CURVATURE DEPENDENT PHASE BOUNDARY MOTION
AND PARABOLIC DOUBLE OBSTACLE PROBLEMS

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Abstract. The use of parabolic double obstacles problems for approximating curvature dependent phase boundary motion is reviewed. It is shown that such problems arise naturally in multi-component diffusion with capillarity. Formal matched asymptotic expansions are employed to show that phase field models with order parameter solving an obstacle problem approximate curvature dependent phase boundary motion. Numerical simulations of surfaces evolving according to their mean curvature are presented.

1. Introduction. Continuum mathematical models of phase transformations are of two basic types being (FB) and (PF). Models consisting of field equations holding in time varying domains in $\mathbb{R}^d$ separated by moving hypersurfaces known as 'free boundaries' we call of type (FB). An example is the classical Stefan problem describing solidification in which the dominating process is heat conduction. The solidifying material occupies the bounded domain $\Omega \subset \mathbb{R}^d$ in such a way that

\begin{equation}
\Omega = \Omega^+(t) \cup \Gamma(t) \cup \Omega^-(t) \quad t > 0,
\end{equation}

\begin{equation}
\partial \Omega^+(t) \cap \partial \Omega^-(t) = \Gamma(t),
\end{equation}

and the temperature field $\theta(x,t)$ satisfies

\begin{equation}
ce \theta_t = k \Delta \theta \quad x \in \Omega^+(t) \cup \Omega^-(t) \quad t > 0.
\end{equation}

\begin{equation}
\Omega^+(t) = \{ x : \theta(x,t) > 0 \},
\end{equation}

\begin{equation}
\Omega^-(t) = \{ x : \theta(x,t) < 0 \}.
\end{equation}

Here $c$ and $k$ respectively denote the specific heat and conductivity. On the interface $\Gamma(t)$ it holds that

\begin{equation}
\theta(x,t) = 0,
\end{equation}

\begin{equation}
[k \nabla \theta]^+ \cdot n = -lV,
\end{equation}

where $[\cdot]^+$ denotes the difference in the limits of the quantity in the brackets as $x$ tends to $\Gamma(t)$ from each side, $l$ is the latent heat, $n$ is the unit normal pointing into $\Omega^-(t)$ and $V$ is the speed of the interface in the direction $n$.

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It is convenient for computational reasons, existence theory in $\mathbb{R}^d$ ($d \geq 2$) and in order to deal with so called ‘mushy’ regions, to introduce the ‘enthalpy’ formulation, c.f. Elliott & Ockendon (1982). Let $u(x, t)$ be an ‘order’ or ‘phase’ parameter which satisfies

\[
u(x, t) \equiv \begin{cases} 
1 & x \in \Omega^+(t), \\
-1 & x \in \Omega^-(t),
\end{cases}
\]

so that in the sense of distributions (1.1–1.3) become

\[
(1.4a) \quad c\theta_t + \frac{1}{2}u_t = k\Delta \theta,
\]

\[
(1.4b) \quad u \in \text{sgn} \theta,
\]

with $\varepsilon := c\theta + \frac{1}{2}u$ being the enthalpy or heat content.

We call (1.4) an example of a (PF) or phase field model. In this case with appropriate restrictions on the data the two formulations coincide, as they should, see Meirmanov (1992). In general what we mean by a (PF) model is a system of partial differential equations holding, perhaps in the sense of distributions rather than classically, in a bounded domain with a distinguishing equation for an order parameter. This phase field variable is used to determine the phase at $(x, t)$. For example the Cahn–Hilliard equation describing ‘up-hill’ diffusion with capillarity in a binary alloy, see Cahn (1961), has $u$ being the difference in mass fractions of the two components of the alloy. The diffusion equation is

\[
(1.5a) \quad u_t = M\Delta u,
\]

\[
(1.5b) \quad w = -\gamma \Delta u + \psi'(u),
\]

where $w$ denotes the chemical potential, $M$ the mobility, $\gamma \ll 1$ is an interfacial energy parameter and $\psi(\cdot)$ is a double well potential with minima, for example, at $u = \pm u_a$. Here the two phases are characterized by the composition values $u_a$.

The domain $\Omega$ can be divided into the three sets $\Omega^+_a(t)$, $\Omega^-_a(t)$ and $\Omega^0_a(t)$ where approximately

\[
(1.6) \quad \begin{aligned}
u(x) &\approx +u_a & x &\in \Omega^+_a(t), \\
u(x) &\approx -u_a & x &\in \Omega^-_a(t), \\
|u(x)| &\approx u_a & x &\in \Omega^0_a(t).
\end{aligned}
\]

The interfacial region $\Omega^0_a(t)$ is not precisely defined and has an expected width of $O(\sqrt{\gamma})$.

We particularly wish to consider (PF) models whose field equations are

\[
(1.7a) \quad c\theta_t + \frac{1}{2}u_t = k\Delta \theta,
\]
\[ \tau u_t = \gamma \Delta u - \psi'(u) + \alpha \theta, \]

where \( \psi : \mathbb{R} \to \mathbb{R}_+ \) is a double well potential. Taking \( c = \tau = 0 \) and \( \alpha = 1 \) we recover the Cahn–Hilliard equation after labelling \( \theta \) as \( w \).

The system (1.7) is one example of a class of models for phase transition phenomena based on Ginzburg–Landau energy functionals; c.f. Hohenberg & Halperin (1977), Caginalp (1986). The associated Ginzburg–Landau energy functional for (1.7) is

\[ E(u, \theta) = \int_{\Omega} \left[ \frac{\gamma}{2} |\nabla u|^2 + \psi(u) + \frac{c\alpha}{l} \theta^2 - 2 \frac{\alpha}{l} \theta \left( c\theta + \frac{l}{2} u \right) \right] \, dx, \]

where \( \theta \) is a prescribed time independent harmonic function. A calculation yields

\[ \frac{dE}{dt} = -\int_{\Omega} \left[ |\nabla u|^2 + \frac{2k\alpha}{l} |\nabla(\theta - \theta_s)|^2 \right] 
\]

\[ + \int_{\partial \Omega} \left[ \gamma \frac{\partial u}{\partial \nu} u_t + \frac{2k\alpha}{l} \frac{\partial \theta}{\partial \nu}(\theta - \theta_s) \right] \, ds. \]

It is clear that \( E \) is a Lyapunov functional under the boundary conditions

\[ \begin{align*}
(1.10) \quad & i) \ u(x, t) = u_0(x) \quad x \in \partial \Omega \quad \text{or} \quad ii) \ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega, \\
(1.11) \quad & i) \ \theta(x, t) = g(x) \quad x \in \partial \Omega \quad \text{or} \quad ii) \ k \frac{\partial \theta}{\partial \nu} + q \theta = gg, \ \text{on} \ \partial \Omega, \ q > 0, 
\end{align*} \]

where in the case of (1.11i) \( \theta_0(x) = g(x) \) on \( \partial \Omega \), in the case of (1.11ii) \( k \frac{\partial \theta}{\partial \nu} + q \theta = gg \) on \( \partial \Omega \) and \( \nu \) is the unit normal on \( \partial \Omega \) pointing out of \( \Omega \).

We refer to Caginalp (1986), Elliott & Zheng (1990), Bates & Zheng (1991) and Brochet, Chen & Hilhorst (1991) for existence and long time behaviour of solutions to (1.7).

A major interest in the (PF) equations lies in their approximation properties when \( \tau, \gamma \) and \( \alpha \) are scaled in appropriate ways with a small parameter \( \epsilon \), c.f. Caginalp (1989). Denoting by \( u^\epsilon(x, t) \) the solution of (1.7) one obtains formally that \( \Omega = \Omega^+(t) \cup \Gamma(t) \cup \Omega^-(t) \) with \( \Gamma(t) = \partial \Omega^+(t) \cap \partial \Omega^-(t) \) where as \( \epsilon \to 0, \)

\[ u^\epsilon(x, t) \to \begin{cases} 
1 \text{ a.e. } x \in \Omega^+(t), \\
-1 \text{ a.e. } x \in \Omega^-(t), 
\end{cases} \]

(1.7a) becomes (1.2a), the jump condition (1.3b) holds on \( \Gamma(t) \) and that (1.3a) becomes

\[ c_2 \theta(x, t) = -(c_0 V + c_1 \kappa^m), \]

(1.12)
where \( c_0, c_1 \) and \( c_2 \) are constants determined from \( \tau, \gamma, \alpha \) and \( \psi(\cdot) \); \( V \) and \( \kappa_m \) are respectively the normal speed and mean curvatures of \( \Gamma(t) \). We shall be more precise in the following sections. In the resultant free boundary problem, sometimes known as the modified Stefan problem, the curvature term is a surface tension effect whereas the velocity term is a ‘relaxation’ effect. Apart from heat conduction in solid/liquid systems and solid-solid diffusional phase transformations, as modelled by the Cahn–Hilliard equation, these equations also arise in Hele–Shaw fluid flow and electrochemistry c.f. Elliott & Ockendon (1982). In certain circumstances the limit problem is purely one of differential geometry, see section 4 and 5; for example if \( c_2 = 0 \) with \( c_0 \) and \( c_1 \) non-zero then we have flow by mean curvature.

In this paper we wish to discuss these aspects of the phase field equations with a new choice of double well \( \psi(\cdot) \). Let \( I_{[-1,1]} \) denote the indicator function of the interval \([-1,1]\) so that \( I_{[-1,1]}(r) \) vanishes for \( |r| \leq 1 \) and is \(+\infty\) for \(|r| > 1\). Then we take \( \psi \) to be

\[
\psi(r) = \frac{1}{2} (1 - r^2) + I_{[-1,1]}(r) \quad r \in \mathbb{R}.
\]

It follows that (1.7b) becomes

\[
(1.14a) \quad -\tau u_t + \gamma \Delta u + \alpha \theta + u \in \partial I_{[-1,1]}(u),
\]

or

\[
(1.14b) \quad \tau u_t - \gamma \Delta u + \alpha \theta = \beta \in \beta(u) := \begin{cases} 
+\infty & u < -1, \\
[-1,0] & u = -1, \\
[u] & |u| < 1, \\
[0,1] & u = +1, \\
-\infty & u > -1.
\end{cases}
\]

We can rewrite (1.14b) as a complementarity problem

\[
(1.15a) \quad (\tau u_t - \gamma \Delta u - \alpha \theta - u)(|u| - 1) = 0,
\]

\[
(1.15b) \quad (\tau u_t - \gamma \Delta u - \alpha \theta - u) \text{ sgn } u \leq 0,
\]

\[
(1.15c) \quad |u| \leq 1.
\]

This is a parabolic double obstacle problem, c.f. Friedman (1982), which for zero Neumann boundary data, can be written as the parabolic variational inequality

\[
(1.16) \quad (\tau u_t - \alpha \theta - u, \eta - u) + \gamma(\nabla u, \nabla \eta - \nabla u) \geq 0 \quad \forall \eta \in K,
\]

where \( K = \{ \eta \in H^1(\Omega) : |\eta| \leq 1 \text{ a.e.} \} \). The use of variational inequalities in phase field models for the Stefan problem with surface tension was first proposed by Visintin (1984, 1988a,b).
The structure of the solution to (1.14) is such that there exist open sets $\Omega^+(t)$, $\Omega^-(t)$ and $\Omega^T(t)$ such that $\Omega = \Omega^+(t) \cup \Omega^-(t) \cup \Omega^T(t)$ where

\begin{equation}
  u(x, t) = \begin{cases} 
    1 & x \in \Omega^+(t), \\
    0 & x \in \Omega^T(t), \\
    -1 & x \in \Omega^-(t).
  \end{cases}
\end{equation}

(1.17)

The interfacial region $\Omega^T(t)$ is expected to be narrow depending on the size of parameters $\tau$, $\gamma$ and $\alpha$.

We find it convenient to introduce some notation and ideas from differential geometry used in the following sections. Let $\{\Gamma(t)\}_{t \geq 0}$ for $t \in [0, T]$ be a family of smooth closed $(d-1)$ dimensional hypersurfaces with bounded principle curvatures. We assume that for each $t$, $\Gamma(t)$ is the boundary of a bounded open set $\Omega^-(t)$ and that $\Omega = \Omega^+(t) \cup \Gamma(t) \cup \Omega^-(t)$ where $\Omega$ is a bounded domain such that $\Gamma(t) \cap \partial \Omega = \emptyset$.

