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THE LIMIT OF THE FULLY ANISOTROPIC DOUBLE-OBSTACLE ALLEN–CAHN EQUATION IN THE NONSMOOTH CASE*

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Abstract. In this paper, we prove that solutions of the anisotropic Allen–Cahn equation in double-obstacle form with kinetic term

$$\varepsilon\beta(\nabla\varphi)\partial_t\varphi - \varepsilon\nabla A'(\nabla\varphi) - \frac{1}{\varepsilon}\varphi = \frac{\pi}{4}u \quad \text{in } [|\varphi| < 1],$$

where A is a convex function homogeneous of degree two and β depends only on the direction of $\nabla\varphi$, converge to an anisotropic mean-curvature flow

$$\beta(N)V_N = -\mathrm{tr}(B(N)D^2B(N)R) - B(N)u.$$

Here N, V_N, and R denote the normal, the normal velocity, and the second fundamental form of the interface, respectively, and $B := \sqrt{2A}$.

Key words. anisotropic Allen–Cahn equation, mean-curvature flow, viscosity solutions, double-obstacle problem

AMS subject classifications. 35R35, 35K22

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1. Introduction.

The Allen–Cahn equation

$$\varepsilon\partial_t\varphi_\varepsilon - \varepsilon\Delta\varphi_\varepsilon + \frac{1}{\varepsilon}W'(\varphi_\varepsilon) = c_W u, \tag{1}$$

where $W(t) := (t^2-1)^2$ is a double well potential and c_W is a certain constant depending only on W, was introduced by Allen and Cahn in [1] as a model for grain boundary motion. It is the L^2-gradient flow of the energy functional

$$F_\varepsilon(\varphi) := \int \frac{\varepsilon}{2}|\nabla\varphi|^2 + \frac{1}{\varepsilon}W(\varphi) - c_W u\varphi. \tag{2}$$

In its double-obstacle form (see Blowey and Elliott in [5]), the Allen–Cahn equation is replaced by a parabolic variational inequality with constraint that reads

$$\begin{gathered}\forall\eta \in \mathrm{L}^2(\mathrm{H}^{1,2}),\ |\eta| \le 1 : \\ \int \varepsilon\partial_t\varphi\ (\varphi-\eta) + \varepsilon\nabla\varphi\ (\nabla\varphi - \nabla\eta) - \frac{1}{\varepsilon}\varphi(\varphi-\eta) - \frac{\pi}{4}u(\varphi-\eta) \le 0, \\ |\varphi| \le 1.\end{gathered} \tag{3}$$

The convergence of solutions of the Allen–Cahn equation to the mean-curvature flow

$$V_N = -\kappa - u \tag{4}$$

was proved by Chen in [6], de Mottoni and Schatzmann in [10], Evans, Soner, and Souganidis in [13], and Bellettini and Paolini in [3]; in double-obstacle form, this was

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proved by Chen and Elliott in [8], Nochetto, Paolini, and Verdi in [17] and Nochetto and Verdi in [18].

Anisotropy is now introduced by replacing the gradient in (2) and considering the functional

$$F_\varepsilon(\varphi) := \int \varepsilon A(\nabla\varphi) + \frac{1}{\varepsilon}W(\varphi) - c_W u\varphi, \tag{5}$$

where A is convex and homogeneous of degree two. Moreover, we do not consider the gradient flow of this functional but introduce a kinetic factor depending on $\nabla\varphi$ in front of $\partial_t\varphi$. Then the Allen–Cahn equation takes the form

$$\varepsilon\beta(\nabla\varphi)\partial_t\varphi_\varepsilon - \varepsilon\nabla A'(\nabla\varphi_\varepsilon) + \frac{1}{\varepsilon}W'(\varphi_\varepsilon) = c_W u, \tag{6}$$

where β is a positive function homogeneous of degree zero. McFadden et al. [15], Wheeler and McFadden [21], and Bellettini and Paolini [4] with $\beta = 1$ used formal asymptotics to provide the conjecture that (6) approximates the anisotropic mean-curvature flow, which reads, in two space dimensions,

$$\beta(N)V_N = -\gamma(\gamma + \gamma'')(\theta(N))\kappa - \gamma(\theta(N))u, \tag{7}$$

where γ is 2π-periodic, θ is the angle, and $\gamma(\theta(N)) := \sqrt{2A(N)}$. This conjecture was proved by the authors in [11] without the kinetic factor in the case when the evolution of the anisotropic mean-curvature flow is smooth. Moreover, it is proved there that the Hausdorff distance between the zero-level set of φ_ε, the solution of (6) in its double-obstacle form, and the interface of the flow is of order $O(\varepsilon^2)$. In the isotropic case, this bound was established by Nochetto, Paolini, and Verdi in [17].

The first difficulty that arises from (6) is how to define a weak solution for this equation. The problem is that, unless β is constant, β is discontinuous at 0. Thus far, it is not clear how to give such a definition. Instead, we consider (6) and its double-obstacle variant in the viscosity sense. For viscosity solutions see, for example, the article [9] of Crandall, Ishi, and Lions. In spite of (6) not admitting a comparison principle in the presence of a nonconstant kinetic factor, we will prove that it has a solution. This will be done by proving uniform convergence of solutions of regularized equations to (6) in section 2.

In section 4, we prove that solutions of (6) in double-obstacle form approximate the corresponding anisotropic mean-curvature flow. The existence of a level-set solution describing this flow was proved by Chen, Giga, and Goto in [7]. The approximation is then meant in the sense that φ_ε, the solution of (6) in double-obstacle form, converges to 1 (respectively, to -1) where the level-set solution is positive (respectively, negative). This is proved by constructing suitable sub- and supersolutions of (6) in double-obstacle form. The construction is similar to that given by Nochetto and Verdi in [18]. Because of the anisotropic nature of the problem, instead of the ordinary distance function, we had to use a distance function that is induced by a Finsler geometry as outlined by Bellettini and Paolini in [4]; see section 3. These sub- and supersolutions are finally compared with the solutions, yielding our result.

The convergence result determines the limit of φ_ε uniquely when the interface of the flow does not develop an interior. Otherwise—that is, in the case called *fattening*—there remains an ambiguity.

For applications, it may be easier to consider equations where the kinetic term does not have a discontinuity at 0. After completing this paper, in [12], we considered

$\beta_\varepsilon \in \mathrm{C}^0(\mathbb{R}^n)$ with $\beta_\varepsilon(p) = \beta(p)$ for $|p| \geq \varepsilon$. The corresponding Allen–Cahn equation reads

$$\varepsilon\beta_\varepsilon(\tilde{\varphi}_\varepsilon)\partial_t\tilde{\varphi}_\varepsilon - \varepsilon\nabla A'(\nabla\tilde{\varphi}_\varepsilon) + \frac{1}{\varepsilon}W(\tilde{\varphi}_\varepsilon) = c_W u.$$

In [12], we proved that the Hausdorff distance between the zero-level set of the solutions of its double-obstacle variant and the interface of the flow is of order $O(\varepsilon^2)$ when this flow is smooth.

2. Existence and comparison.

2.1. Notation. Let $\beta \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^n - \{0\}),\ A \in C^{2,1}_{\mathrm{loc}}(\mathbb{R}^n - \{0\})$, and $u \in W^{2,1}_\infty(\mathbb{R}^n \times [0,T])$ be given. We assume that β is homogeneous of degree zero and that A is homogeneous of degree two. Set $B := \sqrt{2A}$. We assume the following bounds:

$$\begin{gathered} n,\ T,\ \|\beta\|_{C^{0,1}(B_2(0)-B_{1/2}(0))},\ \|B\|_{C^{2,1}(B_2(0)-B_{1/2}(0))}, \\ \|A\|_{C^{2,1}(B_2(0)-B_{1/2}(0))},\ \|u\|_{W^{2,1}_\infty(\mathbb{R}^n\times[0,T])} \leq \Lambda, \\ \Lambda^{-1} \leq \beta(p),\ A(p),\ B(p) \leq \Lambda \quad \text{for } |p| = 1, \quad \text{and} \\ \Lambda^{-1} I \leq D^2 A. \end{gathered} \tag{8}$$

Define $F \in C^0(\mathbb{R}^n \times [0,T] \times (\mathbb{R}^n - \{0\}) \times S(n))$, where $S(n)$ denotes the set of all real, symmetric $n \times n$ matrices endowed with the usual ordering by

$$F(x,t,p,X) := -\beta(p)^{-1}\mathrm{tr}(B(p)D^2B(p)X) - \beta(p)^{-1}B(p)u(x,t). \tag{9}$$

Since B is homogeneous of degree one, it follows that F is geometric in the sense that

$$F(x,t,\lambda p, \lambda X + \sigma(p\otimes p)) = \lambda F(x,t,p,X) \quad \text{for } \lambda > 0,\ \sigma \in \mathbb{R},$$

where $\otimes$ denotes the tensor product $p \otimes q = (p_i q_j)_{i,j}$ on $\mathbb{R}^n$.

Our limit problem, the fully anisotropic mean-curvature flow, is

$$\beta(\nabla\omega)\partial_t\omega - \mathrm{tr}(B(\nabla\omega)D^2B(\nabla\omega)D^2\omega) - B(\nabla\omega)u = 0, \tag{10}$$

with the evolving interface given by $\Gamma_t := [\omega(.,t) = 0]$. Equation (10) was treated by Chen, Giga, and Goto in [7] when u was independent of space, but all of their methods apply to general u. In order to give meaning to (10), we seek a solution in the viscosity sense of

$$\partial_t\omega + F(x,t,\nabla\omega,D^2\omega) = 0.$$

Existence and comparison for (10) were proved in [7].

We recall the definition of viscosity solutions. In the following, we denote by LSC(...) and USC(...) the sets of lower semicontinuous and upper semicontinuous functions.

Definition 2.1. *Let $K \in C^0(\mathbb{R}^n \times [0,T] \times \mathbb{R} \times \mathbb{R} \times (\mathbb{R}^n - \{0\}) \times S(n))$. A function $v : Q \to \mathbb{R}$, where $\emptyset \neq Q \subseteq \mathbb{R}^n \times\ ]0,T[$ is open, is called a* viscosity subsolution *of*

$$K(x,t,v,\partial_t v,\nabla v, D^2 v) = 0 \quad \text{in } Q, \tag{11}$$

written as

$$K(x,t,v,\partial_t v,\nabla v, D^2 v) \leq 0 \quad \text{in } Q,$$

if $v \in \mathrm{USC}(Q)$ *and*

$$\forall (a,p,X) \in \mathrm{P}^{2,+}v(x,t),\ (x,t) \in Q: \qquad K_\star(x,t,v(x,t),a,p,X) \le 0,$$

where $K_\star$ *is the* lower semicontinuous envelope of K*, that is,*

$$K_\star(x,t,r,a,p,X) := \inf\{\liminf_{j\to\infty} K(x_j,t_j,r_j,a_j,p_j,X_j) \mid (x_j,t_j,r_j,a_j,p_j,X_j) \to (x,t,r,a,p,X)\}.$$

$\mathrm{P}^{2,+}$ *is the* set of superdifferentials *and is defined in the next subsection. A* supersolution *is defined analogously by considering* $K^\star$*, the upper semicontinuous envelope of* K*, and the set of subdifferentials* $\mathrm{P}^{2,-}$*.* v *is a* solution of (11) *when it is both a sub- and a supersolution. It is necessary to introduce the semicontinuous envelopes since* K *is not continuous when* $p = 0$*.*

The double-obstacle problem corresponding to (11) *is given by*

$$\max(v-1, \min(v+1, K(x,t,v,\partial_t v, \nabla v, D^2 v))) = 0 \quad \text{in } Q. \tag{12}$$

Equivalently, v *is a* subsolution of (12) *if*

$$v \in \mathrm{USC}(Q),$$
$$v \le 1,$$

and

$$\forall (a,p,X) \in \mathrm{P}^{2,+}v(x,t),\ (x,t) \in Q,\ \text{with } v(x,t) > -1:$$
$$K_\star(x,t,v(x,t),a,p,X) \le 0.$$

In this article, we mainly study parabolic equations and their double-obstacle problems, that is, where K is given by

$$K(x,t,v,\partial_t v,\nabla v, D^2 v) = \partial_t v + H(x,t,v,\nabla v, D^2 v).$$

DEFINITION 2.2. *For* $v : Q \to \mathbb{R}$*, where* $\emptyset \ne Q \subseteq \mathbb{R}^n \times\,]0,T[$ *is open,* $(x_0,t_0) \in Q$*, we define the* sets of superdifferentials

$$\begin{aligned}\mathrm{P}^{2,+}v(x_0,t_0) := \{(a,p,X) \in \mathbb{R}\times\mathbb{R}^n\times S(n) \mid\ & v(x,t) \le v(x_0,t_0) \\ & + a(t-t_0) + p(x-x_0) + \tfrac{1}{2}(x-x_0)^T X(x-x_0) \\ & + o(|t-t_0| + |x-x_0|^2) \text{ as } t \to t_0,\ x \to x_0\},\end{aligned}$$

and

$$\begin{aligned}\overline{\mathrm{P}}^{2,+}v(x_0,t_0) := \{(a,p,X) \in \mathbb{R}\times\mathbb{R}^n\times S(n) \mid\ & \exists (a_j,p_j,X_j) \in \mathrm{P}^{2,+}v(x_j,t_j) : \\ & (a_j,p_j,X_j) \to (a,p,X),\ (x_j,t_j) \to (x_0,t_0), v(x_j,t_j) \to v(x_0,t_0)\}.\end{aligned}$$

The sets of subdifferentials $\mathrm{P}^{2,-}$ *and* $\overline{\mathrm{P}}^{2,-}$ *are defined analogously.*

Remark 2.3. It is seen easily that for $\varphi \in C^{2,1}(Q)$ with $v - \varphi \le (v-\varphi)(x_0,t_0)$ in Q, the triple of derivatives $(\partial_t\varphi, \nabla\varphi, D^2\varphi)(x_0,t_0) \in \mathrm{P}^{2,+}v(x_0,t_0)$.

Conversely, for all superdifferentials $(a,p,X) \in \mathrm{P}^{2,+}v(x_0,t_0)$, there is a $\varphi \in C^{2,1}(Q)$ with $v - \varphi \le (v-\varphi)(x_0,t_0)$ in Q and $(a,p,X) = (\partial_t\varphi, \nabla\varphi, D^2\varphi)(x_0,t_0)$. A proof of the second statement can be found in [19, section 14A].

