty$.

(iii) The limits (59) and Lemma 1.2 give that

$$\left|\int_0^s \left(|\nabla v_k(t)|_\epsilon - |\nabla v(t)|_\epsilon, 1\right) dt\right| \leq \int_0^s \left(|\nabla [v_k(t) - v(t)]|, 1\right) dt$$
$$\leq |\Omega|^{\frac{1}{2}} T^{\frac{1}{2}} \|\nabla v_k - \nabla v\|_{L^2(\Omega_T)} \to 0 \quad \text{as} \quad k \to \infty,$$

$$\left|\int_0^s \left(v_k(t) - v(t), \mathcal{G} f\right) dt\right| \leq T^{\frac{1}{2}} \|f\|_{-1} \|v_k - v\|_{L^2(0,T;\mathcal{F})} \to 0 \quad \text{as} \quad k \to \infty.$$

By an analogous argument to that used to prove (i),

$$\left|\int_0^s \left(v_k(t), \mathcal{G} v_k(t)\right) dt - \int_0^s \left(v(t), \mathcal{G} v(t)\right) dt\right| \to 0 \quad \text{as} \quad k \to \infty.$$

(iv) Since $J_{\lambda,\epsilon}(\cdot)$ is convex (see proof of Theorem 3.1), the limits (55) give that

$$J_{\lambda,\epsilon}(u_\epsilon(t)) \leq \liminf_{k\to\infty} J_{\lambda,\epsilon}(u_{\epsilon,k}(t)),$$

and using Fatou's lemma gives the result.

(v) Limits (43) and (55) and inequality (17) give that for a.e. $s \in [0,T]$,

$$v_k(s) - u_{\epsilon,k}(s) \rightharpoonup v(s) - u_\epsilon(s) \text{ in } \mathcal{F} \quad \text{as} \quad k \to \infty,$$

and using the lower semi-continuity of a norm with respect to weak convergence gives the result.

(vi) The limit (49) and bounds (56), (57) yield

$$\begin{aligned}
&|(v_k(0), \mathcal{G} v_k(0)) - (v(0), \mathcal{G} v(0))| \\
&\leq |(v_k(0), \mathcal{G}[v_k(0) - v(0)])| + |(v_k(0) - v(0), \mathcal{G} v(0))| \\
&\leq \left(\|v_k(0)\|_{-1} + \|v(0)\|_{-1}\right) \|v_k(0) - v(0)\|_{-1} \\
&\leq 2\|v(0)\|_{-1} \|v_k(0) - v(0)\|_{-1} \to 0 \quad \text{as} \quad k \to \infty, \\
&|(v_k(0), \mathcal{G} u_{\epsilon,k}(0)) - (v(0), \mathcal{G} u_{0,\epsilon})| \\
&\leq |(v_k(0) - v(0), \mathcal{G} u_{\epsilon,k}(0))| + |(v(0), \mathcal{G} u_{\epsilon,k}(0) - \mathcal{G} u_{0,\epsilon})| \\
&\leq \|u_{\epsilon,k}(0)\|_{-1} \|v_k(0) - v(0)\|_{-1} + \|v(0)\|_{-1} \|u_{\epsilon,k}(0) - u_{0,\epsilon}\|_{-1} \\
&\quad \to 0 \quad \text{as} \quad k \to \infty, \\
&\|u_{\epsilon,k}(0)\|_{-1}^2 \to \|u_{0,\epsilon}\|_{-1}^2 \, \text{as} k \to \infty.
\end{aligned}$$

□

By Lemma 4.4, passage to the limit $k \to \infty$ of each term in the variational inequality (60) gives that u_ϵ satisfies the variational inequality (31) if $v \in C^1(0,T;C^1(\Omega)) \cap C^1(0,T;\mathcal{V})$. Further, $C^1(\Omega) \cap \mathcal{V}$ is dense in $BV(\Omega) \cap \mathcal{F}$ with respect to strict convergence ([2], p. 132) and $C^1(0,T;BV(\Omega)) \cap C^1(0,T;\mathcal{F})$ is dense in $L^1(0,T;BV(\Omega)) \cap C(0,T;\mathcal{F})$ with respect to norm convergence. It follows that $C^1\left(0,T;C^1(\Omega)\right) \cap C^1(0,T;\mathcal{V})$ is dense in $L^1(0,T;BV(\Omega)) \cap C(0,T;\mathcal{F})$. Hence u_ϵ satisfies the variational inequality (31).