We suppose that

\[ \Gamma(t) = \{ x : x = \phi(s, t), s \in S \}, \]

where $S$ is a smooth compact $(d-1)$ dimensional manifold and for each $t \in [0, T]$, $\phi$ is a diffeomorphism. The evolution of $\Gamma(t)$ is defined by the mapping $\phi : S \times [0, T] \to \mathbb{R}^d$ and since the intrinsic quantities of curvature and normal are independent of the parameterization $\phi$ we restrict our attention to those where

\begin{equation}
  \frac{\partial \phi}{\partial t}(s, t) \perp \frac{\partial \phi}{\partial s_i}(s, t), 1 \leq i \leq d - 1.
\end{equation}

(1.18)

It follows that points on $\Gamma(t)$ move in the normal direction and

\begin{equation}
  \phi_t = V(s, t)n(s, t),
\end{equation}

(1.19)

where $n(s, t)$ is the normal to $\Gamma(t)$ pointing into $\Omega^-(t)$ and $V(s, t)$ is the speed in the normal direction $n$. The boundedness of the principal curvatures of $\Gamma(t)$ implies the existence of a $c_0$ and neighbourhood, $\Omega_r(t)$ of $\Gamma(t)$ such that for each $x \in \Omega_r(t)$, $t \in [0, T]$ there exists a unique $s(x, t)$ satisfying

\begin{equation}
  \text{dist}(\phi(s(x, t), t), x) = \text{dist}(\Gamma(t), x) < c_0.
\end{equation}

(1.20)

It follows that $(s, d, t)$ with $s \in S$, $|d| < c_0$, $t \in [0, T]$ defines a new parameterization of $\bigcup_{t \in [0, T]} \Omega_r(t)$ by

\begin{equation}
  x(s, d, t) = \phi(s, t) - dn(s, t),
\end{equation}

(1.21)

and for each $x \in \Omega_r$, $s(x, t)$ is defined by (1.20) and

\begin{equation}
  d(x, t) = \begin{cases} 
    \text{dist}(\Gamma(t), x) & x \in \Omega^+(t), \\
    -\text{dist}(\Gamma(t), x) & x \in \Omega^-(t).
  \end{cases}
\end{equation}

(1.22)
Differentiating the identity \( d(x, t) = (\phi(s(x, t), t) - x) \cdot n(s(x, t), t) \) with respect to \( t \) and \( x \), using

\[
\frac{\partial \phi(s, t)}{\partial s_i} \cdot n(s, t) = 0 \quad i = 1, \cdots, d - 1,
\]

we obtain that on \( \Gamma(t) \)

\[
d_t = V(s(x, t), t).
\]

Using (1.21), (1.23) and the partial derivative of (1.23) with respect to \( s \), see Giusti (1984), it holds that in a neighbourhood of \( \Gamma(t) \)

\[
n = -\nabla_s d(x, t),
\]

\[
\Delta_s d(x, t) = \kappa^m(s(x, t), t) - d(x, t) \kappa^s(s(x, t), t) + O(d^2),
\]

where \( \kappa^m \) and \( \kappa^s \) are respectively the sum of the principal curvatures and the sum of the squares of the principal curvatures of \( \Gamma(t) \); \( \kappa^m \) is taken to be positive when \( \Omega^-,(t) \) is convex.

It is known that \( \text{Per} \, \Omega^-(t) := \text{meas} \, \Gamma(t) \) and \( \text{meas} \, \Omega^-(t) \) satisfy

\[
\frac{d}{dt} \text{Per} \, \Omega^-(t) = -\int_{\Gamma(t)} \kappa^m V ds,
\]

\[
\frac{d}{dt} \text{meas} \, \Omega^-(t) = -\int_{\Gamma(t)} V ds.
\]

Finally for this section we outline the contents of this paper. In section 2 we begin by remarking that potentials such as (1.13) arise naturally out of a limit problem for the Cahn–Hilliard equation. In section 3 we show formally that (1.14) coupled with (1.7a) approximates the (FB) problem (1.2a), (1.3b) and (1.12). A rigorous result for mean curvature flow is described in section 4. In section 5 a formal asymptotic derivation of mean curvature flow with volume conservation is given. Numerical simulations of the differential geometry problems based on the parabolic double obstacle formulation are also presented in §4 and §5. These numerical experiments have also been used to create a video of simulated flow by mean curvature. The advantage of this approach rather than using a polynomial for \( \psi(\cdot) \), say, is that the relevant free boundary can be approximated by the interfacial region whose definition is precise and whose width can be rigorously estimated, see §4. Computations for the order parameter can in principle be concentrated in the interfacial region since outside it is known exactly.
2 Multi-component diffusion with capillarity. Phase separation of a multi-component alloy is an example of a solid-solid phase transformation driven by diffusion. It is caused by the rapid cooling (quenching) of an alloy with an initially homogeneous composition in a state of thermodynamic equilibrium into the unstable coexistence region of its phase diagram. Unstable small fluctuations in the homogeneity result in a spontaneous decomposition of the alloy into a fine grained mixture of phases characterized by differing composition. This is understood as being due to up-hill diffusion mollified by interfacial energy effects between the phases. Characteristically one has for spinodal decomposition a rich highly interconnected spatial structure which gradually coarsens whereas for nucleation one has sphere like blobs which grow and shrink. Because diffusion is very slow and the quenching is rapid, one can usually assume thermal equilibrium and a constant temperature.

A model for isothermal multi-component diffusion with capillarity was derived and studied by Elliott & Luckhaus (1991) (see also Hoyt (1990a,b) and Eyre (1992)). The formalism used was that of non-equilibrium thermodynamics. For an N-component alloy, let \( \{u_i, \mu_i\} \) denote the mass fraction and chemical potential for each component. By definition \( \sum_{i=1}^{N} u_i = 1 \) and \( 1 \geq u_i \geq 0 \). Let \( L \) be an \( N \times N \) constant symmetric positive semi-definite matrix with a one dimensional kernel

\[
(2.1) \quad Le = 0, \quad e_i = 1 \quad \forall i,
\]

The diffusion equation is

\[
(2.2a) \quad u_i = \Delta L \mu \quad x \in \Omega, \quad t > 0,
\]

with the no mass flux boundary condition

\[
(2.2b) \quad (L \nabla \mu)_i \cdot n = 0 \quad \forall i \quad x \in \Omega.
\]

Let \( A \) be a constant symmetric \( N \times N \) matrix with largest eigenvalue \( \lambda_A > 0 \), and \( \Psi : \mathbb{R}^N \to \mathbb{R} \) be the homogeneous energy function, appropriate for an ideal mixture,

\[
(2.3) \quad \Psi(u) := \frac{\theta}{2} \sum_{i=1}^{N} u_i \ln(u_i) - \frac{1}{2} u^T A u.
\]

The total energy function is

\[
(2.4) \quad \mathcal{E}(u) := \int_{\Omega} \left( \Psi(u) + \frac{1}{2} \nabla u \cdot \nabla u \right) dx,
\]

where \( G \) is a constant symmetric positive definite \( N \times N \) matrix which defines the interfacial or capillary energy. The chemical potential satisfies

\[
(2.5) \quad \mu := \frac{\delta \mathcal{E}(u)}{\delta u},
\]
so that

\[(2.6a) \quad \mu_i := \partial_t \Psi(u) - \Delta (Gu)_t \quad x \in \Omega, \quad t > 0,\]

\[(2.6b) \quad \nabla (Gu)_t \cdot n = 0 \quad x \in \partial \Omega, \quad t > 0.\]

In the binary case, \(N = 2\), we have the following field equations

\[(2.7a) \quad \frac{\partial u_1}{\partial t} = L_{11} \Delta \mu_1 + L_{12} \Delta \mu_2,\]

\[(2.7b) \quad \frac{\partial u_2}{\partial t} = L_{21} \Delta \mu_1 + L_{22} \Delta \mu_2,\]

\[(2.8a) \quad \mu_1 = \frac{\theta}{2} (1 + u_1) - (A_{11} u_1 + A_{12} u_2) - (G_{11} \Delta u_1 + G_{12} \Delta u_2),\]

\[(2.8b) \quad \mu_2 = \frac{\theta}{2} (1 + u_2) - (A_{21} u_1 + A_{22} u_2) - (G_{21} \Delta u_1 + G_{22} \Delta u_2),\]

where, because of the assumption on \(L, A\) and \(G\),

\[L_{11} = L_{22} = -L_{12} = -L_{21} = \frac{M}{2} > 0, \quad G_{12} = G_{21} \quad \text{and} \quad A_{12} = A_{21}.\]

Now let us define

\[\gamma := \frac{G_{11} + G_{22}}{2} - G_{12} > 0 \quad \text{and} \quad \theta_c := \frac{A_{11} + A_{22}}{2} - A_{12} > 0,\]

and assume that

\[A_{11} = A_{22}, \quad G_{11} = G_{22}.\]

Then setting respectively the local concentration and chemical potential to be

\[(2.9) \quad u := u_2 - u_1 \in [-1, 1] \quad \text{and} \quad w = \mu_2 - \mu_1\]

so that \(u_2 = (1 + u)/2, \quad u_1 = (1 - u)/2\),

we find that by considering the equations (2.7, 2.8)

\[(2.10a) \quad \frac{\partial u}{\partial t} = M \Delta w,\]

\[(2.10b) \quad w = \psi_\theta(u) - \gamma \Delta u,\]
\[ (2.10c) \quad \psi_0(u) = \frac{\theta}{2} \left[ (1 + u) \ln(1 + u) + (1 - u) \ln(1 - u) \right] - \frac{1}{2} \theta u^2. \]

Thus we have obtained the generalized diffusion equation proposed by Cahn (1961) for spinodal decomposition in a binary alloy. The minima of \( \psi_0(u) \) are located at the composition values \( \pm u_a \) where

\[ \frac{2\theta}{\theta} = \ln \left( \frac{1 + u_a}{1 - u_a} \right) / u_a. \]

Equations (2.10a,b) have been much studied with \( \psi_0(\cdot) \) replaced by a polynomial with a double well form e.g.

\[ (2.11) \quad \psi_0(u) = \frac{1}{4} (u^2 - u_a^2)^2. \]

The existence theory of Elliott & Luckhaus (1991) is directly applicable to (2.10) and a numerical analysis has been made by Copetti & Elliott (1992). See also Cerezo et al. (1990) for a video of experimental observations of spinodal decomposition and numerical simulations based on the Cahn-Hilliard equation with cubic nonlinearity.

A deep quench corresponds to \( \theta \ll \theta_c \) in which case \( u_a \) is close to 1. The deep quench limit \( \theta \to 0 \) leads to the following parabolic double obstacle problem (scaling time so that \( M = 1 \))

\[ (2.12a) \quad \frac{\partial u}{\partial t} = \Delta w, \]

\[ (2.12b) \quad -\gamma \Delta u - w = \beta \in \beta(u), \]

where \( \beta(\cdot) \) as in (1.14b).

The mathematical and numerical analysis of this binary deep quench limit problem has been studied in Blowey & Elliott (1991, 1992). Using \( \Gamma \)-convergence it can be shown that a sequence of stationary solutions converges as \( \gamma \to 0 \) to a piecewise constant function which satisfies a minimal perimeter criterion and that an appropriate rescaling of the sequence of the chemical potentials satisfies the stationary Gibbs-Thomson relation, see §5.3.

A mathematical model for non-isothermal phase separation has been recently proposed and analysed by Alt & Pawlow (1992).

3 Phase-field asymptotic analysis. In this section we perform a formal asymptotic analysis of the double obstacle version of the phase-field equations along the lines of Caginalp (1989), Fife (1988) and Pego (1989) who considered a smooth double well \( \phi(\cdot) \).