2.2. The equation. We consider the anisotropic Allen–Cahn equation

$$\varepsilon\beta(\nabla\varphi)\partial_t\varphi - \varepsilon\nabla A'(\nabla\varphi) + \frac{1}{\varepsilon}W'(\varphi) - c_W u = 0 \quad \text{in } \mathbb{R}^n \times [0,T], \tag{13}$$

where $W(t) := (t^2-1)^2$. As already pointed out in the introduction, it is not clear how to define a weak solution of (13). Therefore, we treat (13) and its double-obstacle variant in the viscosity sense. To be precise, we define $G_\varepsilon,\ \tilde{G}_\varepsilon \in C^0(\mathbb{R}^n \times [0,T] \times \mathbb{R} \times (\mathbb{R}^n - \{0\}) \times S(n))$ by

(14)

$$G_\varepsilon(x,t,r,p,X) := -\varepsilon\beta(p)^{-1}\mathrm{tr}(D^2A(p)X) - \frac{1}{\varepsilon}\beta(p)^{-1}r - \beta(p)^{-1}\frac{\pi}{4}u(x,t)$$

and

$$\tilde{G}_\varepsilon(x,t,r,p,X) := -\varepsilon\beta(p)^{-1}\mathrm{tr}(D^2A(p)X) + \frac{1}{\varepsilon}\beta(p)^{-1}W'(r) - \beta(p)^{-1}c_W u(x,t).$$

Then in the viscosity formulation, the Allen–Cahn equation (13) reads

$$\partial_t\varphi + \frac{1}{\varepsilon}\tilde{G}_\varepsilon(.,.,\varphi,\nabla\varphi,D^2\varphi) = 0, \tag{15}$$

and its double-obstacle variant is given by

$$\max\left(\varphi - 1, \min\left(\varphi + 1, \partial_t\varphi + \frac{1}{\varepsilon}G_\varepsilon(.,.,\varphi,\nabla\varphi,D^2\varphi)\right)\right) = 0. \tag{16}$$

As already mentioned, existence and comparison for (10) were proved by Chen, Giga, and Goto in [7]. We will state these theorems without proof.

Throughout this article, we will compute on the whole of $\mathbb{R}^n$ and we will consider only space-periodic functions; therefore, we assume u and the initial data to be periodic in space. Here and in the following, *periodic* means *space periodic* and that there are n linearly independent periods.

THEOREM 2.4. *Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be open, $0 < T \leq \Lambda$, and $H \in C^0(\Omega \times [0,T] \times \mathbb{R} \times (\mathbb{R}^n - \{0\}) \times S(n))$ satisfy*

(i) $|H(x,t,r,p,X)| \leq C(\Gamma)$ *for* $|r|, |p|, \|X\| \leq \Gamma$,
(ii) $H(x,t,r,p,X) \geq H(x,t,r,p,Y)$ *when* $X \leq Y$,
(iii) $H(x,t,r,p,X) - H(x,t,s,p,X) \geq -\Lambda(r-s)$ *when* $r \geq s$,
(iv) $|H(x,t,r,p,X) - H(y,t,r,p,X)| \leq \Lambda|x-y|(1+|p|)$, *and*
(v) $H_\star(x,t,r,0,0) = H^\star(x,t,r,0,0)$.

Further, let $v \in \mathrm{USC}(\bar{\Omega} \times [0,T])$ *and* $w \in \mathrm{LSC}(\bar{\Omega} \times [0,T])$ *satisfy*

$$\partial_t v + H_\star(.,.,v,\nabla v,D^2v) \leq 0,$$

and

$$\partial_t w + H^\star(.,.,w,\nabla w,D^2w) \geq 0,$$

either

(a) *in* $\Omega \times]0,T[$

or

(b) *in* $\Omega \times]0,T[$ *in the double-obstacle sense.*

Next, we assume that either

(α) $\Omega = \mathbb{R}^n$ *and* v *and* w *are periodic with the same period*

or

(β) $\Omega \subset\subset \mathbb{R}^n$ *and* $v \leq w$ *on* $\partial\Omega \times [0, T[$.

Then

$$v(.,0) \leq w(.,0) \quad \text{in } \Omega$$

implies

$$v \leq w \quad \text{in } \Omega \times [0, T[.$$

Proof. This comparison principle was proved by Chen, Giga, and Goto in [7]. Properly, they considered H to be independent of x and they did not consider the double-obstacle problem. However, their proof applies to the general case. See also Theorem 2.12, where the ideas of [7] are applied to prove a modified comparison principle for the double-obstacle Allen–Cahn equation. □

THEOREM 2.5. *For periodic, continuous initial data,* (10) *has a unique solution.*

Proof. We again refer to [7]. □

Remark 2.6. When β is constant, one can easily check that G_ε and $\tilde{G}_\varepsilon$ of (14) satisfy the assumptions for H in Theorem 2.4. Therefore, the corresponding parabolic equations admit a comparison principle when they are considered in the viscosity sense. For nonconstant β, a comparison principle of this kind does not hold. This can be seen as follows. We take $u = 0$ and consider solutions which are constant in space. Such solutions w satisfy the ordinary differential equation

$$\varepsilon\beta^\star(t)w'(t) - \frac{1}{\varepsilon}w(t) = 0$$

and the inequality $|w(t)| < 1$, where $\beta^\star$ is any function satisfying $\inf\beta \leq \beta^\star \leq \sup\beta$. Taking $\beta^\star = \inf\beta$ and $\beta^\star = \sup\beta$, comparison is easily contradicted. Since (15) does not admit a comparison principle, the existence of solutions to (15) cannot be proved by Perron's method.

The rest of this section is devoted to proving existence of solutions to (15) and a modified comparison principle in which the sub- or supersolution in Theorem 2.4 satisfies additional conditions.

Existence is proved by approximating (15) through regularized equations. We will establish a uniform bound on the Hölder continuity of solutions of these regularized equations, hence getting a solution of (15). To carry out the limit procedure, we recall a definition of [9, section 6], which is stated below in Definition 2.7.

In section 4, we will construct sub- and supersolutions which admit the additional condition required in the modified comparison principle (Theorem 2.12). Therefore, we will be able to compare these sub- and supersolutions with a solution and conclude the desired convergence as $\varepsilon \to 0$ for these solutions.

DEFINITION 2.7. *For a family of functions* $v_\delta : Q \subseteq \mathbb{R}^N \to \mathbb{R}$, *we define*

$$v := \lim_{\delta \to 0_\star} v_\delta$$

by

$$v(z_0) := \inf\left\{\liminf_{j\to\infty} v_{\delta_j}(z_j) \mid \delta_j \to 0,\ z_j \to z_0\right\}.$$

$\lim_{\delta\to 0}{}^\star v_\delta$ *is defined analogously.*

2.3. Regularization. First, we consider (13) when $\beta \in C^\infty(\mathbb{R}^n),\ A \in C^\infty(\mathbb{R}^n)$, W' is replaced by any $f \in C^\infty(\mathbb{R}),$ and $u \in C^\infty(\mathbb{R}^n \times [0,T])$ is periodic in space. We drop the homogeneity assumptions on β and A, but we demand

$$\begin{aligned} \Lambda^{-1} \leq \beta \leq \Lambda, \\ \Lambda^{-1} I \leq D^2 A, \\ \Lambda^{-1}|p|^2 - \Lambda \leq A(p), \\ \|u\|_{L^\infty(\mathbb{R}^n\times[0,T])},\ \|\nabla u\|_{L^\infty(\mathbb{R}^n\times[0,T])},\ |f(0)| \leq \Lambda, \quad \text{and} \\ f' \geq -\Lambda. \end{aligned} \tag{17}$$

We consider the equation

$$\partial_t\varphi - \beta(\nabla\varphi)^{-1}\mathrm{tr}(D^2A(\nabla\varphi)D^2\varphi) + \beta(\nabla\varphi)^{-1}f(\varphi) - \beta(\nabla\varphi)^{-1}u = 0 \quad \text{in } \mathbb{R}^n \times [0,T] \tag{18}$$

with periodic initial data $\varphi_0 \in C^\infty(\mathbb{R}^n)$ and

$$\|\varphi_0\|_{C^{0,1}(\mathbb{R}^n)} \leq \Lambda. \tag{19}$$

Since (18) admits a comparison principle, we conclude from (19) that a solution of (18) satisfies

$$\|\varphi\|_{L^\infty(\mathbb{R}^n\times[0,T])},\ \|\nabla\varphi\|_{L^\infty(\mathbb{R}^n\times[0,T])} \leq C(\Lambda). \tag{20}$$

With this a priori bound, we find using techniques of [14] that (18) has a unique, periodic solution $\varphi \in C^\infty(\mathbb{R}^n \times [0,T])$.

We write (18) in divergence form as

$$\beta(\nabla\varphi)\partial_t\varphi - \nabla A'(\nabla\varphi) + f(\varphi) - u = 0,$$

multiply by $\partial_t\varphi$, integrate over $K_{t_0} := K \times [0,t_0]$, where K is a periodic cell, and get

$$\int_{K_{t_0}} \beta(\nabla\varphi)|\partial_t\varphi|^2 + A'(\nabla\varphi)\partial_t\nabla\varphi + f(\varphi)\partial_t\varphi = \int_{K_{t_0}} u\partial_t\varphi. \tag{21}$$

We define $F(t) := \int_0^t (f(s) - f(0) + (\Lambda+1)s)ds$ and get $F'(t) = f(t) - f(0) + (\Lambda+1)t,\ F'(0) = 0,$ and $F'' \geq 1$. From (20) and (21), we conclude

$$\begin{aligned} &\Lambda^{-1}\int_{K_{t_0}} |\partial_t\varphi|^2 + \int_K A(\nabla\varphi(t_0)) + F(\varphi(t_0)) \\ &\quad \leq \int_K A(\varphi_0) + F(\varphi_0) + \int_{K_{t_0}} ((\Lambda+1)\varphi - f(0) + u)\partial_t\varphi \\ &\quad \leq \int_K A(\nabla\varphi_0) + F(\varphi_0) + C(\Lambda)|K| + \frac{1}{2\Lambda}\int_{K_{t_0}} |\partial_t\varphi|^2. \end{aligned}$$

Assuming

$$\int_K A(\nabla\varphi_0) + F(\varphi_0) \leq \Lambda \tag{22}$$

we obtain, noting (17),

$$\begin{aligned} &\|\partial_t\varphi\|_{L^2(0,T;L^2(K))},\ \|A(\nabla\varphi)\|_{L^\infty(0,T;L^2(K))}, \\ &\|F(\varphi)\|_{L^\infty(0,T;L^1(K))},\ \|\varphi\|_{L^2(0,T;H^{1,2}(K))} \leq C(\Lambda). \end{aligned} \tag{23}$$

As in [11], the above $L^2(0,T;L^2(K))$ estimate on $\partial_t\varphi$ together with (20) yields

$$\|\varphi\|_{H^{1/n+1,1/2(n+1)}(\mathbb{R}^n\times[0,T])} \le C(\Lambda), \tag{24}$$

where $H^{\alpha,\alpha/2}$ denotes the space of functions that are Hölder continuous of exponent α in space and exponent $\frac{\alpha}{2}$ in time.

Now we approximate β, A, and u of section 2.1 appropriately by β_δ, A_δ, $u_\delta \in \mathrm{C}^\infty$ as in (17). In the case of the smooth well, we take $f_\delta = W'$; for the double-obstacle problem, we take $f_\delta := g' + \frac{1}{\delta}h'$ with $g \ge 0$, $g'(t) = -t$ for $|t| \le 1$, $|g''| \le \Lambda$, $h(t) = 0$ for $|t| \le 1$, $th'(t) > 0$ for $|t| > 1$, and $0 \le h'' \le \Lambda$. Taking the approximation such that $A_\delta \to A$ uniformly on compact subsets of $\mathbb{R}^n$, we have in the smooth case

$$\int_K A_\delta(\nabla\varphi_0) + F_\delta(\varphi_0) \le C(\Lambda) \tag{25}$$

for small δ. Since $|\varphi_0| \le 1$ is required in the double-obstacle problem, we have

$$\int_K \left(g + \frac{1}{\delta}h\right)(\varphi_0) = \int_K g(\varphi_0) \le C(g)|K|,$$

which yields (25) as well. From (22) and (23), we conclude that (24) is satisfied for the unique solution φ_δ of (18) when (β, A, f, u) is substituted by $(\beta_\delta, A_\delta, f_\delta, u_\delta)$. Equation (18) can be considered as a viscosity equation of the form

$$\partial_t\varphi_\delta + H_\delta(.,.,\varphi_\delta, \nabla\varphi_\delta, D^2\varphi_\delta) = 0 \quad \text{in } \mathbb{R}^n \times]0,T[.$$

Choosing β_δ and A_δ as convolutions—that is, $\beta_\delta(p) := \int_{\mathbb{R}} \beta(q)\eta_\delta(p-q)dq$ and $A_\delta(p) := \int_{\mathbb{R}} A(q)\eta_\delta(p-q)dq$, where $\eta_\delta(p) := \delta^{-1}\eta(\frac{p}{\delta})$ with $\eta \in C^\infty(\mathbb{R})$, $\eta \ge 0$, and $\int \eta = 1$—it is easily seen that

$$\lim_{\delta\to 0_\star}{}^{(\star)} (\beta_\delta(p)(a + H_\delta(x,t,r,p,X))) = (\beta(p)(a + \tilde{G}_1(x,t,r,p,X)))_\star^{(\star)}$$

when the smooth well is considered and that

$$\begin{aligned} &\lim_{\delta\to 0_\star} (a + H_\delta(x,t,r,p,X)) \le 0 \\ &\qquad \Rightarrow \max(r-1, \min(r+1, a + G_1(x,t,r,p,X)))_\star \le 0 \end{aligned}$$

and

$$\begin{aligned} &\lim_{\delta\to 0}{}^\star (a + H_\delta(x,t,r,p,X)) \ge 0 \\ &\qquad \Rightarrow \max(r-1, \min(r+1, a + G_1(x,t,r,p,X)))^\star \ge 0 \end{aligned}$$

when the double-obstacle problem is considered. Taking a uniformly convergent subsequence, passing to the limits, and applying [9, Lemma 6.1], we obtain a solution of (15) and its double-obstacle variant. Since only an estimate of $\|\varphi_0\|_{C^{0,1}(\mathbb{R}^n)}$ was required, we have proved the following existence theorem.