As in Feng and Prohl [7], an argument of Lichnewsky and Temam [8] is used to prove that $u_\epsilon(0) = u_{0,\epsilon}$ and $u_\epsilon \in C(0,T;\mathcal{F})$. Indeed, for $\delta > 0$, the following initial value problem is considered:

$$\delta u'_{\epsilon,\delta}(t) + u_{\epsilon,\delta}(t) = u_\epsilon(t) \quad \forall t \in (0,T), \qquad u_{\epsilon,\delta}(0) = u_{0,\epsilon}.$$

This initial value problem is used because its unique solution $u_{\epsilon,\delta}$ is known to belong to $C(0,T;\mathcal{F})$.

Replacing v by $u_{\epsilon,\delta}$ in the variational inequality (31) gives that

$$\begin{aligned}
&\frac{1}{2}\left\|u_{\epsilon,\delta}(s)-u_{\epsilon}(s)\right\|_{-1}^{2} \\
&\leq \int_0^s \left[J_{\lambda,\epsilon}\left(u_{\epsilon,\delta}(t)\right)dt - J_{\lambda,\epsilon}\left(u_{\epsilon}(t)\right)\right]dt - \frac{1}{\delta}\int_0^s \left\|u_{\epsilon,\delta}(t)-u_{\epsilon}(t)\right\|_{-1}^{2}dt \\
&\leq \int_0^s \left[J_{\lambda,\epsilon}\left(u_{\epsilon,\delta}(t)\right)dt - J_{\lambda,\epsilon}\left(u_{\epsilon}(t)\right)\right]dt \\
&= \int_0^s \int_\Omega \left[\left|\nabla u_{\epsilon,\delta}(t)\right|_{\epsilon} - \left|\nabla u_{\epsilon}(t)\right|_{\epsilon}\right] d\boldsymbol{x}dt \\
&\quad + \frac{\lambda}{2}\int_0^s \left[\left\|u_{\epsilon,\delta}(t)-f\right\|_{-1}^{2} - \left\|u_{\epsilon}(t)-f\right\|_{-1}^{2}\right]dt.
\end{aligned}$$

It follows from Lemma 1.2 that

$$\begin{aligned}
\left|\int_0^s \int_\Omega \left[\left|\nabla u_{\epsilon,\delta}(t)\right|_{\epsilon} - \left|\nabla u_{\epsilon}(t)\right|_{\epsilon}\right] d\boldsymbol{x}dt\right| &\leq \int_0^T \int_\Omega \left|\nabla\left[u_{\epsilon,\delta}(t)-u_{\epsilon}(t)\right]\right| d\boldsymbol{x}dt \\
&\leq \left\|u_{\epsilon,\delta}-u_{\epsilon}\right\|_{L^1(0,T;BV(\Omega))}.
\end{aligned}$$

Since $(a-c)^2-(b-c)^2=(a-b)(a+b-2c)$,

$$\begin{aligned}
&\left|\int_0^s \left[\left\|u_{\epsilon,\delta}(t)-f\right\|_{-1}^{2} - \left\|u_{\epsilon}(t)-f\right\|_{-1}^{2}\right]dt\right| \\
&= \left|\int_0^s \left(\nabla\mathcal{G}\left[u_{\epsilon,\delta}(t)-u_{\epsilon}(t)\right], \nabla\mathcal{G}\left[u_{\epsilon,\delta}(t)+u_{\epsilon}(t)-2f\right]\right)dt\right| \\
&\leq \left\|u_{\epsilon,\delta}-u_{\epsilon}\right\|_{L^2(0,T;\mathcal{F})}\left\|u_{\epsilon,\delta}+u_{\epsilon}-2f\right\|_{L^2(0,T;\mathcal{F})} \\
&\leq \left\|u_{\epsilon,\delta}-u_{\epsilon}\right\|_{L^2(0,T;\mathcal{F})}\left(\left\|u_{\epsilon,\delta}-u_{\epsilon}\right\|_{L^2(0,T;\mathcal{F})} + 2\left\|u_{\epsilon}-f\right\|_{L^2(0,T;\mathcal{F})}\right).
\end{aligned}$$