Take the following scaling of the phase field equations, \( \tau = \sigma \epsilon^2 \), \( \gamma = \epsilon^2 \) and \( \alpha = c \sigma \) so that (1.7a) and (1.14) become:

Find \( \theta(t) \) and \( u(t) \) such that

\[ (3.1) \quad c \frac{\partial \theta}{\partial t} + \frac{1}{2} \frac{\partial u}{\partial t} = k \Delta \theta, \]
\[
(3.2) \quad (\sigma_1 e^2 u_t - e^2 \Delta u - u - \sigma_2 \theta, \eta - u) \geq 0 \quad |\eta| \leq 1.
\]

Let the solution \( u(x, t; \epsilon) \) of (3.2) satisfy for \( \epsilon \in [0, \epsilon_0), \ t \in [0, T] \)

\[
\Gamma^\epsilon(t) = \{ x : x = \phi(s, t; \epsilon) \ s \in S \},
\]

\[
(3.3) \quad \Gamma^\epsilon(t) = \partial \Omega^+\epsilon(t) \cap \partial \Omega^-\epsilon(t),
\]

\[
\Omega^+\epsilon(t) = \{ x : u(x, t; \epsilon) > 0 \}, \ \Omega^-\epsilon(t) = \{ x : u(x, t; \epsilon) < 0 \},
\]

where \( S \) is a smooth manifold and \( \phi \) is a diffeomorphism for each \( t \) and \( \epsilon \). We use the notation and results as stated in \( \S 1 \). In particular we shall make use of (1.18-1.28). This methodology is due to Paolini & Verdi (1992) who carried out a similar analysis for the parabolic variational inequality we consider in \( \S 4 \). Note that \( \phi : S \times [0, T] \to \mathbb{R}^d \) describes the evolution of \( \Gamma^\epsilon(t) \) so that

\[
(3.4) \quad u(\phi(s, t), t) = 0, \ \forall s \in S, \ t \in [0, T].
\]

We note that as \( \Gamma^\epsilon(t) \) is a level set, the normal is given by

\[
(3.5) \quad n(s, t) = \frac{-\nabla_x u(\phi(s, t), t)}{\|\nabla_x u(\phi(s, t), t)\|}.
\]

We suppose that the \( \phi, n, V, \kappa_m \) and \( d \) have the asymptotic expansions

\[
(3.6a) \quad \phi(s, t; \epsilon) = \phi^0 + O(\epsilon),
\]

\[
(3.6b) \quad n(s, t; \epsilon) = n^0 + O(\epsilon),
\]

\[
(3.6c) \quad V(s, t; \epsilon) = V^0 + O(\epsilon),
\]

\[
(3.6d) \quad \kappa_m(s, t; \epsilon) = \kappa_m^0 + O(\epsilon),
\]

\[
(3.6e) \quad d(x, t; \epsilon) = d^0(x, t) + O(\epsilon),
\]

where \( \phi^0, \kappa_m^0, n^0, V^0 \) and \( d^0 \) are respectively the parameterization, curvature, normal, speed in the normal direction and distance function all associated with \( \Gamma^0(t) \). By \( \S 1 \) it holds

\[
(3.7a,b,c) \quad \frac{\partial d^0}{\partial t} = V^0(s, t) + O(\epsilon), \ \Delta_x d^0 = \kappa_m^0 + O(\epsilon) \text{ and } \nabla_x d = -n_0 + O(\epsilon).
\]
Outer Expansion

We assume that there exist the following asymptotic expansions in $\epsilon$

\[(3.8) \quad \theta(x, t; \epsilon) = \theta^0(x, t) + \epsilon \theta^1(x, t) + O(\epsilon^2),\]

\[(3.9) \quad u(x, t; \epsilon) = u^0(x, t) + \epsilon u^1(x, t) + O(\epsilon^2),\]

valid for all $x \in \Omega \setminus \Gamma^0(t)$ and $t \in [0, T]$. Substitution into (3.1, 3.2) yields

\[(3.10a) \quad \epsilon \theta^0_t + \frac{1}{2} u^0_t = k \Delta \theta^0,\]

\[(3.10b) \quad -(u^0, \eta - u^0) \geq 0 \quad \forall |\eta| \leq 1.\]

It easily follows from (3.10b) that

\[u^0 = \begin{cases} 
1 & \text{in } \Omega^+_0(t) := \{x \in \Omega : u^0(x, t) > 0\}, \\
-1 & \text{in } \Omega^-_0(t) := \{x \in \Omega : u^0(x, t) < 0\}.
\]

Inner Expansion

In order to proceed we need an inner expansion. We introduce a coordinate in the neighbourhood of the interface, viz

\[(3.11) \quad y := \frac{d}{\tau},\]

and write the inner expansion in terms of the $(y, s)$ coordinate system. Let

\[\theta(x, t; \epsilon) = \Theta(y, s, t; \epsilon) = \Theta^0(y, s, t) + \epsilon \Theta^1(y, s, t) + O(\epsilon^2),\]

\[u(x, t; \epsilon) = U(y, s, t; \epsilon) = U^0(y, s, t) + \epsilon U^1(y, s, t) + O(\epsilon^2).\]

We suppose that $U$ is monotone increasing with $y$ and that there exist $Y^+_\epsilon(s, t)$ and $Y^-_\epsilon(s, t)$ such that for $s \in S$

\[(3.12a) \quad U(0, s, t; \epsilon) = 0,\]

\[(3.12b) \quad U(Y^+_\epsilon(s, t), s, t; \epsilon) = 1; \quad U(Y^-_\epsilon(s, t), s, t; \epsilon) = -1,\]

\[(3.12c) \quad Y^+_\epsilon(s, t) = Y^+_0(s, t) + \epsilon Y^+_1(s, t) + O(\epsilon^2); \quad Y^-_\epsilon(s, t) = Y^-_0(s, t) + \epsilon Y^-_1(s, t) + O(\epsilon^2).\]

One may write the time derivative and Laplacian of $u$ as

\[(3.13a) \quad \frac{\partial u}{\partial t} = U_t + \nabla_s U \frac{\partial s}{\partial t} + \frac{1}{\epsilon} U_s \frac{\partial d}{\partial t},\]
\[ (3.13b) \quad \Delta_x u = \text{Tr} \left( \nabla_x s^T U_x \nabla_x s \right) + \nabla_x U^T \Delta_x s + \frac{1}{\epsilon} U_y \Delta_x d + \frac{1}{\epsilon^2} U_{yy}, \]

where \( \text{Tr} \) is the trace of the matrix. The same relations hold for \( \theta \). Substitution into (3.1,3.2) leads to an asymptotic series in powers of \( \epsilon \) where coefficients are equated to zero.

**Zero Order**

We obtain

\[ (3.14a) \quad \Theta_{yy}^0 = 0, \]

\[ (3.14b) \quad -(U_{yy}^0 + U^0, \chi - U^0) \gtrless 0 \quad |\chi| \leq 1. \]

The solution to (3.14a) is

\[ (3.15) \quad \Theta^0(y, s, t) = a(s, t)y + b(s, t), \]

where \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are to be determined.

In order to proceed we match the inner and outer expansions and use the notation \( f(\Gamma^0_\pm) \) and \( f(\Omega^0_\pm) \) to denote the limit of \( f(\chi) \) as \( \chi \) tends to \( \Gamma^0 \) from \( \Omega^0_+ \) and \( \Omega^0_- \) respectively. Indeed the matching condition

\[ (3.16) \quad \lim_{y \to \pm \infty} \Theta^0(y, s, t) = \Theta^0(\Gamma^0_\pm, t), \]

implies, for boundedness of \( \theta^0 \) on \( \Gamma^0(t) \), that \( a(s, t) = 0 \). This yields another matching condition, viz.,

\[ (3.17) \quad \lim_{y \to \pm \infty} \Theta_1^0(y, s, t) = -\nabla \theta^0(\Gamma^0_\pm, t) \cdot n^0, \]

where \( n^0 \) is the normal pointing into \( \Omega^0_\pm(t) \).

For the inner and outer expansion of \( u \) to match we have

\[ (3.18a) \quad \lim_{y \to \pm \infty} U^0(y, s, t) = u^0(\Gamma^0_\pm, t) = \pm 1 \quad s \in S, \]

and also by (3.11a) we have the interfacial conditions

\[ (3.18b) \quad U^0(0, s, t) = 0, \quad U^1(0, s, t) = 0. \]

Using the assumption that \( U(\cdot, \cdot, \cdot) \) is monotone increasing in \( y \) we have that \( U^0_y \geq 0 \) and (3.14b) is easily seen to have the unique solution, \( t \in [0, T], s \in S \),

\[ (3.19) \quad U^0(y, s, t) := \begin{cases} -1, & \text{if } y < -\pi/2, \\ \sin y, & \text{if } |y| \leq \pi/2, \\ 1, & \text{if } y > \pi/2, \end{cases} \]
so that

\[(3.20a,b) \quad Y^0_0 = \pm \frac{\pi}{2} \text{ and } U^0_y(\pm \frac{\pi}{2}, s, t) = 0.\]

It follows from \((3.12b,c)\) and \((3.20)\) that

\[(3.21) \quad U^1(\pm \frac{\pi}{2}, s, t) = 0.\]

**First Order**

Calculating the \(O(\epsilon)\) term we find

\[(3.22a) \quad k \Theta^0_y n^0 + k \Theta^1_y = V^0(y_0 \theta^0 + \frac{1}{2} U^0_y),\]

and

\[(3.22b) \quad -U^1_y - U^1 = (-\sigma_1 V^0 + \kappa_0^m) U^0_y + \sigma_2 \Theta^0 \quad |y| < \pi/2.\]

Since \(\Theta^0(y, s, t) = b(s, t)\) it follows that \((3.22a)\) reduces to

\[k \Theta^1_y = V^0 \frac{I}{2} U^0_y,\]

and integrating with respect to \(y\) we obtain

\[k \Theta^1_y - V^0 \frac{I}{2} U^0 = C(s, t).\]

It follows that taking the limits \(y \to \pm \infty\) and using \((3.17)\) we obtain

\[-k \nabla \theta^0(\Gamma^0_+.) n^0 - V^0 \frac{I}{2} = -k \nabla \theta^0(\Gamma^0_-.) n^0 + V^0 \frac{I}{2},\]

and

\[k \left[ \nabla \theta^0 \right]^+ n^0 = -V^0 I \quad \text{on } \Gamma^0(t).\]

Multiplying \((3.22b)\) by \(U^0_y\) and integrating by parts from \(y = -\pi/2\) to \(\pi/2\) yields, using the fact \(U^0_y = U^1 = 0\) for \(y = \pm \pi/2\), the solvability condition

\[0 = \int_{-\pi/2}^{\pi/2} \left((-\sigma_1 V^0 + \kappa_0^m)(U^0_y)^2 + \sigma_2 \Theta^0 U^0_y \right) dy = 0,\]

\[= (-\sigma_1 V^0 + \kappa_0^m) \int_{-\pi/2}^{\pi/2} \cos^2 y dy + \sigma_2 b(s, t) \int_{-\pi/2}^{\pi/2} \cos y dy.\]

Thus we obtain the equation

\[\sigma_2 \Theta^0 = \sigma_2 b(s, t) = \frac{\pi}{4} (\sigma_1 V^0 - \kappa_0^m).\]
It follows that
\[ \sigma_2 \theta^0 \bigg|_{\Gamma_0} = \frac{\pi}{4} (\sigma_1 V^0 - \kappa^m_0). \]

Thus we have shown that the zero order term \( \theta^0 \) in the outer expansion (3.8) solves the modified Stefan problem

\[
\begin{align*}
\sigma_2 \theta^0 &= \kappa \Delta \theta^0 & \mathbf{x} \notin \Gamma^0(t), \\
[k \nabla \theta^0]^+ \cdot n^0 &= -IV^0 & \mathbf{x} \in \Gamma^0(t),
\end{align*}
\]

and the order parameter to first order satisfies

\[
\begin{cases} 
1 & \text{in } \Omega^+_0(t), \\
-1 & \text{in } \Omega^-_0(t).
\end{cases}
\]

Now let \( \sigma_1 = \sigma_1(\epsilon) \) where \( \lim_{\epsilon \to 0} \sigma_1(\epsilon) = 0 \) and consider the same scaling of the phase field equation just described. Performing the outer expansion, one arrives at identical equations and so obtains the same outer expansion. However, when calculating the inner expansion \( \sigma_1 \) does not appear up to the first order term, since \( \sigma_1 = o(1) \). Otherwise the inner expansion goes through in the same way and results in (3.23) with \( \sigma_1 \) dropped, this is known as the alternative modified Stefan limit. This concludes the formal asymptotic analysis of the phase field equations. We refer to Luckhaus (1990) for a global existence proof of weak solutions to (3.23) with \( \sigma_1 = 0 \).