THEOREM 2.8. *For every periodic $\varphi_0 \in C^{0,1}(\mathbb{R}^n)$, there are periodic viscosity solutions $\varphi, \tilde{\varphi} \in H^{1/(n+1),1/2(n+1)}(\mathbb{R}^n \times [0,T])$ to the anisotropic double-obstacle Allen–Cahn and its smooth form, that is,*

$$\max\left(\varphi - 1, \min\left(\varphi + 1, \partial_t\varphi + \frac{1}{\varepsilon}G_\varepsilon(.,.,\varphi,\nabla\varphi, D^2\varphi)\right)\right) = 0 \quad \textit{in } \mathbb{R}^n \times\ ]0,T[,$$

$$\partial_t\tilde{\varphi} + \frac{1}{\varepsilon}\tilde{G}_\varepsilon(.,.,\tilde{\varphi},\nabla\tilde{\varphi}, D^2\tilde{\varphi}) = 0 \quad \textit{in } \mathbb{R}^n \times]0,T[,$$

and

$$\varphi(.,0) = \tilde{\varphi}(.,0) = \varphi_0 \quad \text{in } \mathbb{R}^n.$$

Here G_ε and $\tilde{G}_\varepsilon$ are defined in (14).

Remark 2.9. Taking $H = \frac{1}{\varepsilon} G_\varepsilon$ in Theorem 2.4, we observe that it satisfies all conditions except

$$\text{(v)} \quad G_{\varepsilon,\star}(x,t,r,0,0) = G_\varepsilon^\star(x,t,r,0,0).$$

In the next theorem, we impose an additional condition on the sub- or supersolution of Theorem 2.4 for $H = \frac{1}{\varepsilon} G_\varepsilon$ which is sufficient to establish comparison. To prove this theorem, we apply the following theorem, which is given in a more general version in [9, Theorem 8.3]. Since we apply it to the double-obstacle problem, we must state it with one detail slightly changed.

THEOREM 2.10. *Let $v \in \mathrm{USC}(\mathbb{R}^n \times]0,T[)$ and $w \in \mathrm{LSC}(\mathbb{R}^n \times]0,T[)$, and define $\Phi(t,x,y) := v(x,t) - w(y,t) - \alpha|x-y|^2$, where $\alpha \geq 0$. We suppose that $\Phi(t_0,x_0,y_0) = \sup_{]0,T[\,\times\mathbb{R}^n\times\mathbb{R}^n} \Phi$, where $(t_0,x_0,y_0) \in]0,T[\,\times \mathbb{R}^n \times \mathbb{R}^n$. We assume that there is an $r > 0$ such that for $\Gamma > 0$, $(a,p,X) \in \mathrm{P}^{2,+}v(x,t)$,*

$$|x-x_0| + |t-t_0| + |v(x,t) - v(x_0,t_0)| \leq r \quad \text{and} \quad |p| + \|X\| \leq \Gamma$$
$$\text{implies } a \leq C(\Gamma);$$

likewise, for $(a,p,X) \in \mathrm{P}^{2,-}w(y,t)$,

$$|y-y_0| + |t-t_0| + |w(y,t) - w(y_0,t_0)| \leq r \quad \text{and} \quad |p| + \|X\| \leq \Gamma$$
$$\text{implies } a \geq -C(\Gamma).$$

Then there are $a \in \mathbb{R}$ and $X, Y \in S(n)$ such that

$$(a, 2\alpha(x_0-y_0), X) \in \overline{\mathrm{P}}^{2,+}v(x_0,t_0),$$
$$(a, 2\alpha(x_0-y_0), Y) \in \overline{\mathrm{P}}^{2,-}w(y_0,t_0),$$

and

$$X \leq Y.$$

Moreover, when $\alpha = 0$, we get

$$X \leq 0 \leq Y.$$

Remark 2.11. We observe that the condition about the boundedness of the time derivative of v is satisfied if v is a subsolution of a parabolic equation. In this case, it is indeed sufficient to require merely that $|v(x,t)| \leq \Gamma$, as is done in [9, Theorem 8.3]. If v is a solution of a double-obstacle problem, this bound can only be concluded if $v(x,t) > -1$. Therefore, we have to replace $|v(x,t)| \leq \Gamma$ by $|v(x,t) - v(x_0,t_0)| \leq r$ since we will apply the theorem when $v(x_0,t_0) > -1$.

THEOREM 2.12. *We assume that v and w are periodic with the same period, that they are sub- and supersolutions, respectively, of*

$$\max\left(\varphi - 1, \min\left(\varphi + 1, \partial_t\varphi + \frac{1}{\varepsilon}G_\varepsilon(.,.,\varphi,\nabla\varphi,D^2\varphi)\right)\right) = 0 \quad \text{in } \mathbb{R}^n \times]0,T[,$$

and that $v(.,0) \leq w(.,0)$. Moreover, we assume that for $(a,0,X) \in \overline{\mathrm{P}}^{2,-}w(x_0,t_0)$ with $(x_0,t_0) \in \mathbb{R}^n \times]0,T[$ and $w(x_0,t_0) < 1$,

$$\text{(26)} \qquad \limsup_{p\to 0,\ p\neq 0} \left(\varepsilon\beta^\star a - \varepsilon tr(D^2A(p)X) - \frac{1}{\varepsilon}w(x_0,t_0) - \frac{\pi}{4}u(x_0,t_0)\right) \geq 0$$

holds for any $\inf\beta \le \beta^\star \le \sup\beta$*. Then the following comparison inequality holds:*

$$v \le w \quad \textit{in } \mathbb{R}^n \times [0,T[.$$

Proof. We prove the theorem by contradiction. Suppose there is $(\bar{x},\bar{t}) \in \mathbb{R}^n \times]0,T[$ such that $v(\bar{x},\bar{t}) > w(\bar{x},\bar{t})$. Choose T_0 so that $0 < \bar{t} < T_0 < T$. We know that v and w are upper and lower semicontinuous, respectively, so periodicity implies

$$\sup_{y\in\mathbb{R}^n} (v(y,T_0) - w(y,T_0)) < \infty;$$

hence for λ large enough, we have

$$\forall y \in \mathbb{R}^n\colon\ \exp(-\lambda t)(v(x,t) - w(x,t)) > \exp(-\lambda T_0)(v(y,T_0) - w(y,T_0)). \tag{27}$$

We assume additionally that $\lambda > C(\Lambda)\varepsilon^{-2}$.

We define

$$\tilde{v}(x,t) := \exp(-\lambda t)v(x,t)$$

and

$$\tilde{w}(x,t) := \exp(-\lambda t)w(x,t).$$

We conclude from (27) that

$$\sup_{\mathbb{R}^n\times[0,T_0]} (\tilde{v} - \tilde{w}) =: \delta > 0,$$

and, since $\tilde{v}$ and $\tilde{w}$ are upper and lower semicontinuous, respectively, that there exists $(x_0,t_0) \in \mathbb{R}^n \times [0,T_0]$ such that

$$\tilde{v}(x_0,t_0) - \tilde{w}(x_0,t_0) = \sup_{\mathbb{R}^n\times[0,T_0]} (\tilde{v} - \tilde{w}), \tag{28}$$

and, noting that $(\tilde{v} - \tilde{w})(.,0) \le 0$, for any such (x_0,t_0),

$$0 < t_0 < T_0. \tag{29}$$

For proving comparison for viscosity solutions, we proceed with the standard technique of doubling the number of space variables and penalizing this doubling. We define for $\alpha > 0$ and $\eta \in \mathbb{R}^n$ the upper semicontinuous functions

$$\begin{aligned} \Psi(t,x,y) &:= \tilde{v}(x,t) - \tilde{w}(y,t), \\ \Phi_\alpha(t,x,y) &:= \Psi(t,x,y) - \alpha|x-y|^2, \quad \text{and} \\ \Phi_{\alpha,\eta}(t,x,y) &:= \Psi(t,x,y) - \alpha|x-y-\eta|^2. \end{aligned} \tag{30}$$

We distinguish two cases and in each derive a contradiction.

Case (i). There is a $\mu > 0$ such that for all $|\eta| < \mu$ one of the maximum points (t_η, x_η, y_η) of $\Phi_{\alpha,\eta}$, that is

$$\Phi_{\alpha,\eta} \le \Phi_{\alpha,\eta}(t_\eta,x_\eta,y_\eta) \quad \text{in } [0,T_0] \times \mathbb{R}^n \times \mathbb{R}^n,$$

satisfies

$$x_\eta - y_\eta = \eta. \tag{31}$$

We define $f(\eta) := \sup_{x-y=\eta} \Psi$ and get

$$f(\eta) \geq \Psi(t_\eta, x_\eta, y_\eta) = \Phi_{\alpha,\eta}(t_\eta, x_\eta, y_\eta) \geq \sup_{x-y=\xi} \Phi_{\alpha,\eta} = f(\xi) - \alpha|\xi - \eta|^2$$

for all $\xi,\ \eta \in U_\mu(0)$. Therefore, $f : U_\mu(0) \to \mathbb{R}$ is constant and

$$\sup_{x-y=\eta} \Psi = f(\eta) = f(0) = \sup_{x=y} \Psi = \sup_{\mathbb{R}^n \times [0,T_0]} (\tilde{v} - \tilde{w}) = \tilde{v}(x_0, t_0) - \tilde{w}(x_0, t_0)$$

by (28). This yields

$$\forall x, y \in U_{\mu/2}(x_0)\colon \quad \forall t \in [0, T_0]\colon \quad \tilde{v}(x,t) - \tilde{w}(y,t) \leq \tilde{v}(x_0, t_0) - \tilde{w}(x_0, t_0).$$

Applying Theorem 2.10 to $\Phi = \Psi$, we get

$$\begin{aligned} \exists \tilde{a},\ \tilde{X},\ \tilde{Y}\colon \quad & (\tilde{a}, 0, \tilde{X}) \in \overline{\mathrm{P}}^{2,+} v(x_0, t_0), \\ & (\tilde{a}, 0, \tilde{Y}) \in \overline{\mathrm{P}}^{2,-} w(x_0, t_0), \\ & \tilde{X} \leq 0 \leq \tilde{Y}. \end{aligned}$$

Setting $a := \tilde{a} \exp(\lambda t_0)$, $X := \tilde{X} \exp(\lambda t_0)$, and $Y = \tilde{Y} \exp(\lambda t_0)$ and noting that $\exp(\lambda(t - t_0)) v(x_0, t_0) = (1 + \lambda(t - t_0)) v(x_0, t_0) + O(|t - t_0|^2)$, we have

$$\begin{aligned} (a + \lambda v(x_0, t_0), 0, X) &\in \overline{\mathrm{P}}^{2,+} v(x_0, t_0), \\ (a + \lambda w(x_0, t_0), 0, Y) &\in \overline{\mathrm{P}}^{2,-} w(x_0, t_0), \end{aligned}$$

and

$$X \leq 0 \leq Y.$$

Furthermore, the inequality $1 \geq v(x_0, t_0) > w(x_0, t_0) \geq -1$ holds. Since v and w are sub- and supersolutions, respectively, we obtain

$$\begin{aligned} a + \lambda v(x_0, t_0) + \frac{1}{\varepsilon} G_{\varepsilon,\star}(x_0, t_0, v(x_0, t_0), 0, X) \leq 0 \quad \text{and} \\ a + \lambda w(x_0, t_0) + \frac{1}{\varepsilon} G_\varepsilon^\star(x_0, t_0, w(x_0, t_0), 0, Y) \geq 0. \end{aligned} \tag{32}$$

Using the definition of the semicontinuous envelope, we conclude from (32) that

$$\begin{aligned} 0 &\geq \liminf_{p \to 0,\ p \neq 0} \left(\varepsilon \beta(p)(a + \lambda v(x_0, t_0)) - \varepsilon \operatorname{tr}(D^2 A(p) X) - \frac{1}{\varepsilon} v(x_0, t_0) - \frac{\pi}{4} u(x_0, t_0) \right) \\ &\geq \liminf_{p \to 0,\ p \neq 0} \left(\varepsilon \beta(p)(a + \lambda v(x_0, t_0)) - \frac{1}{\varepsilon} v(x_0, t_0) - \frac{\pi}{4} u(x_0, t_0) \right) \end{aligned}$$

since $X \leq 0$. Therefore,

$$0 \geq \varepsilon \beta^\star (a + \lambda v(x_0, t_0)) - \frac{1}{\varepsilon} v(x_0, t_0) - \frac{\pi}{4} u(x_0, t_0) \tag{33}$$

for some $\inf \beta \leq \beta^\star \leq \sup \beta$. Taking this $\beta^\star$ and $(\tilde{a}, 0, \tilde{Y}) \in \overline{\mathrm{P}}^{2,-} w(x_0, t_0)$ in (26),

(34)

$$\begin{aligned} 0 &\leq \limsup_{p \to 0,\ p \neq 0} (\varepsilon \beta^\star (a + \lambda w(x_0, t_0)) - \varepsilon \operatorname{tr} \left(D^2 A(p) Y \right) - \frac{1}{\varepsilon} w(x_0, t_0) - \frac{\pi}{4} u(x_0, t_0) \right) \\ &\leq \varepsilon \beta^\star (a + \lambda w(x_0, t_0)) - \frac{1}{\varepsilon} w(x_0, t_0) - \frac{\pi}{4} u(x_0, t_0) \end{aligned}$$

since $Y \geq 0$. From (33) and (34), we obtain

$$0 \geq \left(\varepsilon\beta^\star\lambda - \frac{1}{\varepsilon}\right)(v(x_0,t_0) - w(x_0,t_0)) > 0,$$

which is a contradiction.