As in [7] and [8],

$$u_{\epsilon,\delta} \to u_{\epsilon} \text{ in } L^2(0,T;\mathcal{F}) \cap L^1(0,T;BV(\Omega)) \quad \text{as} \quad \delta \downarrow 0.$$

It follows that

$$\left\|u_{\epsilon,\delta}-u_{\epsilon}\right\|_{C(0,T;\mathcal{F})} = \sup_{s\in[0,T]}\left\|u_{\epsilon,\delta}(s)-u_{\epsilon}(s)\right\|_{-1} \to 0 \quad \text{as} \quad \delta \downarrow 0,$$

which yields

$$u_{\epsilon} \in C(0,T;\mathcal{F}), \qquad u_{\epsilon}(0) = u_{0,\epsilon} \text{ in } \mathcal{F}.$$

4.7. **Proof of stability estimate** (36)**.** As in Feng and Prohl [7], an argument of Lichnewsky and Temam [8] is used to prove the stability estimate (36). Indeed, let $u_{\epsilon,i}$ $(i=1,2)$ be weak solutions of $(\mathbf{P}^{\epsilon})$ for data $u_{0,\epsilon,i}, f_i$. The function $u_{\epsilon} \in C(0,T;\mathcal{F})$ is defined by

$$u_{\epsilon}(t) := \frac{u_{\epsilon,1}(t)+u_{\epsilon,2}(t)}{2} \quad \forall t\in[0,T) \qquad \left(\Rightarrow u_{\epsilon}(0) = \frac{u_{0,\epsilon,1}+u_{0,\epsilon,2}}{2}\right).$$

Adding the inequalities (31) for $i = 1, 2$ gives that

$$\begin{aligned}
&2\int_0^s \left(v'\left(t\right), \mathcal{G}\left[v\left(t\right) - u_\epsilon\left(t\right)\right]\right) dt \\
&+ \int_0^s \left[2J_{0,\epsilon}\left(v\left(t\right)\right) dt - J_{0,\epsilon}\left(u_{\epsilon,1}\left(t\right)\right) - J_{0,\epsilon}\left(u_{\epsilon,2}\left(t\right)\right)\right] dt \\
&+ \frac{\lambda}{2}\int_0^s \Big[\left\|v\left(t\right) - f_1\right\|_{-1}^2 + \left\|v\left(t\right) - f_2\right\|_{-1}^2 \\
&- \left\|u_{\epsilon,1}\left(t\right) - f_1\right\|_{-1}^2 - \left\|u_{\epsilon,2}\left(t\right) - f_2\right\|_{-1}^2\Big] dt \\
\geq\ & \frac{1}{2}\Big[\left\|v\left(s\right) - u_{\epsilon,1}\left(s\right)\right\|_{-1}^2 + \left\|v\left(s\right) - u_{\epsilon,2}\left(s\right)\right\|_{-1}^2 \\
&- \left\|v\left(0\right) - u_{0,\epsilon,1}\right\|_{-1}^2 - \left\|v\left(0\right) - u_{0,\epsilon,2}\right\|_{-1}^2\Big].
\end{aligned} \tag{61}$$

For $\delta > 0$, $u_{\epsilon,\delta} \in C\left(0, T; \mathcal{F}\right)$ is taken to be the solution of the initial value problem

$$\delta u'_{\epsilon,\delta}\left(t\right) + u_{\epsilon,\delta}\left(t\right) = u_\epsilon\left(t\right) \quad \forall t \in \left(0, T\right), \qquad u_{\epsilon,\delta}\left(0\right) = u_\epsilon\left(0\right).$$

