Chapter 4

Variational Formulation of Boundary Value Problems

4.1 Elements of Function Spaces

4.1.1 Space of Continuous Functions

- $\mathbb{N}$ is a set of non-negative integers.
- 1) An $n$-tuple $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is called a multi-index.
- 2) The length of $\alpha$ is
  \[ |\alpha| := \sum_{j=1}^{n} \alpha_j. \]
- 3) $0 = (0, \cdots, 0)$.
- Set $D^\alpha := (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$.

Example 4.1.1. Assume $n = 3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $u(x_1, x_2, x_3) : \mathbb{R}^3 \to \mathbb{R}$.

What is $\sum_{|\alpha| = 3} D^\alpha u$?

$|\alpha| = 3 \implies \sum_{j=1}^{3} \alpha_j = 3.\\
\implies \alpha = (3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 1, 0), (2, 0, 1), (0, 2, 1), (1, 2, 0), (1, 0, 2), (1, 1, 1).\\
\implies \sum_{|\alpha| = 3} D^\alpha u = \frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_2^3} + \frac{\partial^3 u}{\partial x_1^2 \partial x_2} + \frac{\partial^3 u}{\partial x_2^2 \partial x_1} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2}.

This sort of list can get very long. Hence $D^\alpha$ is useful notation.

Definition 4.1.2. Let $\Omega$ be an open set in $\mathbb{R}^n$. Let $k \in \mathbb{N}$. Define spaces $C^k(\Omega)$, $C^k(\overline{\Omega})$ and $C^\infty(\Omega)$ by

$C^k(\Omega) := \{ u : \Omega \to \mathbb{R} \mid D^\alpha u \text{ is continuous in } \Omega \text{ for all } |\alpha| \leq k \}$.

$C^k(\overline{\Omega}) := \{ u : \overline{\Omega} \to \mathbb{R} \mid D^\alpha u \text{ is continuous in } \Omega \text{ for all } |\alpha| \leq k \}$.

$C^\infty(\Omega) := \{ u : \Omega \to \mathbb{R} \mid D^\alpha u \text{ is continuous in } \Omega \text{ for all } \alpha \in \mathbb{N}^n \}$.

where $\overline{\Omega}$ is the closure of $\Omega$. If $\Omega$ is bounded, $\overline{\Omega} = \Omega \cup \partial \Omega$, where $\partial \Omega$ is the boundary of $\Omega$. We denote $C(\Omega) = C^0(\Omega)$ and $C(\overline{\Omega}) = C^0(\overline{\Omega})$.

Example 4.1.3. Set $I := (0, 1)$ and $u(x) := 1/x^2$ for all $x \in I$. Then clearly for all $k \geq 0$, $u \in C^k(I)$. However, in $I = [0, 1]$ $u$ is not continuous at 0. Thus, $u \notin C^1(I)$.

Definition 4.1.4. For a bounded open set $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $u \in C^k(\overline{\Omega})$, the norm $\| u \|_{C^k(\overline{\Omega})}$ is defined by

$\| u \|_{C^k(\overline{\Omega})} := \sum_{|\alpha| \leq k} \sup_{x \in \overline{\Omega}} |D^\alpha u(x)|$.

Example 4.1.5. Let $I = (0, 1)$, $u(x) := x$, $u \in C^1(I)$. Then, $\sup_{x \in I} |u(x)| = 1$.

Definition 4.1.6. For an open set $\Omega \subset \mathbb{R}^n$ and $u \in C(\Omega)$ the support of $u$ denoted by $\text{supp}(u)$ is defined by $\text{supp}(u) := \{ x \in \Omega \mid u(x) \neq 0 \}$.

Remark. The support of $u$ is the smallest closed subset of $\overline{\Omega}$ such that $u = 0$ in $\Omega \setminus \text{supp}(u)$.
Example 4.1.7. 1) Let $\theta = x_0 < x_1 < \cdots < x_n = 1$ be a partition of $[0,1]$. Define 
$$\phi_j(x) : [0,1] \to \mathbb{R}$$ by 
$$\phi_j(x) := \begin{cases} 
\frac{x - x_{j-1}}{h} & x \in (x_{j-1}, x_j), \\
\frac{x_{j+1} - x}{h} & x \in (x_j, x_{j+1}), \\
0 & \text{elsewhere}.
\end{cases}$$

Then we see $\phi_j \in C(\overline{I})$ and $\text{supp}(\phi_j) = [x_{j-1}, x_{j+1}]$.