Case (ii) For all $\mu > 0$, there is an $|\eta| < \mu$ such that one of the maximum points (t_η, x_η, y_η) of $\Phi_{\alpha,\eta}$, that is,

$$\Phi_{\alpha,\eta} \leq \Phi_{\alpha,\eta}(t_\eta, x_\eta, y_\eta) \quad \text{in } [0,T_0] \times \mathbb{R}^n \times \mathbb{R}^n,$$

satisfies

$$x_\eta - y_\eta \neq \eta. \tag{35}$$

From periodicity and upper semicontinuity, we obtain for a subsequence

$$\begin{gathered}(t_\eta, x_\eta, y_\eta) \to (t_\alpha, x_\alpha, y_\alpha) \in [0,T_0] \times \mathbb{R}^n \times \mathbb{R}^n, \quad \Phi_\alpha \leq \Phi_\alpha(t_\alpha, x_\alpha, y_\alpha) \quad \text{and} \\ \tilde{v}(x_\eta, t_\eta) - \tilde{w}(y_\eta, t_\eta) \to \tilde{v}(x_\alpha, t_\alpha) - \tilde{w}(y_\alpha . t_\alpha).\end{gathered} \tag{36}$$

From [9, Proposition 3.7], for a subsequence $\alpha \to \infty$, we get

$$\begin{gathered}(t_\alpha, x_\alpha, y_\alpha) \to (t_0, x_0, y_0) \in [0,T_0] \times \mathbb{R}^n \times \mathbb{R}^n, \quad x_0 = y_0, \\ \tilde{v}(x_\alpha, t_\alpha) - \tilde{w}(y_\alpha, t_\alpha) \to \tilde{v}(x_0, t_0) - \tilde{w}(x_0, t_0) = \sup_{\mathbb{R}^n \times [0,T_0]} (\tilde{v} - \tilde{w}).\end{gathered} \tag{37}$$

It follows from (28) that

$$\begin{aligned} &0 < t_0 < T_0; \\ \text{hence} \quad &0 < t_\alpha < T_0, \quad v(x_\alpha, t_\alpha) > w(y_\alpha, t_\alpha) \quad \text{for } \alpha \text{ large} \\ \text{and} \quad &0 < t_\eta < T_0, \quad v(x_\eta, t_\eta) > w(y_\eta, t_\eta) \quad \text{for } |\eta| \text{ small.}\end{aligned}$$

Applying Theorem 2.10 to $\Phi = \Phi_{\alpha,\eta}(.,.,. - \eta)$ yields, after multiplying with $\exp(\lambda t_0)$,

$$\begin{aligned} \exists a,\ X,\ Y: \quad &(a + \lambda v(x_\eta, t_\eta), p_\eta, X) \in \overline{\mathrm{P}}^{2,+} v(x_\eta, t_\eta), \\ &(a + \lambda w(y_\eta, t_\eta), p_\eta, Y) \in \overline{\mathrm{P}}^{2,-} w(y_\eta, t_\eta), \\ &X \leq Y, \\ &p_\eta = 2\alpha(x_\eta - y_\eta - \eta)\exp(\lambda t_\eta).\end{aligned}$$

It also holds that $1 \geq v(x_\eta, t_\eta) > w(y_\eta, t_\eta) \geq -1$. Since v and w are sub- and supersolutions, we obtain

$$\begin{aligned} 0 &\geq \varepsilon(a + \lambda v(x_\eta, t_\eta)) + G_\varepsilon(x_\eta, t_\eta, v(x_\eta, t_\eta), p_\eta, X) \\ &\quad - \varepsilon(a + \lambda w(y_\eta, t_\eta)) - G_\varepsilon(y_\eta, t_\eta, w(y_\eta, t_\eta), p_\eta, Y) \\ &\geq \left(\varepsilon\lambda - C(\Lambda)\frac{1}{\varepsilon}\right)(v(x_\eta, t_\eta) - w(y_\eta, t_\eta)) - C(\Lambda)|u(x_\eta, t_\eta) - u(y_\eta, t_\eta)| \\ &\to \left(\varepsilon\lambda - C(\Lambda)\frac{1}{\varepsilon}\right)(v(x_0, t_0) - w(x_0, t_0)) > 0,\end{aligned}$$

which is again a contradiction.

Therefore, $v \leq w$ in $\mathbb{R}^n \times [0, T[$. □

Remark 2.13. The conclusion of the above proposition remains true if (26) is replaced by the analogous condition for v. More precisely, instead of (26), we require for $(a, 0, X) \in \overline{\mathrm{P}}^{2,+}v(x_0, t_0)$ with $(x_0, t_0) \in \mathbb{R}^n \times]0, T[$ and $v(x_0, t_0) > -1$ that

$$\liminf_{p \to 0,\ p \neq 0} \left(\varepsilon\beta^\star a - \varepsilon \mathrm{tr}(D^2 A(p) X) - \frac{1}{\varepsilon} v(x_0, t_0) - \frac{\pi}{4} u(x_0, t_0) \right) \leq 0$$

for any $\inf \beta \leq \beta^\star \leq \sup \beta$.

This modified comparison principle is also valid for the smooth Allen–Cahn equation with obvious changes.

3. The distance function. As already pointed out in the introduction, we use a distance function that is induced by a Finsler metric for the construction of the sub- and supersolutions. This section is devoted to the presentation of some definitions and properties of the Finsler metric (see [4]) and the induced distance function. Most of these properties are known in the isotropic case; see [2]. However, Lemma 3.4 and the viscosity estimate of the time derivative of the distance function in Proposition 3.6 were not required in the isotropic case, but they will be used in section 4 for proving convergence in the anisotropic case.

3.1. The dual. We consider $\beta,\ A,\ B$, and u as in (8). As in [4], we define the dual of B,

$$B^\circ(q) := \sup\{qp \mid B(p) \leq 1\}. \tag{38}$$

B° is convex and homogeneous of degree one. Since $A \in \mathrm{C}^2(\mathbb{R}^\mathrm{n} - \{0\})$ is strictly convex, $B^\circ \in \mathrm{C}^2(\mathbb{R}^\mathrm{n} - \{0\})$ and satisfies

$$\begin{gathered} c_0(\Lambda) \leq B^\circ(q) \leq C(\Lambda) \quad \text{for } |q| = 1, \\ B(\nabla B^\circ(q)) = 1 \quad \text{for } q \neq 0, \\ \nabla B(\nabla B^\circ(q)) D^2 B^\circ(q) = 0 \quad \text{for } q \neq 0, \quad \text{and} \\ \|B^\circ\|_{C^2(B_2(0) - B_{1/2}(0))} \leq C(\Lambda). \end{gathered} \tag{39}$$

3.2. The distance function. We define a nonsymmetric metric d in $\mathbb{R}^n$ by

$$d(x, y) := B^\circ(x - y). \tag{40}$$

We easily obtain for $x, y, z \in \mathbb{R}^n$ that

$$\begin{gathered} d(x, z) \leq d(x, y) + d(y, z), \\ d(x + z, y + z) = d(x, y), \quad \text{and} \\ c_0(\Lambda)|x - y| \leq d(x, y) \leq C(\Lambda)|x - y|. \end{gathered} \tag{41}$$

Let $\omega \in \mathrm{LSC}(\mathbb{R}^n \times [0, T[)$ be a supersolution of

$$\begin{gathered} \beta(\nabla\omega)\partial_t\omega - \mathrm{tr}(B(\nabla\omega) D^2 B(\nabla\omega) D^2\omega) - B(\nabla\omega) u \geq 0 \quad \text{in } \mathbb{R}^n \times]0, T[, \quad \text{or} \\ \partial_t \omega + F^\star(., ., \nabla\omega, D^2\omega) \geq 0, \end{gathered} \tag{42}$$

where F is defined in (14). We define the distance function δ as follows:

$$\delta(x, t) := \inf_{y, \omega(y,t) \leq 0} d(x, y). \tag{43}$$

(41) implies

$$|\delta(x, t) - \delta(y, t)| \leq C(\Lambda)|x - y|. \tag{44}$$

We will prove some properties of δ; most of them are well known.

The next three lemmas were proved by Barles, Soner, and Souganidis in [2] in the isotropic case.

LEMMA 3.1. *$\omega_\infty := \chi_{[\omega>0]}$ is a supersolution of* (42).

Proof. From [7], we know that $\omega_\varepsilon := \min(1, \max(\frac{\omega}{\varepsilon}, 0))$ is a supersolution of (42). Because $\omega_\infty = \lim_{\varepsilon \to 0\star} \omega_\varepsilon$, defined in Definition 2.7, we get from [9, Lemma 6.1] that ω_∞ is a supersolution as well. ∎

LEMMA 3.2. *We define $\delta^k(x,t) := \inf_{y\in\mathbb{R}^n}(k\omega_\infty(y,t) + d(x,y))$. Then*

(i) $\delta^k = \min(\delta, k) \in \mathrm{LSC}(\mathbb{R}^n \times [0,T[)$ *and*

(ii) $\partial_t\delta^k + F^\star(.,.,\nabla\delta^k, D^2\delta^k) + C(\Lambda)|\nabla\delta^k|\delta^k \geq 0$ *in* $\mathbb{R}^n \times]0,T[$ *in the viscosity sense.*

Proof. (i) Trivially, we get

$$0 \leq \delta^k(x,t) \leq k\omega_\infty(x,t) + d(x,x) \leq k.$$

Second, if $\delta(x,t) = d(x,y)$ with $\omega(y,t) \leq 0$, then $\omega_\infty(y,t) = 0$ and

$$\delta^k(x,t) \leq d(x,y) = \delta(x,t);$$

hence

$$0 \leq \delta^k \leq \min(\delta, k).$$

Moreover, $\forall y \in \mathbb{R}^n : \ [\omega(y,t) \leq 0 \Rightarrow d(x,y) \geq \delta(x,t)]$. This yields

$$k\omega_\infty(y,t) + d(x,y) \geq \min(\delta(x,t), k);$$

hence

$$\delta^k \geq \min(\delta, k).$$

Now let $(x_j,t_j) \to (x_0,t_0) \in \mathbb{R}^n \times [0,T[$ and $\delta^k(x_j,t_j) = k\omega_\infty(y_j,t_j) + d(x_j,y_j)$. We obtain $d(x_j,y_j) \leq k$ and, for a subsequence, $y_j \to y_0$. This yields

$$\liminf_{j\to\infty} \delta^k(x_j,t_j) \geq k\omega_\infty(y_0,t_0) + d(x_0,y_0) \geq \delta^k(x_0,t_0).$$

(ii) Let $\varphi \in C^{2,1}(\mathbb{R}^n \times [0,T[)$ with $\delta^k - \varphi \geq (\delta^k - \varphi)(x_0,t_0)$, $0 < t_0 < T$, and $\delta^k(x_0,t_0) = k\omega_\infty(y_0,t_0) + d(x_0,y_0)$. Defining

$$\psi(y,t) := \varphi(y - y_0 + x_0, t), \quad \psi \in C^{2,1}(\mathbb{R}^n \times [0,T[),$$

we get, for $(y,t) \in \mathbb{R}^n \times]0,T[$ and $x := y - y_0 + x_0$,

$$\begin{aligned} k\omega_\infty(y,t) + d(x,y) - \psi(y,t) &\geq \delta^k(x,t) - \varphi(x,t) \\ &\geq \delta^k(x_0,t_0) - \varphi(x_0,t_0) = k\omega_\infty(y_0,t_0) + d(x_0,y_0) - \psi(y_0,t_0). \end{aligned} \tag{45}$$

Since $x - y = x_0 - y_0$, we get $d(x,y) = d(x_0,y_0)$, and (45) together with Lemma 3.1 yields

$$\partial_t\psi(y_0,t_0) + F^\star(y_0,t_0,\nabla\psi(y_0,t_0), D^2\psi(y_0,t_0)) \geq 0.$$

This implies

$$\begin{aligned}
&\partial_t\varphi(x_0,t_0)+F^\star(x_0,t_0,\nabla\varphi(x_0,t_0),D^2\varphi(x_0,t_0))\\
&\quad\ge F^\star(x_0,t_0,\nabla\varphi(x_0,t_0),D^2\varphi(x_0,t_0))-F^\star(y_0,t_0,\nabla\varphi(x_0,t_0),D^2\varphi(x_0,t_0))\\
&\quad=\beta(\nabla\varphi(x_0,t_0))^{-1}B(\nabla\varphi(x_0,t_0))(u(y_0,t_0)-u(x_0,t_0))\\
&\quad\ge -C(\Lambda)|\nabla\varphi(x_0,t_0)||x_0-y_0|\\
&\quad\ge -C(\Lambda)|\nabla\varphi(x_0,t_0)|\delta^k(x_0,t_0)
\end{aligned}$$

since $\delta^k(x_0,t_0)=k\omega_\infty(y_0,t_0)+d(x_0,y_0)\ge d(x_0,y_0)\ge c_0(\Lambda)|x_0-y_0|$. ◻

PROPOSITION 3.3. *δ is lower semicontinuous and is a viscosity supersolution of*

$$\partial_t\delta+F^\star(.,.,\nabla\delta,D^2\delta)+C(\Lambda)|\nabla\delta|\delta\ge 0\quad \textit{in } \mathbb{R}^n\times]0,T[.$$

Proof. Let $(x_j,t_j)\to(x_0,t_0)$ and choose $k>\delta(x_0,t_0)$. We obtain

$$\liminf_{j\to\infty}\delta(x_j,t_j)\ge\liminf_{j\to\infty}\delta^k(x_j,t_j)\ge\delta^k(x_0,t_0)=\delta(x_0,t_0),$$

where we have used the lower semicontinuity of δ^k, established in Lemma 3.2(i).

To prove that δ is a viscosity supersolution, we pass to the limit in Lemma 3.2(ii). According to [9, Lemma 6.1], viscosity supersolutions are preserved under the limit procedure defined in Definition 2.7. Hence it suffices to prove

$$\delta=\lim_{k\to\infty_\star}\delta^k. \tag{46}$$

From Lemma 3.2(i),

$$\delta=\lim_{k\to\infty}\delta^k\ge\lim_{k\to\infty_\star}\delta^k.$$

Conversely, let $(x_j,t_j)\to(x_0,t_0)$ and $k_j\to\infty$ with

$$\limsup_{j\to\infty}\delta^{k_j}(x_j,t_j)\le\left(\lim_{k\to\infty_\star}\delta^k\right)(x_0,t_0)+\tau\le\delta(x_0,t_0)+\tau$$

for some $\tau>0$. For j large, we get $\delta^{k_j}(x_j,t_j)<k_j$; hence

$$\delta(x_0,t_0)\le\liminf_{j\to\infty}\delta(x_j,t_j)=\liminf_{j\to\infty}\delta^{k_j}(x_j,t_j)\le\left(\lim_{k\to\infty_\star}\delta^k\right)(x_0,t_0)+\tau,$$

and (46) is established, concluding the proof. ◻

LEMMA 3.4. *For $x_0\in\mathbb{R}^n$ and $0\le t_0\le t_1<T$, the inequality*

$$\mu(\delta(x_0,t_1))\ge\mu(\delta(x_0,t_0))-C(\Lambda)(t_1-t_0) \tag{47}$$

holds, where $\mu(r):=\int_0^r\frac{s}{1+s}ds$.