4 Motion by mean-curvature. In the zero specific and latent heats limit (1.7a) becomes \( \Delta \theta = 0 \) in \( \Omega \) and with the boundary condition \( \theta = 0 \) on \( \partial \Omega \) we have

\[ \theta(\mathbf{x}, t) = 0 \quad \mathbf{x} \in \Omega, \ t > 0. \]

Substitution into equation (1.7b) for the order parameter yields the Allen–Cahn equation (Allen & Cahn (1979))

\[ \tau u_t = \gamma \Delta u - \psi'(u). \]

The asymptotics as \( \epsilon \to 0 \) with the scaling \( \tau = \gamma = \epsilon^2 \) have been much studied when \( \psi(\cdot) \) has a double well form with equal wells. The solution \( u := u^\epsilon \) converges as \( \epsilon \to 0 \) for each \( (\mathbf{x}, t) \) to one of the two minima of \( \psi \). The interface between the resulting phases evolves according to the geometric problem of the flow by mean curvature, c.f. Rubinstein, Sternberg & Keller (1989), Bronsard & Kohn (1990), De Mottoni & Schatzman (1990), Chen (1991), Evans, Soner & Souganidis (1991), Paolini & Verdi (1992).

In this section we wish to consider (4.1) with \( \psi(\cdot) \) replaced by the energy with infinite walls so that

\[ \tau u_t - \gamma \Delta u \in \beta(u), \]

where \( \beta(\cdot) \) is defined by (1.14b).
Let \{\Gamma(t)\}_{t \geq 0} be as in §1. We say that it forms a motion by mean curvature flow on the time interval \([0, T]\) provided

\begin{equation}
V = \kappa_m.
\end{equation}

It follows from (1.27) that

\[ \frac{d}{dt} \text{Per} (\Omega^-(t)) = - \int_{\Gamma(t)} (\kappa_m)^2 \, ds \leq 0. \]

Thus \(\text{Per}(\Omega^-(t))\) acts as a Lyapunov functional for the geometric motion. It is known that

(a) If \(\Gamma(0)\) is sufficiently smooth, then there exists a unique solution of the motion by mean curvature flow starting from \(\Gamma(0)\) in some time interval \([0, T]\)

(b) If \(d = 2\) and \(\Gamma(0)\) is convex, then the solution can be extended on to the time at which \(\Gamma(\cdot)\) shrinks to a point.


4.1 Rigorous and asymptotic analysis of a parabolic variational inequality. We consider the following initial boundary value problem for (4.2)

(P) Given \(g \in G := \{ \eta \in L^\infty(\Omega) : |\eta| \leq 1 \text{ a.e. } x \in \Omega \} \) and \(T > 0\) find \(u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))')\), \(u(\cdot, t) \in K := \{ \eta \in H^1(\Omega) : |\eta| \leq 1 \text{ a.e. } x \in \Omega \} \) a.e. \(t \in (0, T)\) such that for each \(\eta \in L^2(0, T; H^1(\Omega))\) with \(\eta(t) \in K\) for \(t \in [0, T]\)

\begin{align}
(4.4a) & \quad \int_0^T \{ (\frac{\partial u}{\partial t}, \eta - u) + (\nabla u, \nabla \eta - \nabla u) - \frac{1}{2} (u, \eta - u) \} \geq 0, \\
(4.4b) & \quad u(x, 0) = g(x).
\end{align}

The existence and uniqueness for problem (P) follows from standard methods; see Chen & Elliott (1991) who also consider the long time asymptotic behaviour. The family of solution operators \(\{S(t)\}_{t \geq 0}\) defined by \(S(t)g := u(t)\), is a continuous semi-group acting on \(G \subset L^2(\Omega)\) which possesses a global attractor bounded in \(K\). Furthermore, they prove that if \(\gamma(\epsilon)\) satisfies \(\lim_{\epsilon \to 0} \gamma(\epsilon) = 0\), the signed distance function \(d\) is sufficiently smooth, satisfying the estimate

\[ \sup_{t \in [S, T]} \sup_{|d| \leq c_0} |\nabla(d_t - \Delta d)| \leq D_0, \]

and there exists \(c_0\) such that for \(\epsilon \in (0, c_0),

\[ \frac{\pi}{2} \leq \gamma(\epsilon) \quad \text{and} \quad \frac{\pi}{2} + 2\gamma(\epsilon) \exp^{2D_0 T} \leq c_0, \]
with initial data $g^f \in \mathcal{G} \cap C(\Omega)$ satisfying

\[
\{ x : d(x, 0) > \gamma(\varepsilon) \} \subset \{ x : g^f(x) = 1 \},
\]

\[
\{ x : d(x, 0) < -\gamma(\varepsilon) \} \subset \{ x : g^f(x) = -1 \},
\]

then for all $t \in [0, T]$ the unique solution of (4.4) satisfies

\[
\{ x : d(x, t) \geqslant \gamma(\varepsilon)(1 + 2e^{2\cdot d_0}) \} \subset \{ x : u(x, t) = 1 \},
\]

\[
\{ x : d(x, t) \leqslant -\gamma(\varepsilon)(1 + 2e^{2\cdot d_0}) \} \subset \{ x : u(x, t) = -1 \}.
\]

Paolini & Verdi (1992) have performed an asymptotic analysis on (4.4) proving that, up to $O(\varepsilon^2)$, the level set $\Gamma(t) := \{ x \in \Omega : u(x) = 0 \}$ moves in the normal direction of $\Gamma$ with the speed given by the sum of the principal curvatures. Their methodology was used by us in §3 for the phase field model.

### 4.2 Numerical approximation.

In this section we discuss the numerical approximation of (P). We assume that $\Omega$ is a polygonal domain, where $T^h$ is a triangulation for $\Omega$, so that $\Omega = \cup_{\tau \in \mathcal{T}^h} \tau$. We define the finite element spaces $S^h$ and $K^h$ thus

\[
S^h := \left\{ \chi : \Omega \to \mathbb{R} \text{ such that } \chi|_{\tau} \text{ is linear, } \tau \in \mathcal{T}^h \right\},
\]

\[
K^h := \left\{ \chi \in S^h : -1 \leqslant \chi \leqslant 1 \right\}.
\]

Given $M \in \mathbb{N}$, let $\Delta t := T/M$ be the time step. We analyze the finite element approximation:

(P$^h$) Given $U^0 \in K^h$, for $1 \leqslant n \leqslant M$ find $U^n \in K^h$ such that $\forall \chi \in K^h$

\[
(\frac{U^n - U^{n-1}}{\Delta t}, \chi - U^n)_{L^2} + (\nabla U^n, \nabla (\chi - U^n)) - \frac{1}{\rho}(U^n, \chi - U^n)_{L^2} \geqslant 0
\]

where $\langle \cdot, \cdot \rangle_{L^2}$ is either the $L^2$–inner product or some approximation to the $L^2$–inner product, evaluated according to a vertex quadrature rule such that for $\chi, \phi \in S^h$

\[
|\langle \chi, \phi \rangle - (\chi, \phi)_h| \leqslant C h^{1+r} \| \chi \|_r \| \phi \|_r, \quad (r = 0, 1),
\]

\[
C_1 |\phi|_0^2 \leqslant |\phi|_h^2 \leqslant C_2 |\phi|_0^2,
\]

if $\chi \geqslant 0$ then $(\chi, 1)_h \geqslant 0$,

\[
(1, 1)_h = |\Omega|,
\]

\[
|\chi|_h := [(\chi, \chi)_h]^{1/2}.
\]
Theorem 4.1. For $\Delta t < \varepsilon^2$ there exists a unique sequence $\{U^n\}_{1 \leq n \leq M} \subset K^h$ solving $(P^h)$. Furthermore, $\mathcal{E}^h(\cdot)$, defined by

\begin{equation}
\mathcal{E}^h(\chi) := \frac{1}{2} \| \chi \|^2_h + \frac{1}{2 \Delta t} \left( 1 - \chi^2, 1 \right)^h,
\end{equation}

is a Lyapunov functional for $(P^h)$ satisfying

\begin{equation}
\left( \frac{1}{\Delta t} - \frac{1}{2 \varepsilon^2} \right) \| U^n - U^{n-1} \|^2_h + \frac{1}{2 \Delta t} \| U^n - U^{n-1} \|^2 + \mathcal{E}^h(U^n) \leq \mathcal{E}^h(U^{n-1}),
\end{equation}

and hence the stability estimate holds:

\begin{equation}
\frac{1}{\Delta t} \sum_{n=1}^{M} \| U^n - U^{n-1} \|^2_h + \sum_{n=1}^{M} \| U^n - U^{n-1} \|^2 + \max_{1 \leq n \leq M} \| U^n \|^2 \leq C(U^0).
\end{equation}

Proof. We prove existence and uniqueness for $(P^h)$. Let us fix $n \in [1, M]$ and consider the minimization problem:

Find $U \in K^h$ such that

\[ \mathcal{J}^h(U) := \min_{\chi \in K^h} \mathcal{J}^h(\chi) = \frac{1}{2 \Delta t} \| \chi - U^{n-1} \|^2_h + \mathcal{E}^h(\chi) + (f, \chi)^h. \]

One may easily prove existence of a minimizer by a standard minimization argument and noting that the Euler–Lagrange equation of the minimization problem is (4.5), with $f = 0$. Uniqueness follows according to the usual variational inequality argument.

To prove (4.8) we set $\chi = U^{n-1}$ in (4.5) and using the identity $2a(a - b) = a^2 + (a - b)^2 - b^2$ yields the result. (4.9) results from summing (4.8) from $n = 1$ to $m \in [1, M]$ and noting $\Delta t < \varepsilon^2$. \(\square\)

Adapting the arguments of Blowey & Elliott (1992), for $n \in [1, M]$ one may obtain the error bound

\begin{equation}
|u(t^n) - U^n|^2_h + \varepsilon^2 \sum_{k=1}^{n} \int_{(k-1)\Delta t}^{k\Delta t} |u(s) - U|^2_h ds \leq C \left( h^2 + \frac{h^2}{\varepsilon^2} + \Delta t \right) \exp^{T/\varepsilon^3},
\end{equation}

where $C$ is dependent upon $\mathcal{E}^h(U^0)$ and $\| u \|_{H^1(\Omega)}$. This is clearly not a useful bound for small $\varepsilon$.

4.3 Numerical results and a comparison with the level-set formulation

4.3.1 Numerical technique and construction of data. We briefly describe the numerical technique used to solve (4.5) for $n$ fixed. Let $(\phi_i)_{i=1}^D$ be a piecewise linear basis for $T^h$, where $D$ is the number of vertices of the triangulation, $\phi_i(x_j) = \delta_{ij}$ ($1 \leq i, j \leq D$) and $(x_j)_{j=1}^D$ are the location of the vertices. Define respectively the mass and stiffness matrices to be

\begin{equation}
M_{ij} = (\phi_i, \phi_j)^h \quad \text{and} \quad K_{ij} = (\nabla \phi_i, \nabla \phi_j).
\end{equation}
Let $(\chi, \phi)^h = \int_\Omega I^h(\chi \phi)dx$, so that $M$ is a diagonal matrix, also note that (4.6a-d) are satisfied. Also for any $\chi \in S^h$ we define the vector $\chi \in \mathbb{R}^D$ where $\chi_i = \chi(x_i)$. Hence we rewrite (4.5) in an equivalent vector formulation as $\forall \chi \in \mathbb{R}^D |\chi_i| \leq 1$

\[(\chi - \mathcal{U}^n)^T \left[ \frac{1}{\Delta t} M(\mathcal{U}^n - \mathcal{U}^{n-1}) + K\mathcal{U}^n - \frac{1}{\nu} M\mathcal{U}^n \right] \geq 0.\]

By using the projected SOR method, see Elliott & Ockendon (1982), we easily compute the numerical solution to this variational inequality. The projected SOR technique has an associated relaxation parameter, $\omega$, which was set to be 1.85. We also set the tolerance of convergence to be $1.0 \times 10^{-8}$ in all experiments.