2) Define $u(x) : \mathbb{R}^n \to \mathbb{R}$ by 
$$u(x) := \begin{cases} 
e^{-\frac{1}{2} \left| x \right|^2} & \left| x \right| < 1, \\
0 & \text{otherwise}.
\end{cases}$$

Then we see $u \in C^\infty(\mathbb{R}^n)$ and $\text{supp}u = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$.

Definition 4.1.8. Define $C^0_b(\Omega) \subset C^0(\Omega)$ by 
$$C^0_b(\Omega) := \{u \in C^0(\Omega) \mid \text{support is a bounded subset of } \Omega\}.$$ 

4.1.2 Spaces of Integrable Functions

Definition 4.1.9. Let $\Omega$ denote an open subset of $\mathbb{R}^n$ and assume $1 \leq p < \infty$. We define a space of integrable functions $L^p(\Omega)$ by 
$$L^p(\Omega) := \left\{ v : \Omega \to \mathbb{R} \mid \int_\Omega |v(x)|^p dx < +\infty \right\}.$$ 

The space $L^p(\Omega)$ is a Banach space with norm $\|v\|_{L^p(\Omega)}$ defined by 
$$\|v\|_{L^p(\Omega)} := \left( \int_\Omega |v(x)|^p dx \right)^{1/p}.$$ 

Especially the space $L^2(\Omega)$ is a Hilbert space with inner product $\langle \cdot , \cdot \rangle_{L^2(\Omega)}$ defined by 
$$\langle u, v \rangle_{L^2(\Omega)} := \int_\Omega u(x) \overline{v(x)} dx$$
and norm $\| \cdot \|_{L^2(\Omega)}$ defined by $\|v\|_{L^2(\Omega)} := \sqrt{\langle v, v \rangle_{L^2(\Omega)}}$. 

We have Minkowski’s inequality as follows. For $u, v \in L^p(\Omega)$, $1 \leq p < \infty$ 
$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$ 

We also have Hölder’s inequality. For $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, $1 \leq p, q < \infty$ with $1/p + 1/q = 1$ 
$$\left| \int_\Omega u(x)v(x) dx \right| \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$ 

Now, any two integrable functions are equal if they are equal almost everywhere, that is, they are equal except on a set of zero measure. Strictly speaking, $L^p(\Omega)$ consists of equivalent classes of functions.

Example 4.1.10. Let $u, v : (-1,1) \to \mathbb{R}$ be 
$$u(x) = \begin{cases} 
1 & x \in (0,1), \\
0 & x \in (-1,0),
\end{cases} \quad v(x) = \begin{cases} 
1 & x \in [0,1), \\
0 & x \in (-1,0).
\end{cases}$$

The functions $u$ and $v$ are equal almost everywhere, since the set $\{0\}$ where $u(0) \neq v(0)$ has zero measure in the interval $(-1,1)$. So $u$ and $v$ are equal as integrable functions in $(-1,1)$.

Suppose that $u \in C^0(\Omega)$, where $\Omega$ is an open set of $\mathbb{R}^n$. Let $v \in C^0_b(\Omega)$. Then we see by integration by parts 
$$\int_\Omega D^\alpha u(x)v(x) dx = (-1)^{|\alpha|} \int_\Omega u(x)D^\alpha v(x) dx,$$ 
where $|\alpha| \leq k$.

Definition 4.1.11. A function $u : \Omega \to \mathbb{R}$ is locally integrable if $u \in L^1(U)$ for every bounded open set $U$ such that $\overline{U} \subset \Omega$.

Definition 4.1.12. Suppose $u : \Omega \to \mathbb{R}$ is locally integrable and there is a locally integrable function $w_u : \Omega \to \mathbb{R}$ such that 
$$\int_\Omega w_u(x)\phi(x) dx = (-1)^{|\alpha|} \int_\Omega u(x)D^\alpha \phi(x) dx$$
for all $\phi \in C^0_b(\Omega)$.

Then the weak derivative of $u$ of order $\alpha$ denoted by $D^\alpha u$ is defined by $D^\alpha u = w_u$. 

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Note that at most only one \( w_\alpha \) satisfies (4.1.12) so the weak derivative of \( u \) is well-defined. Indeed, the following DuBois-Raymond lemma shows such \( w_\alpha \) is unique.