Proof. The function μ, used below to define a subsolution for (51), appears in [7].

It suffices to prove the assertion when $\varrho:=\delta(x_0,t_0)>0$. We define $v(x,t):=\mu(\varrho)-\Gamma(t-t_0)-\mu(B^\circ(x_0-x))$ for some positive constant Γ chosen below, and we observe from Lemma 3.1 that $v\in C^{2,1}((\mathbb{R}^n-\{x_0\})\times[0,T[)$. For $x\ne x_0$, we get

$$\begin{aligned}
\partial_tv(x,t)&=-\Gamma,\\
\nabla v(x,t)&=\mu'(B^\circ(x_0-x))\nabla B^\circ(x_0-x)\\
&=B^\circ(x_0-x)\nabla B^\circ(x_0-x)(1+B^\circ(x_0-x))^{-1},\quad\text{and}\\
D^2v(x,t)&=-B^\circ(x_0-x)D^2B^\circ(x_0-x)(1+B^\circ(x_0-x))^{-1}\\
&\quad-\nabla B^\circ(x_0-x)\otimes\nabla B^\circ(x_0-x)(1+B^\circ(x_0-x))^{-1}\\
&\quad+B^\circ(x_0-x)\nabla B^\circ(x_0-x)\otimes\nabla B^\circ(x_0-x)(1+B^\circ(x_0-x))^{-2}.
\end{aligned} \tag{48}$$

We observe that $v \in C^{1,1}(\mathbb{R}^n \times [0,T[)$ and satisfies

(49)
$$\begin{gathered}\nabla v(x_0,t) = 0 \quad \text{and} \\ \|D^2 v(x,t)\| \leq C(\Lambda)(1 + B^\circ(x_0 - x))^{-1},\end{gathered}$$

where we have used (39). We conclude that for $x \neq x_0$,

$$\begin{aligned}&\partial_t v(x,t) + F(x,t,\nabla v(x,t), D^2 v(x,t)) \\ &\quad \leq -\Gamma + (1 + B^\circ(x_0 - x))^{-1} F(x,t,B^\circ(x_0 - x)\nabla B^\circ(x_0 - x), -C(\Lambda)I) \\ &\quad \leq -\Gamma + C(\Lambda)(1 + B^\circ(x_0 - x))^{-1} F(x,t,c_0(\Lambda)B^\circ(x_0 - x)\nabla B^\circ(x_0 - x), -I) \\ &\quad \leq -\Gamma + C(\Lambda) \leq 0\end{aligned}$$

when $\Gamma \geq C(\Lambda)$. Here we have used

(50)
$$F(x,t,p,-I) \leq C(\Lambda)(1 + |p|),$$

which can easily be derived from the definition of F and (8).

To prove that v is a subsolution of

(51)
$$\partial_t v + F_\star(.,.,\nabla v, D^2 v) \leq 0$$

on the whole $\mathbb{R}^n \times]0,T[$, we consider $\psi \in C^{2,1}(\mathbb{R}^n \times\,]0,T[)$ with

$$(v - \psi) \leq (v - \psi)(x_0, s_0)$$

for some $0 < s_0 < T$. We get from (49) that

$$\nabla\psi(x_0,s_0) = \nabla v(x_0,s_0) = 0.$$

Adding $|x - x_0|^4 + |t - s_0|^2$ to ψ, we may assume $(v-\psi)(x,t) < (v-\psi)(x_0,s_0)$ for all $(x,t) \neq (x_0,s_0)$. We define $\psi_\tau(x,t) := \psi(x,t) + \tau\ x * N$ for some $N \neq 0$. As $\tau \to 0$, we get

$$(v - \psi_\tau)(.,s_0) \leq (v - \psi_\tau)(x_\tau, s_0)$$

on a neighborhood $U(x_0)$ of x_0 and $x_\tau \to x_0$. This yields

$$\nabla v(x_\tau,s_0) = \nabla\psi_\tau(x_\tau,s_0) = \nabla\psi(x_\tau,s_0) + \tau N \neq \nabla\psi(x_\tau,s_0);$$

hence $x_\tau \neq x_0$. Furthermore, we have

$$D^2\psi(x_0,s_0) \leftarrow D^2\psi_\tau(x_\tau,s_0) \geq D^2 v(x_\tau,s_0) \geq -(1+B^\circ(x_0 - x_\tau))^{-1} C(\Lambda)I \to -C(\Lambda)I.$$

Using (50), we get

$$\begin{aligned}&\partial_t\psi(x_0,s_0) + F_\star(x_0,s_0,\nabla\psi(x_0,s_0), D^2\psi(x_0,s_0)) \\ &\quad \leq -\Gamma + F_\star(x_0,s_0,0,-C(\Lambda)I) \leq -\Gamma + C(\Lambda) \leq 0;\end{aligned}$$

hence (51) is established.

We define $U := \{x \in \mathbb{R}^n \mid B^\circ(x_0 - x) < \varrho\}$. For $x \in U$, we see that

$$d(x_0,x) = B^\circ(x_0 - x) < \varrho = \delta(x_0,t_0);$$

hence

$$\omega(x, t_0) > 0$$

and

$$v(x, t_0) \leq \mu(\varrho) \leq \mu(\varrho)\omega_\infty(x, t_0).$$

For $x \in \partial U$ and $t \in [t_0, T[$, we get $v(x,t) \leq 0 \leq \mu(\varrho)\omega_\infty(x,t)$. We conclude with the comparison principle (Theorem 2.4),

$$\forall x \in U: \quad \forall t \in [t_0, T[: \quad v(x,t) \leq \mu(\varrho)\omega_\infty(x,t). \tag{52}$$

We choose $y_0 \in [\omega(., t_1) \leq 0]$ such that $\delta(x_0, t_1) = B^\circ(x_0 - y_0)$. If $y_0 \notin U$, we see that $\delta(x_0, t_1) = B^\circ(x_0 - y_0) \geq \varrho = \delta(x_0, t_0)$ and

$$\mu(\delta(x_0, t_1)) \geq \mu(\varrho) \geq \mu(\delta(x_0, t_0)).$$

If $y_0 \in U$, we obtain from (52)

$$0 = \mu(\varrho)\omega_\infty(y_0, t_1) \geq v(y_0, t_1) = \mu(\varrho) - \Gamma(t_1 - t_0) - \mu(B^\circ(x_0 - y_0)).$$

Taking into account that $\varrho = \delta(x_0, t_0)$ and $B^\circ(x_0 - y_0) = \delta(x_0, t_1)$, (47) follows. ☐

PROPOSITION 3.5. *δ is continuous from below; that is, if $(x_j, t_j) \to (x_0, t_0)$ and $t_j \leq t_0 < T$, then*

$$\delta(x_j, t_j) \to \delta(x_0, t_0).$$

Proof. Because of the already established lower semicontinuity of δ, it suffices to show

$$\limsup_{j \to \infty} \delta(x_j, t_j) \leq \delta(x_0, t_0). \tag{53}$$

From (47), we get

$$\mu(\delta(x_0, t_0)) \geq \limsup_{j \to \infty} (\mu(\delta(x_0, t_j)) - C(\Lambda)(t_0 - t_j)) = \mu\left(\limsup_{j \to \infty} \delta(x_0, t_j)\right),$$

where μ is defined in (3.4), since μ is continuous and increasing. Since μ is strictly increasing, it follows that

$$\delta(x_0, t_0) \geq \limsup_{j \to \infty} \delta(x_0, t_j),$$

which yields (53) when taking into account that $|\delta(x_j, t_j) - \delta(x_0, t_j)| \leq C(\Lambda)|x_j - x_0| \to 0$. ☐

PROPOSITION 3.6. *δ is a viscosity supersolution of*

$$\begin{aligned} B(\nabla\delta) &\geq 1, & -B(\nabla\delta) &\geq -1, \\ -\nabla B(\nabla\delta) D^2\delta \nabla B(\nabla\delta) &\geq 0, & -D^2\delta &\geq -C(\Lambda)\delta^{-1} I, \end{aligned}$$

and

$$\partial_t \delta \geq -C(\Lambda)\left(1 + \frac{1}{\delta}\right)$$

in $[\delta > 0] \cap (\mathbb{R}^n \times]0,T[)$.

Proof. Let $\varphi \in C^{2,1}(\mathbb{R}^n \times]0,T[)$, $(x_0,t_0) \in \mathbb{R}^n \times]0,T[$ with $\delta(x_0,t_0) > 0$ and $(\delta - \varphi) \geq (\delta - \varphi)(x_0,t_0) = 0$ in $\mathbb{R}^n \times]0,T[$. We choose $y_0 \in [\omega(.,t_0) \leq 0]$ with

$$0 < \delta(x_0,t_0) = d(x_0,y_0) = B^\circ(x_0 - y_0);$$

note in particular that $x_0 \neq y_0$. For all $x \in \mathbb{R}^n$, we get $B^\circ(x - y_0) \geq \delta(x,t_0)$; hence

$$B^\circ(x - y_0) - \varphi(x,t_0) \geq B^\circ(x_0 - y_0) - \varphi(x_0,t_0).$$

Since $x_0 \neq y_0$ and $B^\circ \in C^2(\mathbb{R}^n - \{0\})$, we get

$$\nabla\varphi(x_0,t_0) = \nabla B^\circ(x_0 - y_0)$$

and

$$D^2\varphi(x_0,t_0) \leq D^2B^\circ(x_0 - y_0) \leq C(\Lambda)I|x_0 - y_0|^{-1} = C(\Lambda)\delta(x_0,t_0)^{-1}I,$$

where we have used the fact that D^2B° is homogeneous of degree -1. From (39), we get

$$B(\nabla\varphi(x_0,t_0)) = 1$$

and

$$-\nabla B(\nabla\varphi)D^2\varphi\nabla B(\nabla\varphi)(x_0,t_0) \geq 0.$$

From (3.4), for $0 \leq t < t_0$, we get

$$\begin{aligned}\mu(\varphi(x_0,t_0)) = \mu(\delta(x_0,t_0)) &\geq \mu(\delta(x_0,t)) - C(\Lambda)(t_0 - t)\\ &\geq \mu(\varphi(x_0,t)) - C(\Lambda)(t_0 - t),\end{aligned}$$

and

$$\begin{aligned}-C(\Lambda) &\leq \frac{\mu(\varphi(x_0,t)) - \mu(\varphi(x_0,t_0))}{t - t_0} \to \mu'(\varphi(x_0,t_0))\partial_t\varphi(x_0,t_0)\\ &= \frac{\varphi(x_0,t_0)}{1 + \varphi(x_0,t_0)}\partial_t\varphi(x_0,t_0).\end{aligned}$$

Since $\varphi(x_0,t_0) = \delta(x_0,t_0)$, this yields

$$\partial_t\varphi(x_0,t_0) \geq -C(\Lambda)(1 + \delta(x_0,t_0)^{-1}). \qquad \square$$

4. Convergence.

Remark 4.1. In this section, we construct sub- and supersolutions for the double-obstacle Allen–Cahn problem which satisfy the additional condition of the modified comparison principle (Theorem 2.12).

We consider β, A, B, and u as in (8) and ω and δ as in section 3.2.

DEFINITION 4.2. *We define the following auxiliary functions, which appear in formal asymptotics for the double-obstacle Allen–Cahn equation:*

$$\psi_0(r) := \begin{cases} 1, & r \geq \dfrac{\pi}{2},\\ \sin(r), & |r| \leq \dfrac{\pi}{2},\\ -1, & r \leq -\dfrac{\pi}{2}\end{cases}$$

and

$$\psi_1(r) := \begin{cases} \frac{1}{2}(r\psi_0(r) - \frac{\pi}{2} + \psi_0'(r)), & |r| \le \frac{\pi}{2}, \\ 0, & |r| \ge \frac{\pi}{2}. \end{cases}$$

We see that $\psi_0,\ \psi_1 \in \mathrm{C}^2([-\frac{\pi}{2}, \frac{\pi}{2}]) \cap \mathrm{C}^{1,1}(\mathbb{R})$. *Moreover,*

$$\begin{aligned} &\psi_0'' + \psi_0 = 0 \quad \text{on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad \text{and} \\ &\psi_1'' + \psi_1 = \psi_0' - \frac{\pi}{4} \quad \text{on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned} \tag{54}$$

We define

$$\psi(r,v) := \psi_0(r) + \varepsilon v \psi_1(r)$$

and

$$\lambda_\varepsilon(x,t) := \frac{\delta(x,t)}{\varepsilon} - \pi - f(t),$$

where $f(t) := \alpha \exp(-\gamma^2 t)$ *with* $1 \le \alpha,\ \gamma \le C(\Lambda)$ *chosen below. We set* $v := u + \varepsilon g$, *where* $g(t) := \alpha\gamma \exp(-\gamma^2 t)$, *and*

$$\psi_\varepsilon := \psi(\lambda_\varepsilon, v).$$

PROPOSITION 4.3. *For* $0 < \varepsilon < \varepsilon_0(\Lambda)$, ψ_ε *is a supersolution of*

$$\max\left(\psi_\varepsilon - 1, \min\left(\psi_\varepsilon + 1, \partial_t \psi_\varepsilon + \frac{1}{\varepsilon} G_\varepsilon^\star(.,.,\psi_\varepsilon, \nabla\psi_\varepsilon, D^2\psi_\varepsilon)\right)\right) \ge 0 \quad \text{in } \mathbb{R}^n \times]0,T[. \tag{55}$$

Moreover, ψ_ε *satisfies the additional condition* (26) *of the modified comparison principle*; *that is, for* $(a,0,X) \in \overline{\mathrm{P}}^{2,-}\psi_\varepsilon(x_0,t_0)$, $0 < t_0 < T$, *and* $\psi_\varepsilon(x_0,t_0) < 1$,

$$\limsup_{p\to 0,\ p\ne 0} \left(\varepsilon\beta^* a - \varepsilon tr(D^2A(p)X) - \frac{1}{\varepsilon}\psi_\varepsilon(x_0,t_0) - \frac{\pi}{4}u(x_0,t_0)\right) \ge 0 \tag{56}$$

holds for any $\inf\beta \le \beta^\star \le \sup\beta$.