All of the experiments described in this section focus on the discrete level set

\[(4.13) \quad \Gamma^h(n\Delta t) := \{ x \in \Omega : U^n(x) = 0 \},\]

which, in light of §4.1, should approximate motion by mean curvature. The initial data $U^0$ was constructed according to the following recipe: Lay down a uniform mesh with nodes $(x_i)_{i=1}^D$. Let $d_i = d(x_i, 0)$ be the value of the signed distance function, defined by (1.22), then we define

\[U^0(x_i) = \begin{cases} 
1 & \text{if } d_i > \pi\epsilon/2, \\
\sin(d_i/\epsilon) & \text{if } |d_i| \leq \pi\epsilon/2, \\
-1 & \text{if } d_i < -\pi\epsilon/2.
\end{cases}\]

### 4.3.2 Two Dimensional experiments

In all experiments we set $\Omega = (-1/2, 1/2) \times (-1/2, 1/2)$.

**Circle**

In the first experiment $\Gamma(0)$ was set to be a circle of radius $3/(8\sqrt{2})$, so that the radius at time $t$ is $\sqrt{5}/128 - 2t$. This was used as a test to obtain appropriate relationships between all of the discretization parameters. We must have a balance between the constraints $\Delta t < \frac{\epsilon^2}{h}$, $h = o(\epsilon)$ and $\Delta t = o(1)$ whilst having a reasonable computing time. We wish to choose $\Delta t$ larger than $\Delta t = O(h^3)$ because the explicit Euler method has to impose $\Delta t = O(h^3)$ for stability and each implicit time step is more costly than an explicit time step. So we took

\[\Delta t = \epsilon^2/4, \quad h = \left(\frac{s}{8}\right)^{5/4} 128^{1/4} \epsilon^s \quad (s = 1.25, 1.5, 1.75, 2).\]

Notice that the time step is reasonably large and clearly $h = o(\epsilon)$. In fact we fixed $h$, where $h = 1/128, 1/160, 1/192, 1/224, 1/256$, and found $\epsilon$, $\Delta t$ accordingly. Furthermore, it holds that

(1) At worst, $\pi \epsilon \geq 7h$, so the discrete initial data has at least seven points across the interface.

(2) The interface of the initial data is always contained within $\Omega$. 

A discretization parameter, $h$, versus the time at which $\Gamma^h(t)$ disappears is shown in figure 4.1a.

One sees that the relationship appears to be best when $s$ is smallest, and the number of computations is at its greatest. Note that when $h$ is fixed and $s$ is large, $\epsilon$ and $\Delta t$ will be big; this may explain the very poor convergence for large $s$.

We performed a second experiment where the same initial data is taken except

$$h = C_s \epsilon^s$$

When $h = 1/128$, $\epsilon$ is fixed for all values of $s$, i.e. the same initial data regardless of $s$. A graph of the mesh parameter, $h$, versus the time $\Gamma^h(n\Delta t)$ disappears is shown in figure 4.1b.

For this new relationship, the time at which the “circle” disappears is more effectively approximated for the larger values of $s$.

In all future computations, we do not consider the relationship between $\epsilon$, $h$ and $\Delta t$. However, it is clear that since $|U^m| = 1$ in a large proportion of $\Omega$ computing on
a uniform mesh is wasteful. In principle the iterative process can be performed in
a narrow region about the interface. But in general efficiency requires non-uniform
meshes changing with time, see Paolini & Verdi (1992).

In the third experiment the level-set formulation of motion by mean curvature
is discretized as in Osher & Sethian (1988), Sethian (1990) and Sethian (1991).
By applying the far-field boundary condition to be the known solution, results were
obtained which, at worse, were better than our best results. In fact when \( h = 1/128 \)
and \( \Delta t = 1 \times 10^{-4} \) the computed critical time, \( t_c \), at which the discrete level surface
disappeared satisfied
\[
|t_c - 9/256| < 4 \times 10^{-5}.
\]

**Disorder to Order**

We perform a linear stability analysis. Given initial data \( g \) let \( \{z_j\}_{j=1}^{\infty} \) be the
orthonormal basis for \( H^1(\Omega) \) consisting of eigenfunctions for
\[
-\Delta z_j + z_j = \mu_j z_j, \quad \frac{\partial z_j}{\partial \nu} = 0,
\]

note that \( \mu_j \geq 1 \) \( (j \geq 1) \). It follows that when \( |u(x, t)| < 1 \), we can express the
solution as
\[
\begin{aligned}
\alpha_j(0) \exp(i/\nu - (\mu_j - 1))t \sum_{j=1}^{\infty} \alpha_j(t) z_j, & \\
\alpha_j(0) \exp(i/\nu - (\mu_j - 1))t \sum_{j=1}^{\infty} \alpha_j(t) z_j, & \\
\end{aligned}
\]

where \( \alpha_j(0) = (g, z_j) \). It is easy to see that in the early stages all but a finite
number of \( \alpha_j(t) \) will decay to zero; clearly the \( \alpha_j(t) \) which grows at the fastest rate
is \( j = 1 \). So for initial random data we take a perturbation about the mean value
of 0, so that \( \alpha_1(0) \approx 0 \) \( (h = 1/128, \epsilon = 7\pi/128, \Delta t = 1 \times 10^{-4} \) and \( \Gamma^h(n\Delta t) \) is
plotted when \( n = 50, 100, \ldots, 3500 \), see figure 4.2).

**Two dimensional Dumbbell**

Let \( \Gamma(0) \) be given by a dumbbell where the inner strip, centred on the x-axis,
is of width 0.15 and the circles at either end of the strip have centres at \( (\pm 0.3, 0) \)
and are of radius 0.15. We took \( h = 1/128, \epsilon = 7/(128\pi) \) and \( \Delta t = \epsilon^2/4. \)

In figure 4.3 \( \Gamma^h(n\Delta t) \) is displayed when \( n = 0, 20, \ldots, 380 \). We see that in the
early stages, no motion takes place on the flat part of the dumbbell and the circles
move inwards. At a later time \( \Gamma^h(n\Delta t) \) takes on an elongated elliptic like form
which, because of convexity, has negative curvature everywhere on the interface.
Hence one expects \( \Gamma^h(n\Delta t) \) to shrink concentrically. This is the case and \( \Gamma^h(n\Delta t) \)
eventually evolves into a circular shape and disappears at a time of 394\Delta t.

**Seven Point Star**

In this experiment we replicate the simulation of Osher & Sethian (1988) and
take the initial level set to be
\[
\Gamma(0) = (0.2 + 0.13 \sin 14\pi s)(\cos 2\pi s, \sin 2\pi s) \quad s \in [0, 1].
\]
Figure 4.2: Disorder to Order.

Figure 4.3: 2-D Dumbbell.

We took $h = 1/128$, $\epsilon = 7/(128\pi)$ and $\Delta t = \epsilon^2/4$. The discrete level $\Gamma^h(0)$ approximates that of a seven point star. In figure 4.4 we plot $\Gamma^h(n\Delta t)$ where
$n = 0, 8, 16 \cdots, 328$. We see that the finger contract inwards, whilst the webbed part of the fingers move outwards, so that at some time $\Gamma^h(n\Delta t)$ take on a circular shape, this then mimics the concentric contraction of a circle with increasing speed as the radius of the circle shrinks, the time at which the circle disappears is $330\Delta t$.

![Figure 4.4: Seven Point Star.](image)

**Wound Spiral**

We now mimic another experiment of Osher & Sethian (1988), where we take

$$\Gamma(0) = (0.5 \exp^{-0.5 \sin 2\pi s + 1} - 0.025 \cos 2\pi s \cos a(s), \sin a(s)),$$

and $a(s) = 25 \text{atan} (1 + 0.5 \sin 2\pi s), s \in [0,1]$. The shape traced out by this curve is a wound spiral; since the spiral wraps around closely upon itself, in order that we approximate the interface well a very narrow interface is taken involving a larger number of mesh points than usual. The following parameters were taken: $h = 1/256$, $\epsilon = 7/(256\pi)$ and $\Delta t = c^2/4$. In figure 4.5a, when $n = 0, 6, \cdots, 42$, the unwinding then slows down so $\Gamma^h(n\Delta t)$ is plotted where $n$ is incremented by 20 for each plot; the spiral continues to unwind and fattens up a little towards the end.

**4.3.3 Three dimensional experiments.** In each three dimensional experiment we took $\Omega = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 0.5, \sqrt{y^2 + z^2} \leq 0.5\}$. Since the initial data was always axisymmetric, our solution is axisymmetric for all time. In order that we utilize this fact, and hence reduce the number of computations, we made the appropriate change of variable so that we seek a solution of the form $U^\alpha(x, r)$, where $r = \sqrt{y^2 + z^2}$. In all future computations in this section, unless explicitly
stated, we take $h = 1/127$, $\epsilon^2 = 4 \times 10^{-4}$ and $\Delta t = \epsilon^2/4$, so that there are at least 7 points across the initial interfacial region.

The graphical representation of the level surface $\Gamma^h(n\Delta t)$ shown is the $x$-$z$ plane.

A Torus of Circular Cross-section

Suppose that $x(\theta, \phi) \in \Gamma(0)$ is given by

$$x(\theta, \phi) := ((r_0 + R \cos \phi) \cos \theta, (r_0 + R \cos \phi) \sin \theta, R \sin \phi),$$

$-\pi < \theta \leq \pi$, $-\pi < \phi \leq \pi$ for some constants $r_0$ and $R$ satisfying $0 < R < r_0$. One may perform elementary calculations to show that for $r < \frac{r_0}{2}$ the torus initially shrinks into itself. Similarly for $r > r_0/2$ there are two points on the circular cross-section which have zero speed and satisfy

$$\cos \theta = -\frac{r_0}{2R}.$$
Soner & Souganidis (1991) give a theoretical value $r^*$ such that for $r < r^*$ the torus remains toroidal in shape. For $r > r^*$ the torus "focuses" at the origin and eventually opens out to topologically become a sphere.

First we take $R = 0.15$ and $r_0 = 0.3$. Initially from the geometrical calculations we expect to see one stationary point on $\Gamma^k(0)$. Indeed as expected in figure 4.6 the point closest to the origin initially appears to be stationary ($\Gamma^k(n\Delta t)$ is plotted when $n = 0, 8, 16, \cdots, 112$). We take the same data in the second experiment except that $r_0 = 0.2$. Now the inner edge moves inwards and joins to topologically become a sphere; $\Gamma^k(n\Delta t)$ is plotted when $n = 0, 8, 16, \cdots, 176$, see figure 4.7 and eventually takes on an ellipsoidal form which shrinks away in a concentric fashion and disappears when $n = 180$. From the geometrical problem, if the two rotational parameters of a torus are rescaled by $s$ ($s > 0$), then the principle curvatures are both rescaled by $1/s$, so the ratio $R/r_0$ is an invariant quantity. It was found that the ratio of radii $R/r_0$ at which the inner edges of the torus met at the same time at which the torus disappeared was approximately 0.65, this is in the range of possible values, $r^* \in (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}})$, estimated by Soner & Souganidis (1991) and is in close agreement with the numerical value found by Paolini & Verdi (1991).

![Figure 4.6: Torus, $R = 0.15$ and $r_0 = 0.3$.](image)

**The Dumbbell**

$\Gamma(0)$ was taken to be a dumbbell where the inner handle was centred around the $x$-axis with a diameter of 0.15 and the spheres stuck on the ends had centres at $(\pm 0.3, 0, 0)$ with a radius of 0.15. Initially the radius at the handle is much smaller than at the spheres on the end, so one expects the handle to shrink at a faster rate so that after some time the handle pinches off. This indeed appears to be the case,
Figure 4.7: Torus, $R = 0.15$ and $r_0 = 0.2$.

see figure 4.8 where $\Gamma^4(n\Delta t)$ is plotted when $n = 0, 2, \ldots, 90$. The handle pinches off when $n = 34$. The two spheres disappear when $n = 78$.