**Lemma 4.1.13.** (DuBois-Raymond) Suppose \( \Omega \) is an open set in \( \mathbb{R}^n \) and \( w : \Omega \to \mathbb{R} \) is locally integrable. If
\[
\int_\Omega w(x) \phi(x) dx = 0
\]
for all \( \phi \in C^\infty_0(\Omega) \), then \( w(x) = 0 \) for a.e \( x \in \Omega \).

We will use \( D \) for both classical and weak derivatives.

**Example 4.1.14.** Let \( \Omega = \mathbb{R} \). Set \( u(x) = (1 - |x|)_+ \), \( x \in \Omega \), where
\[
(x)_+ := \begin{cases} 
  x & x > 0, \\
  0 & x \leq 0.
\end{cases}
\]

Thus,
\[
\begin{align*}
  u(x) &= \begin{cases} 
    0 & x \leq -1, \\
    1 + x & -1 \leq x \leq 0, \\
    1 - x & 0 \leq x \leq 1, \\
    0 & 1 \leq x.
  \end{cases}
\end{align*}
\]

Clearly we see that \( u \) is locally integrable, \( u \in C(\Omega) \) and \( u \notin C^4(\Omega) \). However, it may have a weak derivative. Take any \( \phi \in C^\infty_0(\Omega) \) and \( \alpha \). Then,
\[
(-1)^{|\alpha|} \int_\Omega u(x) D^\alpha \phi(x) dx = -\int_{-\infty}^{\infty} u(x) \phi'(x) dx = -\int_{-1}^{1} (1 - |x|) \phi'(x) dx
\]
\[
= -\int_{-1}^{0} (1 + x) \phi'(x) dx + \int_{0}^{1} (1 - x) \phi'(x) dx
\]
\[
= \int_{-1}^{0} \phi(x) dx + \int_{0}^{1} (1 -) \phi(x) dx
\]
\[
= \int_{\Omega} w(x) \phi(x) dx,
\]
where
\[
\begin{align*}
  w(x) &= \begin{cases} 
    0 & x < -1, \\
    1 & -1 < x < 0, \\
    -1 & 0 < x < 1, \\
    0 & 1 < x.
  \end{cases}
\end{align*}
\]

Here we do not worry about the points \( x = -1, 0, 1 \), since they have zero measure. Thus, \( u \) has its weak derivative \( Du = w \).

**Definition 4.1.15.** Let \( k \) be a non-negative integer and \( p \in [0, \infty) \). The space \( W^{k,p}(\Omega) \) defined by
\[
W^{k,p}(\Omega) := \{ u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \forall |\alpha| \leq k \}
\]
is called a Sobolev space. It is a Banach space with the norm
\[ \lVert u \rVert_{W^{k,p}(\Omega)} := \left( \sum_{|\alpha| \leq k} \lVert D^\alpha u \rVert_{L^p(\Omega)}^p \right)^{1/p}. \]

Especially, when \( p = 2 \), we denote \( H^k(\Omega) \) as \( W^{k,2}(\Omega) \). It is a Hilbert space with the inner product
\[ \langle u, v \rangle_{W^{k,2}(\Omega)} := \sum_{|\alpha| \leq k} \langle D^\alpha u, D^\alpha v \rangle_{L^2(\Omega)}. \]

Of special interest are \( H^1(\Omega) \) and \( H^2(\Omega) \). If \( \Omega = (a, b) \subseteq \mathbb{R} \), we see that
\[
\begin{align*}
(u, v)_{H^1(\Omega)} &= (u, v)_{L^2(\Omega)} + \langle Du, Dv \rangle_{L^2(\Omega)} \\
&= \int_a^b u(x)v(x)dx + \int_a^b Du(x)Dv(x)dx.
\end{align*}
\]

\( (u, v)_{H^2(\Omega)} = (u, v)_{L^2(\Omega)} + \langle Du, Dv \rangle_{L^2(\Omega)} + \langle D^2u, D^2v \rangle_{L^2(\Omega)} \)
\[ = \int_a^b u(x)v(x)dx + \int_a^b Du(x)Dv(x)dx + \int_a^b D^2u(x)D^2v(x)dx. \]