Proof. We take ε so small that $\varepsilon f,\ \varepsilon g,\ \varepsilon|f'|,\ \varepsilon|g'| \le 1$. We have $|\psi_1(r)|,\ |\psi_1'(r)| \le \psi_0'(r)$. For $|v| \le C(\Lambda)$, this yields

$$\psi_r(r,v) = \psi_0'(r) + \varepsilon v \psi_1'(r) = \psi_0'(r)(1 + O_\Lambda(\varepsilon));$$

hence

$$\frac{1}{2}\psi_0'(r) \le \psi_r(r,v) \le 2\psi_0'(r) \tag{57}$$

when $0 < \varepsilon < \varepsilon_0(\Lambda)$. Therefore, $\psi_\varepsilon \in \mathrm{LSC}(\mathbb{R}^n \times [0,T[)$ since $\delta \in \mathrm{LSC}(\mathbb{R}^n \times [0,T[)$. Since $\psi(r,v) = -1$ for $r \le -\frac{\pi}{2}$, we get

$$\psi_\varepsilon \ge -1.$$

Suppose $(\bar{x},\bar{t}) \in \mathbb{R}^n \times]0,T[$ is such that $\psi_\varepsilon(\bar{x},\bar{t}) < 1$. It follows that $\lambda_\varepsilon(\bar{x},\bar{t}) < \frac{\pi}{2}$. From Remark 2.3, we see that for any $(\bar{a},\bar{p},\bar{X}) \in \mathrm{P}^{2,-}\psi_\varepsilon(\bar{x},\bar{t})$, there is a $\varphi \in C^{2,1}(\mathbb{R}^n \times [0,T[)$ such that

$$(58) \qquad \begin{aligned} &(\bar{a},\bar{p},\bar{X}) = (\partial_t\varphi, \nabla\varphi, D^2\varphi)(\bar{x},\bar{t}) \quad \text{and} \\ &\psi_\varepsilon - \varphi \geq (\psi_\varepsilon - \varphi)(\bar{x},\bar{t}), \end{aligned}$$

and, without loss of generality, $(\psi_\varepsilon - \varphi)(\bar{x},\bar{t}) = 0$ and $(\bar{x},\bar{t})$ is a strict minimum. Our aim is to prove that ψ_ε is a supersolution. By Definition 2.1, it remains to show that

$$(59) \qquad \bar{a} + \frac{1}{\varepsilon} G^\star_\varepsilon(\bar{x},\bar{t},\psi_\varepsilon(\bar{x},\bar{t}),\bar{p},\bar{X})) \geq 0.$$

We distinguish two cases.

(i) $\lambda_\varepsilon(\bar{x},\bar{t}) < -\frac{\pi}{2}$. Using the continuity from below in time of δ (see Proposition 3.5), we conclude that

$$\lambda_\varepsilon(x,t) < -\frac{\pi}{2} \quad \text{for } (x,t) \in U(\bar{x},\bar{t}) \text{ and } t \leq \bar{t},$$

and for these (x,t), it follows that $\psi_\varepsilon(x,t) = -1$. This yields

$$\bar{X} = D^2\varphi(\bar{x},\bar{t}) \leq 0, \qquad \bar{a} = \partial_t\varphi(\bar{x},\bar{t}) \geq 0,$$

and

$$\bar{p} = \nabla\varphi(\bar{x},\bar{t}) = 0.$$

For $p \neq 0$ and $\inf\beta \leq \beta^\star \leq \sup\beta$, we obtain

$$(60) \qquad \varepsilon\beta^\star\bar{a} - \varepsilon\mathrm{tr}(D^2A(p)\bar{X}) - \frac{1}{\varepsilon}\psi_\varepsilon(\bar{x},\bar{t}) - \frac{\pi}{4}u(\bar{x},\bar{t}) \geq \left(\frac{1}{\varepsilon} - C(\Lambda)\right) \geq 0$$

when $0 < \varepsilon < \varepsilon_0(\Lambda)$. Taking $\beta^\star = \beta(p)$ and letting p tend to $\bar{p} = 0$, we obtain

$$\begin{aligned} &\bar{a} + \frac{1}{\varepsilon}G^\star_\varepsilon(\bar{x},\bar{t},\psi_\varepsilon(\bar{x},\bar{t}),\bar{p},\bar{X}) \\ &\quad \geq \limsup_{p\to 0, p\neq 0}\left(\bar{a} + \frac{1}{\varepsilon}G_\varepsilon(\bar{x},\bar{t},\psi_\varepsilon(\bar{x},\bar{t}),p,\bar{X})\right) \\ &\quad = \limsup_{p\to 0, p\neq 0}\left(\bar{a} - \frac{1}{\varepsilon\beta(p)}\left(\varepsilon\mathrm{tr}(D^2A(p)\bar{X}) + \frac{1}{\varepsilon}\psi_\varepsilon(\bar{x},\bar{t}) + \frac{\pi}{4}u(\bar{x},\bar{t})\right)\right) \geq 0, \end{aligned}$$

which is (59).

(ii) $-\frac{\pi}{2} \leq \lambda_\varepsilon(\bar{x},\bar{t}) < \frac{\pi}{2}$. In the next two subsections, we will establish the existence of subsequences (x_τ,t_τ) and $\psi^\tau_\varepsilon(x_\tau,t_\tau)$ such that as $\tau \to 0$,

$$(61) \qquad \begin{aligned} &(x_\tau,t_\tau) \to (\bar{x},\bar{t}), \\ &\psi^\tau_\varepsilon(x_\tau,t_\tau) \to \psi_\varepsilon(\bar{x},\bar{t}), \\ &\nabla\varphi(x_\tau,t_\tau) \neq 0, \end{aligned}$$

and

$$R^\tau_\varepsilon := \left(\varepsilon\beta^\star\partial_t\varphi - \varepsilon\mathrm{tr}(D^2A(\nabla\varphi)D^2\varphi) - \frac{1}{\varepsilon}\psi^\tau_\varepsilon - \frac{\pi}{4}u)(x_\tau,t_\tau\right) \geq \varepsilon - \varepsilon^{-1}\varrho_\Lambda(\tau),$$

where $\inf\beta \le \beta^\star \le \sup\beta$ and $\beta^\star = \beta(\nabla\varphi(x_\tau, t_\tau))$. Furthermore, if $|\nabla\varphi(x_\tau, t_\tau)| \le \Lambda\varepsilon$, then the last inequality holds for any $\inf\beta \le \beta^\star \le \sup\beta$. Here $\varrho_\Lambda(\tau) \to 0$.

Taking $\beta^\star = \beta(\nabla\varphi(x_\tau, t_\tau))$ in the definition of R_ε^τ, we find from (61) that

$$\begin{aligned}&\varepsilon\partial_t\varphi(\bar{x},\bar{t}) + G_\varepsilon^\star(\bar{x},\bar{t},\psi_\varepsilon(\bar{x},\bar{t}),\nabla\varphi(\bar{x},\bar{t}),D^2\varphi(\bar{x},\bar{t}))\\ &\quad\ge \limsup_{\tau\to 0}(\varepsilon\partial_t\varphi(x_\tau,t_\tau) + G_\varepsilon(x_\tau,t_\tau,\psi_\varepsilon^\tau(x_\tau,t_\tau),\nabla\varphi(x_\tau,t_\tau),D^2\varphi(x_\tau,t_\tau)))\\ &\quad= \limsup_{\tau\to 0}(\beta(\nabla\varphi(x_\tau,t_\tau))^{-1}R_\varepsilon^\tau) \ge 0,\end{aligned}$$

which is (59). Thus ψ_ε is a supersolution.

We now turn to the proof of (56). First, we observe that for any $(\bar{x},\bar{t}) \in \mathbb{R}^n \times]0,T[$ with $\psi_\varepsilon(\bar{x},\bar{t}) < 1$ and φ satisfying (58) with $|\bar{p}| = |\nabla\varphi(\bar{x},\bar{t})| \le \Lambda\varepsilon$,

$$\limsup_{q\to\bar{p},q\ne 0}\left(\varepsilon\beta^\star\bar{a} - \varepsilon\mathrm{tr}(D^2A(q)\bar{X}) - \frac{1}{\varepsilon}\psi_\varepsilon(\bar{x},\bar{t}) - \frac{\pi}{4}u(\bar{x},\bar{t})\right) \ge 0 \tag{62}$$

for $\inf\beta \le \beta^\star \le \sup\beta$.

To prove (62), we again distinguish two cases. In case (i), $\lambda_\varepsilon(\bar{x},\bar{t}) < -\frac{\pi}{2}$, (62) is an immediate consequence of (60). In case (ii), $-\frac{\pi}{2} \le \lambda_\varepsilon(\bar{x},\bar{t}) < -\frac{\pi}{2}$, it follows from (61), for any $\inf\beta \le \beta^\star \le \sup\beta$, that

$$\begin{aligned}&\limsup_{q\to\bar{p},q\ne 0}\left(\varepsilon\beta^\star\bar{a} - \varepsilon\mathrm{tr}(D^2A(q)\bar{X}) - \frac{1}{\varepsilon}\psi_\varepsilon(\bar{x},\bar{t}) - \frac{\pi}{4}u(\bar{x},\bar{t})\right)\\ &\quad\ge \limsup_{q\to\nabla\varphi(\bar{x},\bar{t}),q\ne 0}\left(\varepsilon\beta^\star\partial_t\varphi(\bar{x},\bar{t}) - \varepsilon\mathrm{tr}(D^2A(q)D^2\varphi(\bar{x},\bar{t})) - \frac{1}{\varepsilon}\psi_\varepsilon(\bar{x},\bar{t}) - \frac{\pi}{4}u(\bar{x},\bar{t})\right)\\ &\quad\ge \limsup_{\tau\to 0} R_\varepsilon^\tau \ge 0,\end{aligned}$$

which is (62).

Now we consider $(a,0,X) \in \overline{\mathrm{P}}^{2,-}\psi_\varepsilon(x_0,t_0)$ with $\psi_\varepsilon(x_0,t_0) < 1$. From the definition of $\overline{\mathrm{P}}^{2,-}$, we get $(a_j,p_j,X_j) \in \mathrm{P}^{2,-}\psi_\varepsilon(x_j,t_j)$, which, as $j\to\infty$, satisfies

$$(a_j,p_j,X_j) \to (a,0,X),$$
$$(x_j,t_j) \to (x_0,t_0),$$

and

$$\psi_\varepsilon(x_j,t_j) \to \psi_\varepsilon(x_0,t_0).$$

We apply (62) to $(\bar{x},\bar{t}) = (x_j,t_j)$ and obtain

$$\limsup_{q\to p_j,\ q\ne 0}\left(\varepsilon\beta^\star a_j - \varepsilon\mathrm{tr}(D^2A(q)X_j) - \frac{1}{\varepsilon}\psi_\varepsilon(x_j,t_j) - \frac{\pi}{4}u(x_j,t_j)\right) \ge 0,$$

which yields the existence of $q_j \ne 0$ with $q_j \to 0$ and

$$\varepsilon\beta^\star a_j - \varepsilon\mathrm{tr}(D^2A(q_j)X_j) - \frac{1}{\varepsilon}\psi_\varepsilon(x_j,t_j) - \frac{\pi}{4}u(x_j,t_j) \ge -\frac{1}{j}.$$

From this we infer (56), concluding the proof. □

4.1. Approximation. In this section, we approximate ψ_ε by a smooth ψ_ε^τ, and we will get $(x_\tau, t_\tau) \to (\bar{x}, \bar{t})$ as in (61).

We take the notation of the preceding subsection, and we assume $-\frac{\pi}{2} \leq \lambda_\varepsilon(\bar{x}, \bar{t}) < \frac{\pi}{2}$.

From the definition of λ_ε, it follows that $\frac{\pi}{2}\varepsilon < \delta(\bar{x}, \bar{t})$, and from the lower semicontinuity of δ (see Proposition 3.3), we get

$$\begin{aligned} \delta &\geq \frac{\pi}{2}\varepsilon, \\ \lambda_\varepsilon &\geq -\pi \end{aligned} \quad \text{in } \overline{U(\bar{x}, \bar{t})} \tag{63}$$

for some neighborhood $U(\bar{x}, \bar{t})$ of $(\bar{x}, \bar{t})$. We take a Dirac sequence $\eta_\tau(r) = \tau^{-1}\eta(\frac{r}{\tau})$ for $\eta \in C_0^\infty(\mathbb{R}),\ \eta \geq 0, \int \eta = 1$, and define

$$\begin{aligned} \psi_0^\tau(r) &:= \int \psi_0(s)\eta_\tau(r-s)ds, \\ \psi_1^\tau(r) &:= \int \psi_1(s)\eta_\tau(r-s)ds. \end{aligned}$$

We get $(\psi_0^\tau)',\ \psi_1^\tau \in C_0^\infty(\mathbb{R})$, and

$$|\psi_1^\tau|,\ |(\psi_1^\tau)'| \leq C(\Lambda)(\psi_0^\tau)'. \tag{64}$$

We define

$$\psi^\tau(r, v) := \psi_0^\tau(r) + \varepsilon v \psi_1^\tau(r) + \tau r,$$

and for $|v| \leq C(\Lambda),\ 0 < \varepsilon < \varepsilon_0(\Lambda)$, we get

$$\psi_r^\tau(r, v) \geq \tau > 0. \tag{65}$$

We choose $u^\tau \in C^\infty(\mathbb{R}^n \times [0, T])$ with

$$\|u^\tau\|_{C^{2,1}(\mathbb{R} \times [0,T])} \leq C(\Lambda)$$

and

$$\|u^\tau - u\|_{L^\infty(\mathbb{R}^n \times [0,T])} \leq \varrho(\tau) \to 0 \quad \text{for } \tau \to 0.$$