Replacing v by $u_{\epsilon,\delta}$ in inequality (61) yields

$$\begin{aligned}
&\int_0^s \left[2J_{0,\epsilon}\left(u_{\epsilon,\delta}\left(t\right)\right) - J_{0,\epsilon}\left(u_{\epsilon,1}\left(t\right)\right) - J_{0,\epsilon}\left(u_{\epsilon,2}\left(t\right)\right)\right] dt \\
&+ \frac{\lambda}{2}\int_0^s \Big[\left\|u_{\epsilon,\delta}\left(t\right) - f_1\right\|_{-1}^2 + \left\|u_{\epsilon,\delta}\left(t\right) - f_2\right\|_{-1}^2 \\
&- \left\|u_{\epsilon,1}\left(t\right) - f_1\right\|_{-1}^2 - \left\|u_{\epsilon,2}\left(t\right) - f_2\right\|_{-1}^2\Big] dt \\
\geq\ & \frac{1}{2}\Big[\left\|u_{\epsilon,\delta}\left(s\right) - u_{\epsilon,1}\left(s\right)\right\|_{-1}^2 + \left\|u_{\epsilon,\delta}\left(s\right) - u_{\epsilon,2}\left(s\right)\right\|_{-1}^2\Big] - \frac{1}{4}\left\|u_{0,\epsilon,2} - u_{0,\epsilon,1}\right\|_{-1}^2.
\end{aligned} \tag{62}$$

As in Section 4.6,

$$u_{\epsilon,\delta} \to u_\epsilon \text{ in } L^2\left(0, T; \mathcal{F}\right) \cap L^1\left(0, T; BV\left(\Omega\right)\right) \text{ and } u_{\epsilon,\delta}\left(s\right) \to u_\epsilon\left(s\right) \text{ in } \mathcal{F} \text{ as } \delta \downarrow 0.$$

The convexity of $J_{0,\epsilon}$ ([1], Theorem 2.4) implies that

$$2J_{0,\epsilon}\left(u_\epsilon\left(t\right)\right) \leq J_{0,\epsilon}\left(u_{\epsilon,1}\left(t\right)\right) + J_{0,\epsilon}\left(u_{\epsilon,2}\left(t\right)\right).$$

Letting $\delta \downarrow 0$ in inequality (62) and using

$$\left(\frac{a+b}{2} - c\right)^2 + \left(\frac{a+b}{2} - d\right)^2 - \left(a-c\right)^2 - \left(b-d\right)^2 \leq \frac{1}{2}\left(d-c\right)^2 \ \forall\, a, b, c, d \in \mathbb{R} \tag{63}$$

yields

$$\left\|u_{\epsilon,2}\left(s\right) - u_{\epsilon,1}\left(s\right)\right\|_{-1}^2 \leq \lambda s\left\|f_2 - f_1\right\|_{-1}^2 + \left\|u_{0,\epsilon,2} - u_{0,\epsilon,1}\right\|_{-1}^2.$$

4.8. **Proof of existence of a weak solution of (P).** For $\epsilon > 0$, take u_ϵ to be the weak solution of **($\mathbf{P}^\epsilon$)** with

$$u_{0,\epsilon} = u_0. \tag{64}$$

It follows from the bounds (53), (54) that there exists $C \equiv C\left(u_0, f, \lambda, T, \epsilon, \Omega\right)$, which remains bounded as $\epsilon \downarrow 0$, such that

$$\left\|u_\epsilon\right\|_{L^\infty\left(0,T;\mathcal{F}\right)},\ \left\|u'_\epsilon\right\|_{L^2\left(0,T;\mathcal{F}\right)},\ \left\|u_\epsilon\right\|_{L^\infty\left(0,T;BV\left(\Omega\right)\right)} \leq C. \tag{65}$$

Hence there exist a subsequence of $\{u_\epsilon\}_{\epsilon>0}$, still denoted by $\{u_\epsilon\}_{\epsilon>0}$, and $u \in L^\infty(0,T;BV(\Omega)) \cap L^\infty(0,T;\mathcal{F})$ such that $u' \in L^2(0,T;\mathcal{F})$ and as $\epsilon \downarrow 0$,

$$\begin{aligned} &u_\epsilon \rightharpoonup u \text{ in } L^2(0,s;\mathcal{F}) \ \forall s \in (0,T], \\ &\left.\begin{aligned} &u_\epsilon(s) \rightharpoonup u(s) \text{ in } \mathcal{F} \\ &u_\epsilon(s) \to u(s) \in BV(\Omega) \text{ in } L^1(\Omega) \end{aligned}\right\} \text{for a.e. } s \in [0,T]. \end{aligned} \tag{66}$$