Figure 4.8: 3-D Dumbbell.
5. Motion by mean-curvature with mass conservation. The evolution equation (1.7a) with the boundary condition (1.11ii) with $q = 0$ has the conservation law,
\[ \frac{d}{dt} \int c \theta + \frac{1}{2} u = 0. \]
Setting $c = 0$ yields
\[ \int u(x, t) = \int u(x, 0). \]
(5.1)
Furthermore if the conductivity is very large compared to the latent heat $l$, i.e. $l/k \ll 1$ it is reasonable to consider the limit
\[ 0 = \Delta \theta. \]
The boundary condition $\partial \theta / \partial v = 0$ then implies
\[ \theta(x, t) = \lambda(t). \]
(5.2)
Substituting into the field equation (1.7b) for the order parameter now results in the modified Allen–Cahn equation
\[ \tau u_t = \gamma \Delta u - \psi'(u) + \alpha \lambda(t), \]
and taking the boundary condition for the order parameter to be
\[ \frac{\partial u}{\partial v} = 0, \]
(5.3b)
and imposing the constraint (5.1) we obtain
\[ \alpha \lambda(t) = \int \psi'(u) dx. \]
(5.3c)
This problem has been studied by Rubinstein & Sternberg (1991) for $\psi(\cdot)$ being a smooth double well potential with the scaling $\tau = \epsilon^2$ and $\gamma = \epsilon^2$. They show that formally in the asymptotic $\epsilon = 0$ limit one obtains a modification of mean curvature flow. They show that the motion of the interface is in the normal direction with speed $V$ given by
\[ V = \kappa^n - \frac{1}{|\Gamma|} \int \kappa^m ds. \]
(5.4)
We note that
1. This geometric flow decreases the measure of the interface and preserves the volume enclosed within the interface as time evolves, viz using (1.27,1.28)
\[ \frac{d}{dt} \text{Per} (\Omega^-(t)) = - \int \kappa^m V = - \int (\kappa^m)^2 + \frac{1}{|\Gamma|} \left( \int \kappa^m \right)^2 \leq 0, \]
so Per $((\Omega^-(t))$ acts as a Lyapunov functional and $\frac{d}{dt} \text{meas } (\Omega^-(t)) = 0$.

(2) Given a collection of balls with radii and centres $\{R_i, x_i\}$ which evolve by this flow and $\max_{i,j} |x_i - x_j|$ is sufficiently large, one finds that there is no coalescence and that the fittest survive, Rubinstein & Sternberg (1991). As a consequence a configuration of two identical balls represents an unstable equilibrium.

In the remainder of this section we consider the parabolic double obstacle version of (5.3). A formal asymptotic analysis, similar to that of §3, a numerical scheme and Γ-convergence result and numerical experiments are all presented.

5.1 Mathematical and asymptotic analysis. We consider the parabolic variational inequality:

Given $g \in L^2(\Omega)$, find $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)) \cap L^\infty(\delta, T; H^2(\Omega)) \cap H^1(\delta, T; L^2(\Omega)) (\forall \delta > 0)$, $|u(t)| \leq 1$ such that for all $\eta \in K$ and a.e. $t \in (0, T)$

(a) $(u_t, \eta - u) + (\nabla u, \nabla \eta - \nabla u) - \frac{1}{\epsilon} (u, \eta - u) \geq \frac{\lambda(t)}{\epsilon} (1, \eta - u)$

(b) $u(x, 0) = g(x)$

(c) $(u(t), 1) = (g, 1) = m$

Existence and uniqueness of $u$ solving (5.5a–c) are proven by penalizing the problem and taking a standard Galerkin approximation. We follow the asymptotic analysis of §3 using the same notation and making the same assumptions. We assume that $\partial u/\partial v = 0$ on $\partial \Omega$ so that integration by parts yields

$\delta u - \Delta u - \frac{1}{\epsilon} (u, \eta - u) \geq 0$.

**Outer Expansion**

Using the notation and results of §1 and §3, we formally expand the variables to obtain the outer expansion for $u$ as

$u(x, t; \epsilon) = u^0(x, t) + \epsilon u^1(x, t) + O(\epsilon^2),$

and the expansion

$\lambda(t; \epsilon) = \lambda^0(t) + O(\epsilon),$

which are valid for all $x \in \Omega \setminus \Gamma^0(t)$ and $t \in [0, T]$. Substitution into (5.5a,5.5c) and respectively calculating the $O(1)$ term yields

$-(u^0, \eta - u^0) \geq 0 \quad |\eta| \leq 1,$
(5.8b) \( (u^0, 1) = m. \)

A simple calculation yields that \( u^0 = \pm 1 \) and

\[
u^0(x) = \begin{cases} 1, & x \in \Omega^0_+(t) := \{ x \in \Omega : u^0(x, t) > 0 \}, \\ -1, & x \in \Omega^0_-(t) := \{ x \in \Omega : u^0(x, t) < 0 \}, \end{cases}\]

where \( |\Omega^0_+| = (|\Omega| + m)/2. \)

**Inner Expansion**

As in §3 we define the new variable

\[
y := \frac{d}{\epsilon},
\]

which stretches the transition layer, and write the inner expansion in terms of the \((y, s)\) coordinate system. Let

\[
u(x, t; \epsilon) := U(y, s, t) = U^0(y, s, t) + \epsilon U^1(y, s, t) + O(\epsilon^2).
\]

Hence substituting into (5.5a) and calculating the \(O(1)\) term we obtain

\[-(u^0_{yy} + U^0, \eta - U^0) \geq 0 \quad |\eta| \leq 1.\]

By matching the inner and outer expansions, c.f. (3.18a), and using the definition of the interface of \( \Gamma(t) \), we obtain

\[
\lim_{y \to \pm \infty} U^0(y, s, t) = u^0(\Gamma^0_{\pm}, t) = \pm 1, \quad U^0(0, s, t) = 0,
\]

where \( \Gamma^0_+ := \partial \Omega^0_+ \cap \Omega \) and \( \Gamma^0_- := \partial \Omega^0_- \cap \Omega \). A calculation yields that

\[
u^0(y, y, t) := \begin{cases} -1, & y < -\pi/2, \\ \sin y, & |y| \leq \pi/2, \quad t \in [0, T], \\ 1, & y > \pi/2, \end{cases}
\]

Calculating the \(O(\epsilon)\) term, using (3.13) and (3.7), we find that \( U^1 = 0 \) for \( |y| \geq \pi/2 \) and when \( |y| < \pi/2, \)

\[-U^1_{yy} - U^1 = \lambda^0 + U^0_\gamma (\kappa^m_0 - V^0),\]

so that using the interfacial condition

\[
u^1(0, s, t) = 0,
\]

we find that \( U^1 \) exists if and only if \( U^0_\gamma \) is orthogonal to \( U^0_\gamma (\kappa^m_0 - V^0) - \lambda^0 \). We obtain

\[
0 = \int_{-\pi/2}^{\pi/2} (\lambda^0 + U^0_\gamma (\kappa^m_0 - V^0)) U^0_\gamma dy = (-V^0 + \kappa^m_0 - 4\lambda^0/\pi) \frac{\pi}{2},
\]
so that \( V^0 = \kappa_m^m - 4\lambda^0/\pi. \) Since \(|\Omega^0| = (|\Omega| - m)/2\), by applying the identity (1.28)

\[
0 = \partial_t |\Omega^0| = \int_{\Gamma^0} V^0(s,t)ds,
\]

it is easy follows that

\[
\lambda^0 = \frac{\pi}{4} \frac{1}{|\Gamma^0|} \int_{\Gamma^0} \kappa^m ds
\]

\[
V^0(s,t) = \kappa^m_0 - \frac{1}{|\Gamma^0|} \int_{\Gamma^0} \kappa^m(s,t)ds,
\]

which concludes the asymptotic analysis.

Luckhaus & Modica (1989), with a different double well potential function, and Blowey & Elliott (1991) have rigorously established (5.12), i.e.

\[
\lim_{t \to 0} \lambda_t = \frac{\pi}{4} \kappa^m,
\]

for the stationary variant of (5.5a,c), where \( \kappa^m \) is the constant mean curvature of the smooth hypersurface solving the variational problem of theorem 5.3(ii) below. This is known as the Gibbs-Thomson relation.

**5.2 Numerical approximation.** In this subsection we discuss the numerical approximation of (Q). We make the same assumptions that were made in §4.2 and consider the finite element approximation:

(Q^h) Given \( U^0 \in K^h_m \), for \( 1 \leq n \leq M \) find \( U^n \in K^h_m \) such that \( \forall \chi \in K^h \)

\[
\left( \frac{U^n - U^{n-1}}{\Delta t}, \chi - U^n \right)_h + \left( \nabla U^n, \nabla \chi - \nabla U^n \right)_h - \frac{1}{2} \left( U^n, \chi - U^n \right)_h \geq \frac{\lambda^h}{\epsilon} (1, \chi - U^n)_h,
\]

where \( K^h_m := \{ \chi \in K^h : (\chi, 1)^h = m \} \).

As in §4.2, it is easy to prove the following theorem

**Theorem 5.1.** For \( \Delta t < \epsilon^2 \) there exists a unique sequence \( \{ U^n \} \) \( 1 \leq n \leq M \subset K^h_m \) solving (Q^h). Furthermore, \( \mathcal{E}^h(\cdot) \) as given by (4.7) is a Lyapunov functional for (Q^h) satisfying

\[
\left( \frac{\Delta t}{2} - \frac{1}{2\epsilon} \right) |U^n - U^{n-1}|^2 + \frac{1}{2} |U^n - U^{n-1}|_h^2 + \mathcal{E}^h(U^n) \leq \mathcal{E}^h(U^{n-1}).
\]

**Proof.** Set \( f = \frac{1}{\epsilon^2} \) in (4.9) where \( \lambda \in (\lambda_L, \lambda_R), \lambda_L = -2\epsilon/\Delta t + 1 \) and \( \lambda_R = 2\epsilon/\Delta t - 1 \). It is easy to show that \( g : [\lambda_L, \lambda_R] \to [-|\Omega|, |\Omega|] \) defined by

\[
g(\lambda) = (U^1, 1)^h,
\]

where \( U_\lambda \) is the minimizer to (4.9), is monotone and continuous. Hence the intermediate value theorem yields the existence of \( \lambda \in (\lambda_L, \lambda_R) \) satisfying \( (U^1, 1)^h = m \), thus proving existence. Uniqueness for \( U^n \) follows easily. The stability estimate follows by setting \( \chi = U^{n-1} \) in (5.14) and rearranging. \( \square \)
5.3 $\Gamma$-convergence. We now prove that any sequence of minimizers $(u_{\epsilon,h})$ of $\mathcal{E}_{\epsilon,h}(-):=\epsilon\mathcal{E}^h(-)$, with an appropriate relation between $\epsilon$ and $h$, converges in $L^1(\Omega)$ to $u_*$ which is a piecewise constant function and that $\lim_{\epsilon,h\to 0} \mathcal{E}_{\epsilon,h}(u_{\epsilon,h}) = \frac{\pi}{2} P_\partial(A_*) \leq \frac{\pi}{2} P_\partial(A)$ for all $A$ such that $|A| = (|\Omega| - m)/2$. Here $P_\partial(A)$ is the perimeter of $A$ and $A_* = \{ x \in \Omega : u_*(x) = -1 \}$. Previously Bellettini, Paolini & Verdi (1990) introduced the notion of $\Gamma$-convergence for finite element approximations and in particular proved convergence for surfaces with prescribed mean curvature. We use the notation $BV(\Omega)$ to mean the space of functions with bounded variation, $D$ to mean the generalized gradient, where $Dv = \nabla v$ if $v \in H^1(\Omega)$, and $\mathcal{H}^{d-1}$ to mean the $(d-1)$-Hausdorff dimensional measure. All of these definitions, along with that of the perimeter, may be found in Giusti (1984).