**Remark.** 1) By using H"older’s inequality, we can prove Cauchy-Schwarz inequality for the inner product of \( H^1(\Omega) \) as follows.
\[
\lvert \langle u, v \rangle_{H^1(\Omega)} \rvert \leq \sum_{|\alpha| \leq 1} \lVert D^\alpha u \rVert_{L^2(\Omega)} \lVert D^\alpha v \rVert_{L^2(\Omega)} \\
\leq \sum_{|\alpha| \leq 1} \lVert D^\alpha u \rVert_{L^2(\Omega)} \lVert D^\alpha v \rVert_{L^2(\Omega)} ^{1/2} \lVert D^\alpha v \rVert_{L^2(\Omega)} ^{1/2} \\
\leq \left( \sum_{|\alpha| \leq 1} \lVert D^\alpha u \rVert_{L^2(\Omega)} ^2 \right) ^{1/2} \left( \sum_{|\alpha| \leq 1} \lVert D^\alpha v \rVert_{L^2(\Omega)} ^2 \right) ^{1/2} \lVert u \rVert_{H^1(\Omega)} \lVert v \rVert_{H^1(\Omega)}.
\]

2) Let \( \Omega = (a, b) \subseteq \mathbb{R} \) and \( u \in H^1(\Omega) \). Then \( u \in C(\overline{\Omega}) \). In higher space dimensions this statement is no longer true.

### 4.2 One Dimensional Problem: Dirichlet condition

Let \( \Omega = (0, 1) \), \( p(\cdot), q(\cdot) \in C(\overline{\Omega}) \) and \( f(\cdot) \in L^2(\Omega) \). Note that \( \partial \Omega = \{ x = 0 \} \cup \{ x = 1 \} \). We consider the following problem.

Find \( u : \Omega \to \mathbb{R} \) such that
\[
\begin{cases}
- \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + qu = f, & x \in \Omega, \\
\frac{du}{dx}(0) = 0, & x \in \partial \Omega.
\end{cases}
\]

Specifying the value of \( u \) at boundary points is said to be a Dirichlet boundary condition. Now the methodology is

1) multiply the equation by a test function, integrate by parts and use boundary conditions appropriately.
2) identify \( V, a(\cdot, \cdot) \) and \( l(\cdot) \).
3) verify, if possible, the assumptions of Lax-Milgram.

\[ \implies \text{Unique existence to the variational formulation of the BVP.} \]

Let \( \phi : \overline{\Omega} \to \mathbb{R} \) be sufficiently smooth. We will call \( \phi \) our test function. Let us follow the methodology.

1)
\[
\int_\Omega f(x)\phi(x)dx = \int_\Omega \left( -\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) \phi(x) + qu(x)\phi(x) \right) dx \\
= \int_0^1 \left( p(x) \phi(x) \frac{du}{dx}(x) \right) dx + \int_0^1 \left( p(x) \frac{du}{dx}(x) \phi(x) + q(x)u(x)\phi(x) \right) dx.
\]

We want to eliminate the term \( p(x)\frac{du}{dx}(x)\phi(x) \) so that we can suppose that the test function \( \phi \) satisfies the same Dirichlet conditions as \( u \), i.e., \( \phi(0) = \phi(1) = 0 \).

Then we have that
\[
\int_0^1 \left( p(x) \frac{du}{dx} \phi(x) - q(x)u(x)\phi(x) \right) dx = \int_0^1 f(x)\phi(x)dx
\]
for any test function \( \phi \). We want \( u, v \) to be from the same space. For the term \( \int_0^1 u(x)dx \) to make sense, we need \( u, \phi \in L^2(\Omega) \). For the derivatives \( du/dx, d\phi/dx \) to make sense, we take this further, so \( u, \phi \in H^1(\Omega) \).

2) Let us choose \( V := \{ \phi \in H^1(\Omega) | \phi(0) = \phi(1) = 0 \} \).
where
\[ H^1(\Omega) = \{ \phi \in L^2(\Omega) \mid D\phi \in L^2(\Omega) \}. \]

We equip the inner product \( \langle \cdot, \cdot \rangle : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \). Let us define
\[ a(u, v) := \int_{\Omega} (p Du Dv + quv) \, dx, \]
\[ l(v) := \int_{\Omega} f v \, dx. \]

Moreover, assume that \( p(x) \geq p_0 > 0, q(x) \geq q_0 > 0 \) for all \( x \in \Omega \).