We define $v^\tau := u^\tau + \varepsilon g$ and get

$$\|v^\tau\|_{C^{2,1}(\mathbb{R}^n \times [0,T])} \leq C(\Lambda).$$

We set

$$\psi_\varepsilon^\tau := \psi^\tau(\lambda_\varepsilon, v^\tau).$$

Using the above approximation properties in τ and (63), we obtain $\varrho_\Lambda(\tau) \to 0$ as $\tau \to 0$ such that

$$\begin{aligned} \psi_\varepsilon^\tau - \psi_\varepsilon \geq & -\|\psi_0^\tau - \psi_0\|_{L^\infty(\mathbb{R})} - \varepsilon\|v^\tau - v\|_{L^\infty(\mathbb{R}^n \times [0,T])}\|\psi_1^\tau\|_{L^\infty(\mathbb{R})} \\ & - \varepsilon\|v\|_{L^\infty(\mathbb{R}^n \times [0,T])}\|\psi_1^\tau - \psi_1\|_{L^\infty(\mathbb{R})} - \tau\pi \\ \geq & -\varrho_\Lambda(\tau) \end{aligned}$$

on $\overline{U(\bar{x}, \bar{t})}$ and

$$|\psi_\varepsilon^\tau(\bar{x}, \bar{t}) - \psi_\varepsilon(\bar{x}, \bar{t})| \leq \varrho_\Lambda(\tau).$$

Therefore, for small τ,

$$\begin{aligned} &\exists (x_\tau, t_\tau) \in U(\bar{x}, \bar{t}): \quad \psi_\varepsilon^\tau - \varphi \geq (\psi_\varepsilon^\tau - \varphi)(x_\tau, t_\tau) \quad \text{on } U(\bar{x}, \bar{t}), \\ &(x_\tau, t_\tau) \to (\bar{x}, \bar{t}) \quad \text{as } \tau \to 0, \quad \text{and} \\ &(\psi_\varepsilon^\tau - \varphi)(x_\tau, t_\tau) =: \nu_\tau \to (\psi_\varepsilon - \varphi)(\bar{x}, \bar{t}) = 0. \end{aligned} \tag{66}$$

The last convergence yields

$$\begin{aligned} 1 > \psi_\varepsilon(\bar{x}, \bar{t}) &= \lim_{\tau \to 0} \psi_\varepsilon^\tau(x_\tau, t_\tau) \\ &\geq \limsup_{\tau \to 0} \psi(\lambda_\varepsilon(x_\tau, t_\tau), v(x_\tau, t_\tau)) \geq \psi\left(\limsup_{\tau \to 0} \lambda_\varepsilon(x_\tau, t_\tau), v(\bar{x}, \bar{t})\right); \end{aligned}$$

hence

$$\lambda_\varepsilon(x_\tau, t_\tau) < \frac{\pi}{2} \quad \text{for } \tau \text{ small.} \tag{67}$$

Because of (65) and since ψ^τ is smooth, there is $\delta^\tau \in C^{2,1}(U(\bar{x}, \bar{t}))$ such that

$$\begin{aligned} \varphi(x, t) &= \psi^\tau(\lambda_\varepsilon^\tau(x, t), v^\tau(x, t)) - \nu_\tau \quad \text{and} \\ \lambda_\varepsilon^\tau(x, t) &= \frac{\delta^\tau(x, t)}{\varepsilon} - \pi - f(t), \end{aligned} \tag{68}$$

where ν_τ is defined in (66). For $(x, t) \in U(\bar{x}, \bar{t})$, we obtain

$$\psi^\tau(\lambda_\varepsilon, v^\tau) - \varphi(x, t) = (\psi_\varepsilon^\tau - \varphi)(x, t) \geq \nu_\tau = (\psi^\tau(\lambda_\varepsilon^\tau, v^\tau) - \varphi)(x, t),$$

and equality holds for $(x, t) = (x_\tau, t_\tau)$. From (65), we conclude

$$\begin{aligned} \delta &\geq \delta^\tau \quad \text{in } U(\bar{x}, \bar{t}) \quad \text{and} \\ \delta(x_\tau, t_\tau) &= \delta^\tau(x_\tau, t_\tau) \geq \frac{\pi}{2}\varepsilon, \end{aligned} \tag{69}$$

where we have used (63).

4.2. Computation. In this section, we will carry out the computations to establish the fourth line of (61). We again use the notation of the preceding sections.

We continue from (69). Observing Remark 2.3, we have that

$$(\partial_t \delta^\tau, \nabla \delta^\tau, D^2 \delta^\tau)(x_\tau, t_\tau) \in \mathrm{P}^{2,-}\delta(x_\tau, t_\tau),$$

and from the definition of supersolutions, using Proposition 3.6, we obtain

$$\begin{aligned} B(\nabla \delta^\tau(x_\tau, t_\tau)) = 1, \qquad c_0(\Lambda) &\leq |\nabla \delta^\tau(x_\tau, t_\tau)| \leq C(\Lambda), \\ -\nabla B(\nabla \delta^\tau) D^2 \delta^\tau \nabla B(\nabla \delta^\tau)&(x_\tau, t_\tau) \geq 0, \\ D^2 \delta^\tau(x_\tau, t_\tau) \leq C(\Lambda) \delta(x_\tau, t_\tau)^{-1} I &\leq C(\Lambda) \varepsilon^{-1} I, \quad \text{and} \\ \partial_t \delta^\tau(x_\tau, t_\tau) \geq -C(\Lambda)(1 + \delta(x_\tau, t_\tau)^{-1}) &\geq -C(\Lambda) \varepsilon^{-1}. \end{aligned} \tag{70}$$

Further, from Proposition 3.3, we get

$$(\beta(\nabla \delta^\tau) \partial_t \delta^\tau - \operatorname{tr}(B(\nabla \delta^\tau) D^2 B(\nabla \delta^\tau) D^2 \delta^\tau) - B(\nabla \delta^\tau) u + C(\Lambda) |\nabla \delta^\tau| \delta)(x_\tau, t_\tau) \geq 0,$$

and, since $D^2 A = B D^2 B + B' \otimes B'$, we conclude

$$(\beta(\nabla \delta^\tau) \partial_t \delta^\tau - \operatorname{tr}(D^2 A(\nabla \delta^\tau) D^2 \delta^\tau) - u + C(\Lambda) \delta)(x_\tau, t_\tau) \geq 0. \tag{71}$$

From (68), we compute

$$\nabla\varphi = \varepsilon^{-1}\psi_r^\tau \nabla\delta^\tau + \varepsilon\psi_1^\tau \nabla v^\tau = \varepsilon^{-1}\psi_r^\tau(\nabla\delta^\tau + O_\Lambda(\varepsilon^2)). \tag{72}$$

From (65) and $|\nabla\delta^\tau(x_\tau, t_\tau)| \geq c_0(\Lambda)$, we get

$$\nabla\varphi(x_\tau, t_\tau) \neq 0, \tag{73}$$

which is the third line of (61).

Differentiating again, we get

$$\begin{aligned} D^2\varphi &= \varepsilon^{-2}\psi_{rr}^\tau \nabla\delta^\tau \otimes \nabla\delta^\tau + \varepsilon^{-1}\psi_r^\tau D^2\delta^\tau \\ &\quad + (\psi_1^\tau)'(\nabla v^\tau \otimes \nabla\delta^\tau + \nabla\delta^\tau \otimes \nabla v^\tau) + \varepsilon\psi_1^\tau D^2 v^\tau \\ &= \varepsilon^{-2}\psi_{rr}^\tau \nabla\delta^\tau \otimes \nabla\delta^\tau + \varepsilon^{-1}\psi_r^\tau D^2\delta^\tau + O_\Lambda(1) \end{aligned} \tag{74}$$

since $|\nabla\delta^\tau(x_\tau, t_\tau)| \leq C(\Lambda)$.

Further, we obtain

$$\varepsilon\partial_t\varphi = \psi_r^\tau(\partial_t\delta^\tau - \varepsilon f') + \varepsilon^2\psi_1^\tau\partial_t v^\tau = \psi_r^\tau(\partial_t\delta^\tau - \varepsilon f' + O_\Lambda(\varepsilon^2)). \tag{75}$$

We recall the definition of R_ε^τ in (61),

$$R_\varepsilon^\tau := \left(\varepsilon\beta^\star\partial_t\varphi - \varepsilon\,\mathrm{tr}(D^2A(\nabla\varphi)D^2\varphi) - \frac{1}{\varepsilon}\psi_\varepsilon^\tau - \frac{\pi}{4}u\right)(x_\tau, t_\tau) \tag{76}$$

for $\inf\beta \leq \beta^\star \leq \sup\beta$ and $\beta^\star = \beta(\nabla\varphi(x_\tau, t_\tau))$ or $|\nabla\varphi(x_\tau, t_\tau)| \leq \Lambda\varepsilon$.

Using (72), (74), and (75), we get

$$\begin{aligned} R_\varepsilon^\tau \geq &-\varepsilon^{-1}(\psi_{rr}^\tau \nabla\delta^\tau D^2A(\nabla\varphi)\nabla\delta^\tau + \psi^\tau) \\ &+ \psi_r^\tau\beta^\star\partial_t\delta^\tau - \frac{\pi}{4}u \\ &- \psi_r^\tau\mathrm{tr}(D^2A(\nabla\varphi)D^2\delta^\tau) \\ &- \varepsilon\psi_r^\tau\beta^\star f' - C(\Lambda)\varepsilon. \end{aligned} \tag{77}$$

Using the homogeneity of A and β, (8), and (72), we obtain

$$\begin{aligned} D^2A(\nabla\varphi) &= D^2A(\nabla\delta^\tau + O_\Lambda(\varepsilon^2)) = D^2A(\nabla\delta^\tau) + O_\Lambda(\varepsilon^2), \\ \nabla\delta^\tau D^2A(\nabla\delta^\tau)\nabla\delta^\tau &= 2A(\nabla\delta^\tau) = B(\nabla\delta^\tau)^2 = 1, \quad \text{and} \\ \beta(\nabla\varphi) &= \beta(\nabla\delta^\tau + O_\Lambda(\varepsilon^2)) = \beta(\nabla\delta^\tau) + O_\Lambda(\varepsilon^2). \end{aligned} \tag{78}$$

From (77) and (78), it follows that

$$\begin{aligned} R_\varepsilon^\tau \geq &-\varepsilon^{-1}(\psi_{rr}^\tau + \psi^\tau) \\ &+ \psi_r^\tau(\beta^\star\partial_t\delta^\tau - \mathrm{tr}(D^2A(\nabla\varphi)D^2\delta^\tau) - u) \\ &+ \left(\psi_r^\tau - \frac{\pi}{4}\right)u \\ &- \varepsilon c_0(\Lambda)\psi_r^\tau f' - C(\Lambda)\varepsilon, \end{aligned} \tag{79}$$

where again $|\nabla\delta^\tau(x_\tau, t_\tau)| \leq C(\Lambda)$ was used.

In the following lemma, we will prove that

$$\psi_r^\tau(\beta^\star\partial_t\delta^\tau - \mathrm{tr}(D^2A(\nabla\varphi)D^2\delta^\tau) - u) \geq -C(\Lambda)(\delta + \varepsilon). \tag{80}$$

From (63) and (67), we know that $|\lambda_\varepsilon(x_\tau,t_\tau)| \le \pi$. This yields $\delta(x_\tau,t_\tau) \le \varepsilon(\lambda_\varepsilon(x_\tau,t_\tau) + \pi + f(t_\tau)) \le C(\Lambda)\varepsilon(1+f(t_\tau))$. Plugging (80) in (79), we obtain

$$
\begin{aligned}
R_\varepsilon^\tau \ge & -\varepsilon^{-1}(\psi_{rr}^\tau + \psi^\tau) \\
& + \left(\psi_r^\tau - \frac{\pi}{4}\right) u \\
& - \varepsilon \psi_r^\tau c_0(\Lambda) f' - C(\Lambda)\varepsilon(1+f) \\
\ge & -\varepsilon^{-1}(\psi_{rr}^\tau + \psi^\tau) \\
& + \left((\psi_0^\tau)' - \frac{\pi}{4}\right) u \\
& - \varepsilon (\psi_0^\tau)' c_0(\Lambda) f' - C(\Lambda)\varepsilon(1+f)
\end{aligned} \tag{81}
$$

since $|\varepsilon f'|, |\varepsilon f| \le 1$.

Setting $r := \lambda_\varepsilon(x_\tau,t_\tau)$, we compute

$$
\begin{aligned}
&-\varepsilon^{-1}(\psi_{rr}^\tau + \psi^\tau) + \left((\psi_0^\tau)' - \frac{\pi}{4}\right) v^\tau \\
&= \int_{\mathbb{R}} \left(-\varepsilon^{-1}(\psi_0'' + \psi_0)(s) - (\psi_1'' + \psi_1)(s) v^\tau + \left(\psi_0' - \frac{\pi}{4}\right)(s) v^\tau\right) \eta_\tau(r-s)\,ds - \varepsilon^{-1}\tau r \\
&= \int_{|s| \ge \frac{\pi}{2}} \left(-\varepsilon^{-1}(\psi_0'' + \psi_0)(s) - (\psi_1'' + \psi_1)(s) v^\tau + \left(\psi_0' - \frac{\pi}{4}\right)(s) v^\tau\right) \eta_\tau(r-s)\,ds - \varepsilon^{-1}\tau r,
\end{aligned}
$$

where we have used (54). We know from (67) and the definition of η_τ that $\eta_\tau(r-s) = 0$ when $s \ge \frac{\pi}{2}$, $r = \lambda_\varepsilon(x_\tau,t_\tau)$, and τ is small since $\limsup_{\tau\to 0} \lambda_\varepsilon(x_\tau,t_\tau) < \frac{\pi}{2}$. Therefore, the term above is estimated for small τ by

$$
\begin{aligned}
&\int_{-\infty}^{-\frac{\pi}{2}} \left(-\varepsilon^{-1}(\psi_0'' + \psi_0)(s) + \left(-\psi_1'' - \psi_1 + \psi_0 - \frac{\pi}{4}\right)(s) v^\tau\right) \eta_\tau(r-s)\,ds - \varepsilon^{-1}\tau r \\
&\qquad \ge \int_{-\infty}^{-\frac{\pi}{2}} (\varepsilon^{-1} - C(\Lambda))\eta_\tau(r-s)\,ds - \varepsilon^{-1}\tau r \\
&\qquad \ge -\varepsilon^{-1}\varrho_\Lambda(\tau).
\end{aligned}
$$

Using the above computations, we get

$$
\begin{aligned}
R_\varepsilon^\tau \ge & \left((\psi_0^\tau)' - \frac{\pi}{4}\right)(u - v^\tau) \\
& - c_0(\Lambda)\varepsilon(\psi_0^\tau)' f' - C(\Lambda)\varepsilon(1+f) - \varepsilon^{-1}\varrho_\Lambda(\tau) \\
\ge & \, \varepsilon(\psi_0^\tau)'(-c_0(\Lambda) f' - g) \\
& + \varepsilon\left(\frac{\pi}{4} g - C(\Lambda) - C(\Lambda) f\right) - \varepsilon^{-1}\varrho_\Lambda(\tau),
\end{aligned} \tag{82}
$$

where we have used $v^\tau = u^\tau + \varepsilon g$.