In Section 4.6, Lemma 4.4 was used to pass to the limit $k \to \infty$ in the variational inequality (60) (a corollary of the Galerkin problem (48)), yielding the variational inequality (31). For v in a dense subspace of the test space in inequality (31) and suitably chosen v_k, Lemma 4.4 identified the $k \to \infty$ limit of each term in the variational inequality (60) as being (up to inequality) the corresponding term in the variational inequality (31). Analogously, Lemma 4.5 below identifies the $\epsilon \downarrow 0$ limit of each term in the variational inequality (31) as being (up to inequality) the corresponding term in the variational inequality (30).

By the same density argument as that in Section 4.6, it is sufficient to consider $v \in C^1\left(0,T;C^1(\Omega)\right) \cap C^1(0,T;\mathcal{V})$.

Lemma 4.5. *For any* $v \in C^1\left(0,T;C^1(\Omega)\right) \cap C^1(0,T;\mathcal{V})$,

$$\begin{aligned} &(i) && \lim_{\epsilon\downarrow 0} \int_0^s (v'(t), \mathcal{G}u_\epsilon(t))\,dt = \int_0^s (v'(t), \mathcal{G}u(t))\,dt, \\ &(ii) && \lim_{\epsilon\downarrow 0} \int_0^s J_{\lambda,\epsilon}(v(t))\,dt = \int_0^s J_\lambda(v(t))\,dt, \\ &(iii) && \liminf_{\epsilon\downarrow 0} \int_0^s J_{\lambda,\epsilon}(u_\epsilon(t))\,dt \geq \int_0^s J_\lambda(u(t))\,dt, \\ &(iv) && \liminf_{\epsilon\downarrow 0} \|v(s) - u_\epsilon(s)\|_{-1}^2 \geq \|v(s) - u(s)\|_{-1}^2, \\ &(v) && \lim_{\epsilon\downarrow 0} \|v(0) - u_{0,\epsilon}\|_{-1}^2 = \|v(0) - u_0\|_{-1}^2. \end{aligned}$$

Proof. The proofs of **(i)** and **(iv)** are analogous to (and simpler than, since it is not necessary to pass to the limit of $\{v_k\}_{k\in\mathbb{N}}$) those of parts (ii) and (v) of Lemma 4.4 given in Section 4.6.

(ii) follows from $\left||\boldsymbol{p}|_\epsilon - |\boldsymbol{p}|\right| \leq \epsilon$:

$$\left|\int_0^s (|\nabla v(t)|_\epsilon - |\nabla v(t)|, 1)\,dt\right| \leq \epsilon T|\Omega| \to 0 \quad \text{as } \epsilon \downarrow 0.$$

The proof of **(iii)** is similar to that of part (iv) of Lemma 4.4 given in Section 4.6: the convexity of $J_\lambda(\cdot)$ (see proof of Theorem 3.1), the inequality $J_\lambda(\cdot) \leq J_{\lambda,\epsilon}(\cdot)$ and the limits (66) yielding

$$J_\lambda(u(t)) \leq \liminf_{\epsilon\downarrow 0} J_\lambda(u_\epsilon(t)) \leq \liminf_{\epsilon\downarrow 0} J_{\lambda,\epsilon}(u_\epsilon(t)).$$

(v) follows from the initial condition (64). □

5. **Convergence of weak solutions to minimizers of energies.**

5.1. **Statement of result.**

Theorem 5.1 (c.f. [6], Theorem 2.2)**.** *Let $2 \leq d \leq 3$, u_0, $u_{0,\epsilon} \in BV(\Omega) \cap \mathcal{F}$, $f \in \mathcal{F}$ and $\overline{u}$, $\overline{u}_\epsilon$ be the minimizers of $J_\lambda(\cdot)$, $J_{\lambda,\epsilon}(\cdot)$. The weak solutions u, u_ϵ of **(P)**, **(P$^\epsilon$)** satisfy*

$$\forall\, p \in \left[1, \frac{d}{d-1}\right), \qquad u(t) \underset{L^p(\Omega)}{\to} \overline{u} \quad \text{and} \quad u_\epsilon(t) \underset{L^p(\Omega)}{\to} \overline{u}_\epsilon \quad \text{as} \quad t \to \infty. \tag{67}$$