3) We will verify the assumptions of Lax-Milgram’s theorem.

i) For \( \phi \in V \) and \( f \in L^2(\Omega) \), we see by Cauchy-Schwarz inequality that
\[ |I(\phi)| = |\int_{\Omega} f \phi \, dx| \]
\[ \leq \| f \|_{L^2(\Omega)} \| \phi \|_{L^2(\Omega)} \]
\[ \leq \| f \|_{L^2(\Omega)} (\| \phi \|_{L^2(\Omega)}^2 + \| D\phi \|_{L^2(\Omega)}^2) \]
\[ = \alpha \| \phi \|_V, \]
where we have set \( \alpha := \| f \|_{L^2(\Omega)} \). Thus, \( I : V \to \mathbb{R} \) is bounded. Clearly \( I \) is linear, i.e., \( I(\alpha \phi + \beta \psi) = \alpha I(\phi) + \beta I(\psi) \) for any \( \phi, \psi \in V \) and \( \alpha, \beta \in \mathbb{R} \).

ii) Obviously \( a(\cdot, \cdot) : V \times V \to \mathbb{R} \) is bilinear. Moreover \( a(\cdot, \cdot) \) is bounded. Indeed,
\[ |a(\phi, \psi)| \leq \int_{\Omega} p D\phi D\psi \, dx + \int_{\Omega} q \phi \psi \, dx \]
\[ \leq \max_{x \in \Omega} |p(x)| \int_{\Omega} |D\phi D\psi| \, dx + \max_{x \in \Omega} |q(x)| \int_{\Omega} |\phi \psi| \, dx \]
\[ \leq \max_{x \in \Omega} |p(x)| \| D\phi \|_{L^2(\Omega)} \| D\psi \|_{L^2(\Omega)} + \max_{x \in \Omega} |q(x)| \| \phi \|_{L^2(\Omega)} \| \psi \|_{L^2(\Omega)} \]
\[ \leq C \sqrt{\| D\phi \|_{L^2(\Omega)}^2 + \| \phi \|_{L^2(\Omega)}^2} \sqrt{\| D\psi \|_{L^2(\Omega)}^2 + \| \psi \|_{L^2(\Omega)}^2} \]
\[ = C \| \phi \|_{H^1(\Omega)} \| \psi \|_{H^1(\Omega)}, \]
where we have set
\[ C := \max_{x \in \Omega} \max_{x \in \Omega} |p(x)|. \]

The bilinear form \( a(\cdot, \cdot) \) is coercive, since for all \( \phi \in V \)
\[ a(\phi, \phi) = \int_{\Omega} p |D\phi|^2 \, dx + \int_{\Omega} q |\phi|^2 \, dx \]
\[ \geq p_0 \int_{\Omega} |D\phi|^2 \, dx + q_0 \int_{\Omega} |\phi|^2 \, dx \]
\[ = C \| \phi \|^2_V, \]
where we have set \( C := \min\{p_0, q_0\} \).

We can now apply Lax-Milgram’s theorem to see that there uniquely exists a solution to the following problem
\[ (P) \text{ Find } u \in V \text{ such that} \]
\[ \int_{\Omega} (p Du D\phi + qu \phi) \, dx = \int_{\Omega} f \phi \, dx, \]
for any \( \phi \in V \).

Remark. For \( V = H^1_0(\Omega) \) we can use the norm \( \| \cdot \|_V \) defined by
\[ \| \phi \|^2_V = \int_{\Omega} |D\phi|^2 \, dx, \]
since the following Poincare inequality holds: there exists \( C > 0 \) such that
\[ \int_{\Omega} |\phi|^2 \, dx \leq C \int_{\Omega} |D\phi|^2 \, dx \text{ for } \forall \phi \in V. \]

By using this inequality we can prove the unique existence of the solution solving
\[ (P) \] with \( q = 0 \) in the same way as above.

4.3 One Dimensional Problem: Neumann condition

Let \( \Omega = (0, 1) \), \( p(x), q(x) \in C(\Omega) \) and \( f(x) \in L^2(\Omega) \). We consider the following problem.