Since $f(t) = \alpha \exp(-\gamma^2 t)$ and $g(t) = \alpha\gamma \exp(-\gamma^2 t)$, we obtain

$$
R_\varepsilon^\tau \ge \varepsilon - \varepsilon^{-1}\varrho_\Lambda(\tau) \tag{83}
$$

when $\gamma \ge C(\Lambda)$ and $\alpha \ge \exp(\gamma^2 T)$, which is the fourth line of (61).

Proof of (80). We now prove (80), that is,

$$
\psi_r^\tau(\beta^\star \partial_t \delta^\tau - \mathrm{tr}(D^2 A(\nabla\varphi) D^2 \delta^\tau) - u) \ge -C(\Lambda)(\delta + \varepsilon).
$$

We again take the notation of the previous subsection.

Since $\psi_r^\tau > 0$, we need only consider the situation when

$$
\beta^\star \partial_t \delta^\tau - \mathrm{tr}(D^2 A(\nabla\varphi) D^2 \delta^\tau) - u \le 0.
$$

From (70), we know

$$\partial_t \delta^\tau \geq -C(\Lambda)\varepsilon^{-1}$$

and

$$D^2\delta^\tau \leq C(\Lambda)\varepsilon^{-1} I.$$

When $\partial_t \delta^\tau \geq 0$, since $0 \leq D^2 A \leq C(\Lambda) I$, we infer

$$c_0(\Lambda)\partial_t\delta^\tau \leq \beta^\star \partial_t \delta^\tau \leq \operatorname{tr}(D^2 A(\nabla\varphi) D^2\delta^\tau) + u \leq C(\Lambda)\varepsilon^{-1};$$

hence in any case,

$$|\partial_t \delta^\tau| \leq C(\Lambda)\varepsilon^{-1}.$$

Since $D^2\delta^\tau$ is symmetric, there is an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ and $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ such that

$$D^2\delta^\tau = \sum_{i=1}^n \gamma_i v_i \otimes v_i$$

and

$$\gamma_i \leq C(\Lambda)\varepsilon^{-1}.$$

We define

$$\alpha_i^\varphi := v_i^T D^2 A(\nabla\varphi) v_i$$

and

$$\alpha_i^\delta := v_i^T D^2 A(\nabla\delta^\tau) v_i.$$

We know

$$c_0(\Lambda) \leq \alpha_i^\varphi, \qquad \alpha_i^\delta \leq C(\Lambda)$$

since $c_0(\Lambda) I \leq D^2 A \leq C(\Lambda) I$. From (78), we obtain $|\alpha_i^\varphi - \alpha_i^\delta| \leq C(\Lambda)\varepsilon^2$ and

$$\exp(-C(\Lambda)\varepsilon^2) \leq \frac{\alpha_i^\varphi}{\alpha_i^\delta} \leq \exp(C(\Lambda)\varepsilon^2)$$

when $0 < \varepsilon < \varepsilon_0(\Lambda)$.

We compute

$$\begin{aligned}
&\operatorname{tr}(D^2 A(\nabla\varphi) D^2\delta^\tau) \\
&\quad = \sum_{i=1}^n \gamma_i \alpha_i^\varphi = \sum_{\gamma_i > 0} \gamma_i \alpha_i^\varphi + \sum_{\gamma_i < 0} \gamma_i \alpha_i^\varphi \\
&\quad \leq \exp(C(\Lambda)\varepsilon^2) \sum_{\gamma_i > 0} \gamma_i \alpha_i^\delta + \exp(-C(\Lambda)\varepsilon^2) \sum_{\gamma_i < 0} \gamma_i \alpha_i^\delta \\
&\quad \leq \exp(-C(\Lambda)\varepsilon^2) \sum_{i=1}^n \gamma_i \alpha_i^\delta + C(\Lambda)\varepsilon = \exp(-C(\Lambda)\varepsilon^2) \operatorname{tr}(D^2 A(\nabla\delta^\tau) D^2\delta^\tau) + C(\Lambda)\varepsilon.
\end{aligned}$$

Multiplying by $\exp(C(\Lambda)\varepsilon^2)$ yields

$$\begin{aligned}&\operatorname{tr}(D^2A(\nabla\varphi)D^2\delta^\tau)\\&\quad\le \operatorname{tr}(D^2A(\nabla\delta^\tau)D^2\delta^\tau)+C(\Lambda)\varepsilon+(1-\exp(C(\Lambda)\varepsilon^2))\operatorname{tr}(D^2A(\nabla\varphi)D^2\delta^\tau).\end{aligned}$$

Taking into account the fact that $\operatorname{tr}(D^2A(\nabla\varphi)D^2\delta^\tau)\ge\beta^\star\partial_t\delta^\tau-u\ge -C(\Lambda)\varepsilon^{-1}$, we obtain

$$\operatorname{tr}(D^2A(\nabla\varphi)D^2\delta^\tau)\le \operatorname{tr}(D^2A(\nabla\delta^\tau)D^2\delta^\tau)+C(\Lambda)\varepsilon. \tag{84}$$

In the case where $\beta^\star=\beta(\nabla\varphi)$, we observe from (78) that

$$|\beta(\nabla\varphi)\partial_t\delta^\tau-\beta(\nabla\delta^\tau)\partial_t\delta^\tau|\le C(\Lambda)\varepsilon.$$

On the other hand, when $|\nabla\varphi|\le\Lambda\varepsilon$, we conclude from (72) that $\psi_r^\tau\le C(\Lambda)\varepsilon^2$; hence

$$|\psi_r^\tau(\beta^\star\partial_t\delta^\tau-\beta(\nabla\delta^\tau)\partial_t\delta^\tau)|\le C(\Lambda)\varepsilon.$$

Together with (71) and (84), we obtain

$$\begin{aligned}&\psi_r^\tau(\beta^\star\partial_t\delta^\tau-\operatorname{tr}(D^2A(\nabla\varphi)D^2\delta^\tau)-u)\\&\quad\ge\psi_r^\tau(\beta(\nabla\delta^\tau)\partial_t\delta^\tau-\operatorname{tr}(D^2A(\nabla\delta^\tau)D^2\delta^\tau)-u)-C(\Lambda)\varepsilon\\&\quad\ge -C(\Lambda)(\varepsilon+\delta),\end{aligned}$$

which concludes the proof. □

Remark 4.4. Subsolutions can be constructed in an analogous way.

With these sub- and supersolutions and the modified comparison principle (Theorem 2.12), we are now able to prove the convergence result.

We consider the following situation. Let $D\subset\mathbb{R}^n$ be an open, periodic subset with $\emptyset\ne D$, $\overline{D}\ne\mathbb{R}^n$, and define $\Gamma_0:=\partial D$. We denote by δ_0 the signed distance function of Γ_0, positive on D, induced by the metric d of section 3.2, that is,

$$\delta_0(x):=\begin{cases}\inf\limits_{y\in\Gamma_0} d(x,y) & \text{for } x\in D,\\ -\inf\limits_{y\in\Gamma_0} d(x,y) & \text{for } x\notin D.\end{cases}$$

δ_0 is periodic, bounded, and Lipschitz continuous,

$$|\nabla\delta_0|\le C(\Lambda).$$

Since $\emptyset\ne D$, $\overline{D}\ne\mathbb{R}^n$, for Λ large enough, we have

$$\begin{gathered}|\delta_0|\le\Lambda,\\ \sup\delta_0\ge\Lambda^{-1},\quad\text{and}\\ \inf\delta_0\le-\Lambda^{-1}.\end{gathered}$$

From Theorem 2.5, we obtain the existence of a unique periodic $\omega\in C(\mathbb{R}^n\times[0,T[)$ which solves

$$\begin{gathered}\partial_t\omega+F(.,.,\nabla\omega,D^2\omega)=0\quad\text{in }\mathbb{R}^n\times]0,T[\quad\text{and}\\ \omega(.,0)=\delta_0,\end{gathered} \tag{85}$$

where F is defined in section 2.1. We assume that for $0\le t<T$,

$$\sup\omega(.,t)\ge\Lambda^{-1}$$

and

$$\inf \omega(.,t) \leq -\Lambda^{-1}.$$

From Theorem 2.8, we get a periodic viscosity solution $\varphi_\varepsilon \in C(\mathbb{R}^n \times [0,T[)$ of the double-obstacle Allen–Cahn problem,

(86)

$$\max\left(\varphi_\varepsilon - 1, \min\left(\varphi_\varepsilon + 1, \partial_t\varphi_\varepsilon + \frac{1}{\varepsilon}G_\varepsilon(.,.,\varphi_\varepsilon, \nabla\varphi_\varepsilon, D^2\varphi_\varepsilon)\right)\right) = 0 \quad \text{in } (\mathbb{R}^n \times]0,T[).$$

For the initial conditions, we assume

(87)
$$\begin{aligned} &\varphi_\varepsilon(.,0) = 1 \quad \text{for } \delta_0 \geq C(\Lambda)\varepsilon \quad \text{and} \\ &\varphi_\varepsilon(.,0) = -1 \quad \text{for } \delta_0 \leq -C(\Lambda)\varepsilon; \end{aligned}$$

for example, $\varphi_\varepsilon(.,0) = \varphi_{\varepsilon,0} = \max(-1, \min(1, \frac{\delta_0}{\varepsilon}))$.

The convergence theorem can now be stated.

THEOREM 4.5.

$$\varphi_\varepsilon \to 1 \quad \textit{pointwise on } [\omega > 0]$$

and

$$\varphi_\varepsilon \to -1 \quad \textit{pointwise on } [\omega < 0].$$

Moreover, this convergence is uniform on compact subsets of $[\omega > 0]$*, respectively,* $[\omega < 0]$.

Proof. We define $\omega_\varepsilon^+ := \omega + \Gamma\varepsilon$ for $\Gamma = C(\Lambda)$ chosen below. According to [7], ω_ε^+ is a supersolution of (42). As in section 3.2 and Definition 4.2, we define

$$\begin{aligned} \delta_\varepsilon^+(x,t) &:= \inf_{y,\ \omega_\varepsilon^+(y,t)\leq 0} d(x,y), \\ \lambda_\varepsilon^+(x,t) &:= \frac{\delta_\varepsilon^+(x,t)}{\varepsilon} - \pi - f(t), \quad \text{and} \\ \psi_\varepsilon^+ &:= \psi(\lambda_\varepsilon^+, v). \end{aligned}$$

For $0 < \varepsilon < \varepsilon_0(\Lambda)$, we claim that

(88)
$$\psi_\varepsilon^+ \geq \varphi_\varepsilon.$$

From Theorem 2.12 and Proposition 4.3, it suffices to verify that

(89)
$$\psi_\varepsilon^+(.,0) \geq \varphi_\varepsilon(.,0).$$

When $\varphi_\varepsilon(x,0) = -1$, the inequality is satisfied since $\psi_\varepsilon^+ \geq -1$.

Now we assume $\varphi_\varepsilon(x,0) > -1$. From (87), we get $\omega(x,0) = \delta_0(x) \geq -C(\Lambda)\varepsilon$ and

$$\omega_\varepsilon^+(x,0) \geq -C(\Lambda)\varepsilon + \Gamma\varepsilon \geq \frac{\Gamma}{2}\varepsilon > 0$$

when $\Gamma \geq C(\Lambda)$. Therefore, there is a $y \in [\omega_\varepsilon^+(.,0) \leq 0]$ such that $\delta_\varepsilon^+(x,0) = d(x,y) > 0$. This yields

$$\begin{aligned} \frac{\Gamma}{2}\varepsilon &\leq \omega_\varepsilon^+(x,0) - \omega_\varepsilon^+(y,0) = \omega(x,0) - \omega(y,0) \\ &= \delta_0(x) - \delta_0(y) \leq C(\Lambda)|x-y| \leq C(\Lambda)d(x,y) = C(\Lambda)\delta_\varepsilon^+(x,0). \end{aligned}$$

We conclude

$$\lambda_\varepsilon^+(x,0) = \frac{\delta_\varepsilon^+(x,0)}{\varepsilon} - \pi - \alpha \geq c_0(\Lambda)\Gamma - \pi - \alpha.$$

Since $\alpha \leq C(\Lambda)$, we can choose $\Gamma \geq C(\Lambda)$ to get

$$\lambda_\varepsilon^+(x,0) \geq \frac{\pi}{2};$$

hence

$$\psi_\varepsilon^+(x,0) = 1 \geq \varphi_\varepsilon(x,0),$$

establishing (89) and therefore (88).

We take (x_0,t_0) with $\omega(x_0,t_0) < 0$. There is $\tau > 0$ and a neighborhood $U(x_0,t_0)$ such that

$$\omega \leq -\tau \quad \text{on } U(x_0,t_0);$$

hence

$$\omega_\varepsilon^+ \leq -\frac{\tau}{2} \quad \text{on } U(x_0,t_0) \qquad \text{for } 0 < \varepsilon < \varepsilon_0(\Lambda,\tau).$$

On $U(x_0,t_0)$, we have

$$\delta_\varepsilon^+ = 0;$$

hence

$$\lambda_\varepsilon^+ \leq -\pi$$

and, finally,

$$\varphi_\varepsilon \leq \psi_\varepsilon^+ = -1 \quad \text{on } U(x_0,t_0). \qquad \square$$

Remark 4.6. When $H^n([\omega = 0]) = 0$, then the limit of φ_ε is uniquely determined in $L^1(\mathbb{R}^n \times [0,T[)$ by Theorem 4.5.

When fattening occurs—that is, $H^n([\omega = 0]) > 0$—there remains an ambiguity.

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