5.2. **Technical Lemma.**

Lemma 5.2. *Let u_0, $u_{0,\epsilon} \in BV(\Omega) \cap \mathcal{F}$ and $f \in \mathcal{F}$. Then for all $s_0 \in (0, T)$, the weak solutions u, u_ϵ of **(P)**, **(P$^\epsilon$)** satisfy*

$$\int_{s_0}^{s} \langle u'(t), \mathcal{G}[w(t) - u(t)]\rangle\, dt + \int_{s_0}^{s} [J_\lambda(w(t)) - J_\lambda(u(t))]\, dt \geq 0$$
$$\forall\, w \in L^1(0, T; BV(\Omega)) \cap L^2(0, T; \mathcal{F}),\ s \in [s_0, T]; \tag{68}$$

$$\int_{s_0}^{s} \langle u_\epsilon'(t), \mathcal{G}[w(t) - u_\epsilon(t)]\rangle\, dt + \int_{s_0}^{s} [J_{\lambda,\epsilon}(w(t)) - J_{\lambda,\epsilon}(u_\epsilon(t))]\, dt \geq 0$$
$$\forall\, w \in L^1(0, T; BV(\Omega)) \cap L^2(0, T; \mathcal{F}),\ s \in [s_0, T]. \tag{69}$$

Hence u, u_ϵ satisfy

$$\langle u'(t), \mathcal{G}[w - u(t)]\rangle + J_\lambda(w) - J_\lambda(u(t)) \geq 0 \qquad \forall\, w \in BV(\Omega) \cap \mathcal{F} \text{ and a.e. } t \in (s_0, T), \tag{70}$$

$$\langle u_\epsilon'(t), \mathcal{G}[w - u_\epsilon(t)]\rangle + J_{\lambda,\epsilon}(w) - J_{\lambda,\epsilon}(u_\epsilon(t)) \geq 0 \qquad \forall\, w \in BV(\Omega) \cap \mathcal{F} \text{ and a.e. } t \in (s_0, T). \tag{71}$$

Proof. The inequalities (68) and (70) are proved (the inequalities (69) and (71) can be proved analogously). Choose $s_0 \in (0, T)$, and take $w \in L^1(0, T; BV(\Omega)) \cap L^2(0, T; \mathcal{F})$. The inequality (68) follows from taking $s \in [s_0, T]$ and

$$v(t) = \begin{cases} u(t) & \text{for } t \in [0, s_0] \\ w(t) & \text{for } t \in (s_0, s], \end{cases}$$

in inequality (28). The inequality (70) follows from inequality (68) by the Lebesgue differentiation theorem (see [2]). □

5.3. **Proof of Theorem 5.1.** The proof of the result for **(P)** is given (the result for **(P$^\epsilon$)** can be proved analogously). Choose $s_0 > 0$ such that $u(s_0) \in BV(\Omega) \cap \mathcal{F}$. Taking $w(t) = u(t - \tau)$ for $0 < \tau < s_0$ in inequality (68) with $s = T$, dividing the resulting inequality by $-\tau$ and passing to the limit $\tau \downarrow 0$ yields

$$\int_{s_0}^{T} \|u'(t)\|_{-1}^2\, dt + J_\lambda(u(T)) \leq J_\lambda(u(s_0)) < \infty \quad \forall\, T \in [s_0, \infty).$$

Hence there exists a sequence $\{t_j\}_{j\in\mathbb{N}}$ and a constant $C \equiv C(\Omega)$ such that $t_j \to \infty$ as $j \to \infty$ and

$$\|u'(t_j)\|_{-1} \to 0 \quad \text{as} \quad j \to \infty \qquad \text{and} \qquad \|u(t_j)\|_{BV(\Omega)}, \|u(t_j)\|_{-1} \leq C \quad \forall j \in \mathbb{N}.$$