Find \( u : \Omega \to \mathbb{R} \) such that
\[ \begin{cases} \frac{d}{dx} \left( \frac{du}{dx} \right) + qu = f, & x \in \Omega, \\ \frac{d}{dx} u(x) = 0, & x \in \partial \Omega. \end{cases} \]
Specifying the value of $du/dx$ at boundary points is said to be a Neumann boundary condition. We assume the same conditions for $p, q$ as before, i.e. $p(x) \geq p_0 > 0$, $q(x) \geq q_0 > 0$ in $\Omega$. Let us derive the variational form. Take a sufficient smooth test function $\phi$, multiply (NBVP) by $\phi$ and integrate.

$$
\int_\Omega f(x)\phi(x)dx = \int_\Omega \left( \frac{d}{dx} \left( p(x) \frac{du(x)}{dx} \right) \phi(x) + q(x)u(x)\phi(x) \right)dx
$$

We have eliminated the term $[p(x)du(x)/dx]_{x=1}^{x=1}$ by taking into account the Neumann boundary conditions $du(x)/dx = 0$ for $x = 0, 1$. Let us choose the functional

$$
a(u, v) := \int_\Omega (pDuDv + quv)dx,
$$

The corresponding variational problem is that:

\begin{align*}
\text{(P)} \quad & \text{Find } u \in V \text{ such that } \\
& \int_\Omega (pDuD\phi + quv)dx = \int_\Omega f\phi dx,
\end{align*}

for any $\phi \in V$.

Again the linear form $l(\cdot) : V \to \mathbb{R}$ and the bilinear form $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ satisfy the assumptions of Lax-Milgram’s theorem, hence the problem (P) has the unique solution.

Remark. Consider (NBVP) with $q = 0$. Then we see

$$
\int_\Omega f(x)dx = -\int_\Omega \frac{d}{dx} \left( p(x) \frac{du(x)}{dx} \right)dx = p(x)\frac{du(x)}{dx}\bigg|_{x=0}^{x=1} = 0.
$$

Thus, we need to assume $\int_\Omega f(x)dx = 0$ as a compatibility condition in this case. In order to prove the unique existence of the solution, we need to modify the functional space. Let us define $H^1_{n}(\Omega)$ by

$$
H^1_{n}(\Omega) := \{ \phi \in H^1(\Omega) \mid \int_\Omega \phi dx = 0 \},
$$

and equip the same inner product as $H^1(\Omega)$. Again Poincaré’s inequality is available for this space $H^1_n(\Omega)$, i.e. for all $\phi \in H^1_n(\Omega)$

$$
\int_\Omega |\phi|^2dx \leq C \int_\Omega |\phi|^2dx,
$$

where $C > 0$ is a constant. By using this inequality we can prove the unique existence of the solution solving (P) with $q = 0$ and $V = H^1_n(\Omega)$ in the same way as above on the assumption $\int_\Omega f(x)dx = 0$.

4.4 One Dimensional Problem: Robin/Newton Condition

Let $\Omega = (0, 1)$, $p(x), q(x) \in C(\bar{\Omega})$, $f(x) \in L^2(\Omega)$, $\delta, g_0, g_1 \in \mathbb{R}$ be constants. We consider the following problem. Find $u : \overline{\Omega} \to \mathbb{R}$ such that

\begin{align*}
-\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u &= f, & x \in \Omega, \\
-\frac{d}{dx} u(x) + \delta u(x) &= g_0, \\
p(x)\frac{du}{dx}(1) + \delta u(1) &= g_1, & \text{(RNVBP)}
\end{align*}

Let us derive the variational form. Take sufficiently smooth $\phi$. This kind of boundary condition is said to be a Robin/Newton boundary condition. We assume the same conditions for $p, q$ as before, i.e. $p(x) \geq p_0 > 0$, $q(x) \geq q_0 > 0$ in $\Omega$ and that $\delta \geq 0$.

\begin{align*}
\int_\Omega f(x)\phi(x)dx &= \int_\Omega \left( -\frac{d}{dx} \left( p(x) \frac{du(x)}{dx} \right) \phi(x) + q(x)u(x)\phi(x) \right)dx \\
&= \left[ -p(x)\frac{du(x)}{dx} \phi(x) \right]_{x=0}^{x=1} + \int_\Omega \left( p(x)\frac{du(x)}{dx}\phi(x) + q(x)u(x)\phi(x) \right)dx \\
&= (-g_1 + \delta u(1))\phi(1) - (g_0 - \delta u(0))\phi(0) \\
&+ \int_\Omega \left( p(x)\frac{du(x)}{dx}\phi(x) + q(x)u(x)\phi(x) \right)dx,
\end{align*}

for all $\phi \in H^1_n(\Omega)$.
which is equal to
\[ \int_{\Omega} \left( p(x) \frac{du(x)}{dx} \frac{d\phi(x)}{dx} + q(x) u(x) \phi(x) \right) dx + \delta u(1) \phi(1) + \delta u(0) \phi(0) \]
\[ = \int_{\Omega} f(x) \phi(x) dx - g_1 \phi(1) + g_0 \phi(0). \]