Define $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(s) = \frac{1}{2} (1 - s^2),$$

for $\epsilon > 0$, $\chi \in C^0$ we note that

$$\mathcal{E}_{\epsilon,h}(\chi) = \frac{\epsilon}{2} |\chi|^2 + \frac{\epsilon}{2e}(\psi(\chi),1)^h,$$

and

$$\int_{-1}^1 \psi^{1/2}(s) ds = \frac{\pi}{2\sqrt{2}}.$$

**Proposition 5.1.** Let $(v_{\epsilon,h})$ be any sequence in $K^h_m$ which converges strongly in $L^1(\Omega)$ to $v_*$ as $\epsilon,h \to 0$. Let $h = o(\epsilon)$ then

$$\frac{\pi}{2} P_\partial(A) \leq \liminf_{\epsilon,h \to 0} \mathcal{E}_{\epsilon,h}(v_{\epsilon,h}),$$

where $A = \{ x \in \Omega : v_*(x) = -1 \}$. Furthermore, if the right-hand side of (5.19) is finite then $|v_*| = 1$ a.e in $\Omega$, $(v_*,1) = m$ and

**Proof.** Clearly (5.19) is trivial if the right-hand side is infinite, so assume that $\lim_{\epsilon,h \to 0} \mathcal{E}_{\epsilon,h}(v_{\epsilon,h}) < \infty$ from which we deduce that

$$\epsilon |v_{\epsilon,h}|^2 \leq C \quad \text{and} \quad \frac{1}{\epsilon} \int \Omega (1 - v_{\epsilon,h}^2) dx \leq C.$$

Now

$$\mathcal{E}_{\epsilon,h}(v_{\epsilon,h}) = \left\{ \frac{\epsilon}{2} |v_{\epsilon,h}|^2 + \frac{1}{2e} \int \Omega (1 - v_{\epsilon,h}^2) dx \right\} + \frac{1}{2e} \int \Omega (v_{\epsilon,h}^2 - \epsilon h v_{\epsilon,h}^2) dx,$$

and

$$=: \mathcal{E}_*(v_{\epsilon,h}) + \mathcal{A}(v_{\epsilon,h}).$$

Using (4.6a) with $r = 1$ and (5.20a)

$$|\mathcal{A}(v_{\epsilon,h})| \leq \frac{1}{2e} Ch^2 \|v_{\epsilon,h}\|_2^2 \leq \frac{Ch^2}{\epsilon^2}.$$
So taking $h = o(\epsilon)$, we deduce that $\lim_{\epsilon \to 0} |A(v_{\epsilon, h})| = 0$. Combining (5.20b) and (5.22) we find $\lim_{\epsilon \to 0} \int_\Omega (1 - v_{\epsilon, h}^2) dx = 0$, so that Fatou’s lemma yields $|v_*| = 1$ a.e. and since

$$
(v_{\epsilon, h}, 1)^h = \int_\Omega I(h, v_{\epsilon, h}) dx = \int_\Omega v_{\epsilon, h} dx = m,
$$

for all $\epsilon, h$, it follows that $(v_*, 1) = m$. For $t \in [-1, 1]$ set

$$
\phi(t) := \int_{-1}^1 \psi^{1/2}(s) ds,
$$

so that $\phi(-1) = 0$ and $\phi(1) = \pi/(2\sqrt{2})$. For $\epsilon > 0$ define $w_{\epsilon, h} = \phi(v_{\epsilon, h})$ and $w_* = \phi(u_*)$. Obviously $|w_{\epsilon, h}|_{L^1(\Omega)} \leq \frac{\pi}{\sqrt{2}} |\Omega|$ and

$$
\int_\Omega |\nabla w_{\epsilon, h}| dx = \frac{\sqrt{2}}{\sqrt{2}} \int_\Omega |\psi^{1/2}(v_{\epsilon, h})| \frac{1}{\sqrt{2}} |\nabla v_{\epsilon, h}| dx,
$$

$$
\leq \frac{1}{\sqrt{2}} \int_\Omega \left\{ \frac{\epsilon}{2} |\nabla v_{\epsilon, h}|^2 + \frac{1}{\epsilon} \psi(v_{\epsilon, h}) \right\} dx,
$$

$$
= \frac{1}{\sqrt{2}} \mathcal{E}_{\epsilon, h}(v_{\epsilon, h}) - \frac{1}{\sqrt{2}} A(v_{\epsilon, h}) \leq \frac{1}{\sqrt{2}} \mathcal{E}_{\epsilon, h}(v_{\epsilon, h}) + \frac{C \epsilon^2}{\epsilon}.
$$

As $(w_{\epsilon, h})$ is equibounded and $\phi \in C^1$, it follows that $w_{\epsilon, h}$ converges to $w_*$ in $L^1(\Omega)$ and

$$
\frac{\pi}{2\sqrt{2}} P_0(A) = \int_\Omega |Dw_*| \leq \liminf_{\epsilon \to 0} \int_\Omega |\nabla w_{\epsilon, h}|,
$$

see Blowey & Elliott (1991) for the details and further references. Hence

$$
\frac{\pi}{2\sqrt{2}} P_0(A) = \int_\Omega |D\phi(v_*)| \leq \liminf_{\epsilon \to 0} \int_\Omega |D\phi(v_{\epsilon, h})| \leq \frac{1}{\sqrt{2}} \liminf_{\epsilon \to 0} \mathcal{E}_{\epsilon, h}(v_{\epsilon, h}).
$$

**Proposition 5.2.** Let $v_* \in L^1(\Omega)$ such that $A = \{ x \in \partial A : v_*(x) = -1 \}$ is an open set and $\partial A$ is a non-empty, compact, smooth hypersurface. Let $h = o(\epsilon)$, then there exists $(v_{\epsilon, h}) \in K^\delta$ converging to $v_*$ in $L^1(\Omega)$ such that

$$
\frac{\pi}{2} P_0(A) \geq \limsup_{\epsilon \to 0} \mathcal{E}_{\epsilon, h}(v_{\epsilon, h}) \quad \text{and} \quad (v_{\epsilon, h}, 1)^h \to (v_*, 1), \quad \text{as} \ \epsilon \to 0
$$

where and $\mathcal{H}_{d-1}(\partial A \cap \partial \Omega) = 0$.

**Proof.** Following Blowey & Elliott (1991), one may construct a family of Lipschitz continuous function $(v_\epsilon)_{\epsilon > 0} \in H^1(\Omega)$ with the following properties:

1. For every $\epsilon > 0$, $|v_\epsilon| \leq 1$,
2. $v_\epsilon \to v_*$ in $L^1(\Omega)$,
3. $(v_\epsilon, 1) = (v_*, 1)$ for $\epsilon > 0$,
4. $\limsup_{\epsilon \to 0^+} \mathcal{E}_\epsilon(v_\epsilon) \leq \frac{\pi}{2} P_0(A)$,
5. $|\nabla v_\epsilon(x)| \leq 1/\epsilon$
6. $\int_\Omega |v_\epsilon - v_*| dx \leq 2\pi \epsilon \xi,$
where \( \lim_{\epsilon \to 0} \xi_\epsilon = P_h(A) \). Now let us define \((v_{\epsilon, A})\) according to the rule \( v_{\epsilon, A} = I_h v_\epsilon \) and analyze the difference,

\[
\int_\Omega |v_\epsilon - I_h v_\epsilon| \, dx \leq \sum_{\tau \in \mathcal{T}_h} \int_{\text{int} \tau} \sum_{i \in \mathcal{I}_\tau} |v_\epsilon(x) - v_\epsilon^i| \, \phi_i(x) \, dx,
\]

where \( x_i \) are the nodes of the element \( \tau \), \( \phi_i \) and \( v_\epsilon^i \) are the basis functions and values of \( v_\epsilon \) associated with the node \( x_i \). Since \( \sum_{i \in \mathcal{I}_\tau} \phi_i(x) \equiv 1 \) and \( v_\epsilon \) is Lipschitz continuous, with Lipschitz constant \( 1/\epsilon \), it follows that

\[
\int_\Omega |v_\epsilon - I_h v_\epsilon| \, dx \leq \sum_{\tau \in \mathcal{T}_h} \int_{\text{int} \tau} \sum_{i \in \mathcal{I}_\tau} |(x - x_i)^T \nabla_x v_\epsilon(x_i) \phi_i(x)| \, dx \leq \frac{h(\Omega)}{\epsilon},
\]

where \( \xi_\epsilon \) lies between \( x \) and \( x_i \), hence \((v_{\epsilon, A}, 1)^h \to (v_\epsilon, 1) \) and \( v_{\epsilon, A} \to v_\epsilon \) in \( L^1(\Omega) \), thus from (6) \( v_{\epsilon, A} \to v_\epsilon \).

Noting that \( |\nabla I_h v_\epsilon| \leq |\nabla v_\epsilon| \) a.e. in \( \Omega \),

\[
\mathcal{E}_{\epsilon, A}(v_{\epsilon, A}) \leq \mathcal{E}_\epsilon(v_\epsilon) + \frac{1}{2\epsilon} \int_\Omega \left( v_\epsilon^2 - I_h(v_\epsilon^2) \right) \, dx = \mathcal{E}_\epsilon(v_\epsilon) + B
\]

\[
\leq \sum_{\tau \in \mathcal{T}_h} \int_{\text{int} \tau} \sum_{i \in \mathcal{I}_\tau} \left( (v_\epsilon^i)^2 - (v_\epsilon(x))^2 \right) \phi_i(x) \, dx,
\]

\[
\leq \sum_{\tau \in \mathcal{T}_h} \int_{\text{int} \tau} \sum_{i \in \mathcal{I}_\tau} |(x - x_i)^T (2v_\epsilon \nabla_x v_\epsilon(x_i)) \phi_i(x)| \, dx \leq \frac{2h(\Omega)}{\epsilon}, \quad 0 \text{ as } \epsilon \to 0.
\]

Hence \( \limsup_{\epsilon \to 0} \mathcal{E}_{\epsilon, A}(v_{\epsilon, A}) \leq \frac{\pi}{2} P_h(A) \). \Box

**Theorem 5.3.** For \( \epsilon > 0 \), \( m \in (-|\Omega|, |\Omega|) \), let \( h = o(\epsilon) \) and suppose that \( u_{\epsilon, A} \) is a solution to

\[
(5.28) \quad \mathcal{E}_{\epsilon, A}(u_{\epsilon, A}) \leq \mathcal{E}_{\epsilon, A}(v) \quad \forall \, v \in K^h_m.
\]

Then there exists a subsequence \( \{u_{\epsilon, A}\} \), which we denote as \( \{u_{\epsilon, A}\} \), such that

(i) As \( \epsilon \to 0 \), \( u_{\epsilon, A} \) converges to \( u_\ast \) in \( L^1(\Omega) \) where \( |u_\ast(x)| = 1 \) for a.e. \( x \in \Omega \).

(ii) The set \( \Omega_\ast = \{ x \in \Omega : u_\ast(x) = -1 \} \) is a solution of the variational problem

\[
P_h(\Omega_\ast) = \min \left\{ P_h(F) : F \subseteq \Omega, |F| = \frac{|\Omega| - m}{2} \right\}.
\]

(iii) \( \lim_{\epsilon \to 0} \mathcal{E}_{\epsilon, A}(u_{\epsilon, A}) = \frac{\pi}{2} P_h(\Omega_\ast) \).

Note that \( P_h(\Omega_\ast) = P_h(\Omega^\ast) \) where \( \Omega^\ast = \{ x \in \Omega : u_\ast(x) = 1 \} \).