It follows that there exists $\hat{u} \in BV(\Omega) \cap \mathcal{F}$ and a subsequence of $\{u(t_j)\}_{j\in\mathbb{N}}$ (still denoted by $\{u(t_j)\}_{j\in\mathbb{N}}$) such that

$$u(t_j) \underset{BV-w^*}{\overset{*}{\rightharpoonup}} \hat{u}, \qquad u(t_j) \underset{\mathcal{F}}{\rightharpoonup} \hat{u}, \qquad u(t_j) \underset{L^1(\Omega)}{\to} \hat{u} \qquad \text{as} \qquad j \to \infty;$$

(see [2], p. 125). Taking $t = t_j$ in (70), letting $j \to \infty$ and using the convexity of $J_\lambda(\cdot)$ (see proof of Theorem 3.1) gives that

$$J_\lambda(w) \geq \liminf_{j\to\infty} J_\lambda(u(t_j)) \geq J_\lambda(\hat{u}) \quad \forall w \in BV(\Omega) \cap \mathcal{F}.$$

The uniqueness of the solution of the minimization problem implies that $\hat{u} = \overline{u}$ and that the whole sequence $\{u(t)\}_{t>0}$ satisfies the limit in (67). ☐

6. **Rate of convergence of u_ϵ to u as $\epsilon \downarrow 0$.** It follows from the proof of the existence of a weak solution to **(P)** given in Section 4.8 that if $u_{0,\epsilon} = u_0$ for $\epsilon > 0$ and $f \in \mathcal{F}$, then the weak solutions u, $\{u_\epsilon\}_{\epsilon>0}$ of **(P)**, **(P$^\epsilon$)** for data u_0 and f, $\{u_{0,\epsilon}\}_{\epsilon>0}$ and f satisfy

$$u_\epsilon \overset{\star}{\rightharpoonup} u \quad \text{in} \quad L^\infty(0,T;\mathcal{F}) \quad \text{as} \quad \epsilon \downarrow 0.$$

We prove Theorem 6.1 concerning the rate of this convergence.

Theorem 6.1 (c.f. [6], Theorem 3.1)**.** *Suppose that $2 \leq d \leq 3$, $u_0 \in BV(\Omega) \cap \mathcal{F}$, $\{u_{0,\epsilon}\}_{\epsilon>0} \subset BV(\Omega) \cap \mathcal{F}$ and $f \in \mathcal{F}$. Let u, $\{u_\epsilon\}_{\epsilon>0}$ be the weak solutions of **(P)**, **(P$^\epsilon$)** for data u_0 and f, $\{u_{0,\epsilon}\}_{\epsilon>0}$ and f. Then*

$$\|u - u_\epsilon\|_{C(0,T;\mathcal{F})} \leq \|u_0 - u_{0,\epsilon}\|_{-1} + 2\sqrt{\epsilon T |\Omega|}.$$

Hence, if $u_{0,\epsilon} = u_0$ for $\epsilon > 0$,

$$u_\epsilon \to u \quad in \quad C(0,T;\mathcal{F}) \quad as \quad \epsilon \downarrow 0.$$

Proof. The inequality $||\boldsymbol{p}|_\epsilon - |\boldsymbol{p}|| \leq \epsilon$ gives that $|J_{\lambda,\epsilon}(\cdot) - J_\lambda(\cdot)| \leq \epsilon |\Omega|$. Hence taking $v = u$ in the inequality (29) for **(P$^\epsilon$)** and $v = u_\epsilon$ in the inequality (28) for **(P)** and adding the resulting inequalities gives the desired result. ☐

7. **Conclusions.** Functional analytic techniques have been used to show the existence of a unique suitably defined weak solution to partial differential equation arising from the H^{-1} gradient flow of the energy consisting of TV regularization plus H^{-1} fidelity. A regularized version of the energy was considered, which gave rise to a regularized partial differential equation. The existence of a unique weak solution to this regularized problem was used to show the existence of a unique weak solution to the original problem. The convergence of each weak solution to the minimizer of the corresponding energy as time $t \to \infty$ was established. Further, a result for the rate of convergence of the weak solution of the regularized problem to that of the original one as the regularization parameter $\epsilon \downarrow 0$ was established.

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