This suggests that we should define \( a(\cdot, \cdot) \), \( l(\cdot) \) and the functional space \( V \) as following.

\[ a(u, v) := \int_{\Omega} \left( pDuDv + quv \right) dx + \delta u(1)v(1) + \delta u(0)v(0), \]
\[ l(v) := \int_{\Omega} fvdx - g_1 v(1) + g_0 v(0), \]
\[ V := H^1(\Omega). \]

As usual we need to show that the \( (b) \)-linear forms \( a, l \) are bounded and \( a \) is coercive to establish the unique solvability of \((RNBVP)\). Let us assume the following inequality holds true for a while:
\[ |\phi(x)| \leq C|\phi|_{H^1(\Omega)} \quad (4.1) \]
for all \( \phi \in H^1(\Omega) \). Then it is easy to see that \( a(\cdot, \cdot), l(\cdot) \) are bounded. Now
\[ a(\phi, \phi) \geq \min(p_0, q_0) |\phi|_2^2 + \delta (\phi(1)^2 + \phi(0)^2) \]
\[ \geq \min(p_0, q_0) |\phi|_2^2 \]
\[ = a|\phi|_2^2, \]
where \( \alpha = \min(p_0, q_0) \). Hence the bilinear form \( a \) becomes coercive and Lax-Milgram’s theorem assures the unique existence of the solution.

### 4.5 One Dimensional \( H^1 \) inequalities

Here we derive some inequalities in the one dimensional case \( \Omega = (a, b) \). We define a function space \( H^1_{\alpha}(\Omega) \) by
\[ H^1_{\alpha}(\Omega) := \{ \phi \in H^1(\Omega) \mid \phi(a) = 0 \}. \]

The following inequality is one example of Poincaré–Friedrichs inequality.

### Proposition 4.5.1. For all \( \phi \in H^1_{\alpha}(\Omega) \),
\[ \|\phi\|_{L^2(\Omega)} \leq \frac{1}{\sqrt{2}}(b-a)^{-\frac{1}{2}}\|D\phi\|_{L^2(\Omega)}. \]

**Proof.** We can write that for \( a \leq x \leq b \)
\[ \phi(x) = \int_a^x D\phi(y)dy. \]

Then we see that
\[ \|\phi\|^2_{L^2(\Omega)} = \int_a^b \phi(x)^2dx \]
\[ = \int_a^b \left( \int_a^x \phi''(y)dy \right)^2 dx \]
\[ \leq \int_a^b \left( \int_a^x \phi''(y)^2dy \right) dx \]
\[ = \int_a^b \left( \int_a^x \phi''(y)^2dy \right) dx \]
\[ \leq \int_a^b (x-a) \int_a^b \phi''(y)^2dydx \]
\[ = \frac{1}{2}(b-a)^2 \|D\phi\|^2_{L^2(a,b)}. \]

### Example 4.5.2. We can apply this inequality to prove the unique solvability of the Dirichlet problem with a functional space \( V \), a bilinear form \( a(\cdot, \cdot) \) and a linear functional \( l(\cdot) \) defined by
\[ V := \{ \phi \in H^1(0, 1) \mid \phi(0) = \phi(1) = 0 \}, \]
\[ a(u, v) := \int_0^1 pDuDv \text{ for all } u, v \in V, \]
\[ l(u) := \int_0^1 fudx \text{ for all } u \in V. \]

We only check that the bilinear form \( a \) is bounded and coercive. We see that
\[ |a(u, v)| \leq \sup_{x \in (0,1)} |p(x)| \|Du\|_{L^2(\Omega)} \|Dv\|_{L^2(\Omega)} \leq \sup_{x \in (0,1)} |p(x)| \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \]
and
\[ a(v, v) \geq p_0 \| Dv \|^2_{L^2(\Omega)} \]
\[ = \frac{p_0}{2} \| Dv \|^2_{L^2(\Omega)} + \frac{p_0}{2} \| Dv \|^2_{L^2(\Omega)} \]
\[ \geq \frac{p_0}{4} \left( \| Dv \|^2_{L^2(\Omega)} - \frac{2 \| v \|^2_{L^2(\Omega)}}{(b - a)^2} \right) \]
\[ \geq \frac{\alpha}{4} \| v \|^2_{L^2(\Omega)}, \]
where \( \alpha := p_0 \min(1, 2/(b - a)^2)/2 \).