**Proof.** The details in the previous two theorems along with the proof of the continuous version of this theorem, as proved in Blowey & Elliott (1991), make it sufficient to prove that the sequence \( v_{\epsilon, A} = \phi(u_{\epsilon, A}) \) is relatively compact in \( L^1(\Omega) \).
First we show that $E_{\epsilon,h}(u_{\epsilon,h})$ is bounded independently of $\epsilon$ and $h$. As comparison functions for $u_{\epsilon,h}$ we construct the following piecewise affine functions $w_{\epsilon,h}$, depending on the first variable $x_1$. Let $w_{\epsilon}$ be defined by

$$w_{\epsilon} = \begin{cases} 
-1 & \text{if } x_1 \leq t^* - \epsilon, \\
\frac{x_1 - t^*}{\epsilon} & \text{if } t^* - \epsilon < x_1 < t^* + \epsilon, \\
1 & \text{if } x_1 \geq t^* + \epsilon,
\end{cases}$$

with $t^*$ chosen so that $(w_{\epsilon,1})^h = m$; define $w_{\epsilon,h} = I^h w_{\epsilon}$ so that $(w_{\epsilon,h,1})^h = m$. If we let $T_{\epsilon} = \{x \in \Omega : \ t^* - \epsilon \leq x_1 \leq t^* + \epsilon\}$, then by the boundedness of $\Omega$, it follows that $|T_{\epsilon}| \leq C \epsilon$ for every $\epsilon > 0$ and some suitable constant $C$. Hence by the minimizing property of $u_{\epsilon,h}$ and Lipschitz continuity of $w_{\epsilon}$,

$$E_{\epsilon,h}(u_{\epsilon,h}) \leq E_{\epsilon,h}(w_{\epsilon,h}) \leq E_{\epsilon}(w_{\epsilon}) + \frac{1}{\epsilon} \int_{\Omega} (w_{\epsilon}^2 - I^h w_{\epsilon,h}^2) dx$$

(5.29)

so that $\lim_{\epsilon \to 0^+} E_{\epsilon,h}(u_{\epsilon,h}) = C$ where $C$ is independent of $\epsilon$ and $h$.

We now prove the existence of a subsequence such that $u_{\epsilon,h} \to u_*$ as $\epsilon,h \to 0$, where $h = o(\epsilon)$. For $\epsilon > 0$ define $v_{\epsilon,h} = \phi(u_{\epsilon,h})$. As $0 \leq \sqrt{2}\phi'(s) \leq 1$ for $s \in [-1,1]$, $(v_{\epsilon,h})$ is bounded in $L^1(\Omega)$. However, recalling (5.25) and (5.29)

$$\int_{\Omega} |Du_{\epsilon,h}| dx \leq \frac{1}{\sqrt{2}} E_{\epsilon,h}(u_{\epsilon,h}) + \frac{Ch^2}{\epsilon^2} \leq C \ \forall \ \epsilon > 0.$$

Now from compactness of $W^{1,1}(\Omega)$ in $BV(\Omega)$, see Giusti (1984), there is a sequence $\{\epsilon\}$ converging to zero such that $u_{\epsilon,h} \to u_*$ in $L^1(\Omega)$. We now return to the sequence $\{u_{\epsilon,h}\}$; as $\phi$ is strictly monotone increasing and continuous on $[-1,1]$, $\phi^{-1}$ is well defined, bounded and uniformly continuous on $[0,\pi/(2\sqrt{2})]$. Define $u_\epsilon(x) = \phi^{-1}(v_{\epsilon,h}(x))$, by the uniform continuity of $\phi^{-1}$ we conclude that $u_{\epsilon,h} = \phi^{-1}(v_{\epsilon,h})$ converges to $u_\epsilon$ in $L^1(\Omega)$. Having proven the existence of a convergent subsequence of $\{u_{\epsilon,h}\} \subset K_h$, converging strongly to $u_\epsilon$ in $L^1(\Omega)$, proposition 5.1 applies and (i) follows. By repeating the arguments of Blowey & Elliott (1991) and using propositions 5.1 and 5.2, (ii) and (iii) follows. 

5.4 Numerical technique and simulations. We briefly sketch the numerical method used to solve (5.14). Clearly as the mapping $g$, defined in the proof of theorem 5.1, is monotone and continuous, we could solve (5.14) iteratively by defining a sequence $\{f_{\epsilon,h}\}_{\epsilon=1}^\infty$ as follows. Seek $U_k \in K_h$ solving

$$\left( \frac{\partial U_k}{\partial x} \right)_{x=0} = m - (U_k,1)^h, \ \chi - U_k \in (\nabla U_k, \nabla \chi - \nabla U_k) - \frac{1}{h} (U_k, \chi - U_k)^h \geq \frac{\lambda_k}{h} (1, \chi - U_k)^h,$$

\forall $\chi \in K_h$ where $\lambda_1 = \lambda_L$, $\lambda_2 = \lambda_R$ and $\{\lambda_{k+1}\}_{k=2}^\infty$ is constructed by the secant method, that is

$$\lambda_{k+1} = \lambda_k + \frac{(\lambda_k - \lambda_{k-1}) (m - (U_k,1)^h)}{(U_k,1)^h - (U_{k-1},1)^h).$$
However, given $\lambda_1$ and $\lambda_2 = \lambda_1 \pm \lambda_{\text{step}}$, where $\lambda_{\text{step}}$ was added if $(U_1, 1) - m < 0$ and similarly for subtraction, the method was found to be satisfactory with $(\lambda_{\text{step}} = 1 \times 10^{-3} (\frac{1}{\Delta t} - 1))$. $U_{k+1}$ solving (5.30) was found using the projected SOR method described in §4.2. We judged that the outer iteration had converged when $|\lambda^k - \lambda^{k+1}| < 1 \times 10^{-6}$. In all experiments the outer iteration converged successfully in at most 4 iterations.

5.4.1 Two dimensional experiments

In the first two experiments which are described, some simulations are performed in which the initial data does not approximate a level set. In each we set $\Omega = (-1/2, 1/2) \times (-1/2, 1/2)$, $h = 1/128$, $\epsilon = 7/(128\pi)$ and $\Delta t = \epsilon^2/4$.

Order to Disorder

In these experiments we let $U^0$ be a random perturbation with a small amplitude of 0.05 about some mean value $m$; in the first experiment $m = 0$ and in the second one $m = -0.704$. Using the same notation as in §4.3.2, one may show that when $|u(x,t)| < 1$

$$u(x,t) = \frac{m}{|\Omega|} \sum_{j=2}^{\infty} \alpha_j(0) \exp(1/j^2 - (n_j - 1)^t) x_j,$$

where $\lambda = -m/(\epsilon |\Omega|))$. All but a finite number of modes in the expansion decay away. For the first experiment, the level set $\Gamma^h(n\Delta t)$ is plotted in figure 5.1 for $n = 50, 100, \ldots, 850$. During the early stages there are several “blobs” which appear and then quickly disappear. The motion of the $\epsilon$-interfacial region is towards a strip approximating a line, being the minimal interface dividing $\Omega$ into two equal measure sets. When $m = -0.704$, see figure 5.2 where $\Gamma^h(n\Delta t)$ is plotted for $n = 50, 100, \ldots, 1250$, one sees fewer blobs appear and $\Gamma^h(n\Delta t)$ quickly moves towards the shape of a quarter circle, which for this value of $m$ approximates the minimal interface. In both of these experiments one can see that the length of $\Gamma^h(n\Delta t)$ approximately decreases as $n$ increases.

Survival of the Fattest

We took six balls, labelled in anti-clockwise rotation by $i$ $(i = 1, 6)$, with centres at

$$x_1, y_1 = (0.2, 0.2), \quad (x_2, y_2) = (0.4, 0.4), \quad (x_3, y_3) = (0.6, 0.2),$$

$$x_4, y_4 = (0.8, 0.5), \quad (x_5, y_5) = (0.5, 0.7) \text{ and } (x_6, y_6) = (0.2, 0.7),$$

where the radii of each ball was given by

$$r_1 = 0.063, \quad r_2 = 0.042, \quad r_3 = 0.070, \quad r_4 = 0.077, \quad r_5 = 0.056 \text{ and } r_6 = 0.077.$$

In figure 5.3 we plot $\Gamma^h(n\Delta t)$ for $n = 0, 40, 80 \ldots, 1160$. We see that balls 1, 2, 3, and 5 quickly decay away giving their masses to balls 4 and 6 which remain of equal size. This represents an unstable configuration. Numerical perturbation upsets this unstable equilibrium and ball 4 eventually survives.
Two dimensional Dumbbell

The parameterization of $\Gamma(0)$ and the whole set of initial data was precisely the same as described in the §4.2 two dimensional dumbbell experiment. The average of the mean curvature over the whole of $\Gamma(0)$ is negative. Thus initially $\Gamma(t)$ will move outwards on the flat part and inwards on the circles at the end of the strip. Initially at the corners $\Gamma^k(n\Delta t)$ moves rapidly. In figure 5.4 we plot $\Gamma^k(n\Delta t)$ for
Figure 5.3: Survival of the Fattest.

$n = 0, 40, \ldots, 880$. The final stationary level set is that of a circle with the correct area.

Figure 5.4: 2-D Dumbbell.
5.4.2 Three dimensional experiments

The Torus with Circular Cross-section

We now consider a torus with circular cross-section. From (4.16), it is easy to calculate the average curvature over the whole torus \( \overline{\kappa} \) and find it is given by

\[
\overline{\kappa} = \frac{1}{|\Gamma|} \int_\Gamma \kappa_1 + \kappa_2 = \frac{r}{(r_0^2 - R^2)^{1/2} \left( r_0 + (r_0^2 - R^2)^{1/2} \right)} - \frac{1}{R},
\]

Initially the maximum speed is attained at the point on the circular cross-section which is closest to the origin with motion towards the origin. There are two values of \( \phi, \phi^1 \) and \( \phi^2 \), where \( \pi/2 < \phi^1 < \pi \) and \( \phi^2 = 2\pi - \phi^1 \), which satisfy

\[
\cos \phi = \frac{R}{(r_0^2 - R^2)^{1/2} + r_0},
\]

and these points have the initial speed of zero. The points where \( \phi^1 < \phi < \phi^2 \) have positive curvature, all other points have negative curvature. Hence initially one expects the front closest to the origin to move towards the origin, thus increasing its curvature and hence speed. It is for this reason that we conjecture that a stationary torus with regular cross-section, if it exists, is unstable since we may perturb the edge closest to the origin to have positive curvature, it should then move closer to the origin thus increasing its curvature. Also note that as the whole front moves towards the centre then the area of the cross-section will increase to conserve the volume contained within the torus.
The Stationary Torus

It is easy to construct a toroidal shape which on the edge closest to the origin has mean curvature less than the average mean curvature. In this experiment we piece together two different halves of an ellipse where the initial surface $\Gamma(0)$ is given by

$$
\Gamma(0) = \begin{cases} 
\frac{1}{2}(r - 0.5)^2 + 2z^2 = 0.1 & \text{if } r \leq 0.5, \\
\frac{4}{2}(r - 0.5)^2 + 2z^2 = 0.1 & \text{if } r \geq 0.5,
\end{cases}
$$

$$
\Omega = \left\{(x, y, z) : |x| \leq 0.75, \sqrt{x^2 + y^2} \leq 0.75\right\}, \quad h = 1.5/191, \quad \epsilon = 10.5/(191\pi) \quad \text{and} \quad \Delta t = \epsilon^3/\lambda. \quad \text{In the experiment, see figure 5.5a where } \Gamma^h(n\Delta t) \text{ is plotted for } n = 25, 50, \cdots, 400 \text{ and figure 5.5b where } n = 500, 600, \cdots, 2200, \text{ the outer edge appears to be almost stationary and the movement of the inner edge is slow too. However, given enough time } \Gamma^h(n\Delta t) \text{ moved towards a spherical shape.}
$$

The Dumbbell

We take the precisely the same initial data for the three dimensional dumbbell experiment as described in §4.3. In figure 5.6 ($\Gamma^h(n\Delta t)$ is plotted for $n = 0, 40, \cdots, 880$) we see that the handle moves inwards and the spheres on the end contract towards the centre. After some time the handle moves outwards and $\Gamma^h(n\Delta t)$ eventually takes on the stationary shape of a sphere. We reduce the width of the handle to 0.1 and plot $\Gamma^h(n\Delta t)$ for $n = 0, 10, \cdots, 500$ and 510, $\cdots, 1050$, see figures 5.7a, b. Notice that the handle pinches off after 20 steps. There is then a long time whilst the unstable two spheres compete. Because of numerical perturbation there is a lack of symmetry, so the left sphere wins and the right sphere disappears after 1080 time steps.
Figures 5.5a,b: Stationary Torus?
Figure 5.6: 3-D Dumbbell, handle width=0.15.

Figure 5.7a,b: 3-D Dumbbell, handle width=0.10.

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