Proposition 4.5.3. The following Agmon's inequality holds. For all \( \phi \in H^1_0(\Omega) \)
\[ \max_{x \in \Omega} |\phi(x)|^2 \leq 2 \| \phi \|_{L^2(\Omega)} \| \phi \|_{H^1(\Omega)}. \]

Proof. Let \( \phi(x) = \sum_{i=1}^{N} \phi_i \xi_i(x) \) be a solution to the elliptic equation. Then
\[ \phi(x)^2 = \int_{\Omega} \frac{d\phi(x)}{dx} d\eta = 2 \int_{\Omega} \phi(x) \frac{d\phi(x)}{dx} d\eta \]
\[ \leq 2 \left( \int_{\Omega} \phi(x)^2 d\eta \right)^{1/2} \left( \int_{\Omega} \frac{d\phi(x)}{dx} d\eta \right)^{1/2} \]
\[ \leq 2 \left( \int_{\Omega} \phi(x)^2 d\eta \right)^{1/2} \left( \int_{\Omega} \phi(x)^2 d\eta \right)^{1/2} \]
\[ \leq 2 \| \phi \|_{L^2(\Omega)} \| \phi \|_{H^1(\Omega)}, \]
which gives the inequality. \( \Box \)

4.6 Weak Solutions to Elliptic Problems

The simplest elliptic equation is Laplace's equation:
\[ \Delta u = 0, \quad (4.2) \]
where \( \Delta := \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \) is the Laplace operator. A general second order elliptic equation is given a bounded open set \( \Omega \subset \mathbb{R}^n \) find \( u \) such that:
\[ \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f(x) \quad \forall x \in \Omega, \quad (4.3) \]
where classically \( a_{ij} \in C^1(\Omega), i, j = 1, \ldots, n; b_i \in C(\Omega), i = 1, \ldots, n; c \in C(\Omega); f \in C(\Omega). \) For the equation to be elliptic we require
\[ \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \tilde{C} \sum_{i=1}^{n} \xi_i^2 \quad \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \quad (4.4) \]
where \( \tilde{C} > 0 \) is independent of \( x, \xi. \) Condition (4.4) is called uniform ellipticity.

The equation is usually supplemented with boundary conditions - Dirichlet, Neumann, or a mixed Dirichlet/Neumann boundary.

In the case of the homogeneous Dirichlet problem \( u = 0 \) on \( \partial \Omega \) \( u \) is said to be a classical solution provided \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \). Elliptic theory tells us that there exists a unique classical solution provided \( a_{ij}, b_i, c, f \) and \( \Omega \) are sufficiently smooth. However we are only interested in problems where the data is not smooth, for example \( f = \sin(1/2 \pi \xi \cdot x), \Omega = (-1, 1). \) This problem can’t have \( u \in C^2(\Omega) \) because \( \Delta u \) has a jump discontinuity are \( |\xi| = 1/2. \) With the help of functional analysis the existence/uniqueness theory for 'weak', 'variational' solutions turn out to be easy and is good for FEM.

4.7 Variational Formulation of Elliptic Equation: Neumann Condition

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Let \( p, q \in C(\overline{\Omega}) \) such that
\[ p(x) \geq p_0 > 0, \quad q(x) \geq q_0 > 0 \quad \forall x \in \overline{\Omega}, \]
and \( f \in L^2(\Omega) \).

Find \( u : \Omega \rightarrow \mathbb{R} \) such that

\[
\begin{align*}
-\nabla \cdot (p\nabla u) + qu &= f, \quad x \in \Omega, \\
\frac{\partial u}{\partial n} &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( n \) is the unit outward normal to the boundary \( \partial \Omega \). Note that

\[
\nabla \cdot (p\nabla u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{\partial u}{\partial x_i} p \right).
\]

\[
= \sum_{i=1}^{n} \left( \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial p}{\partial x_i} \frac{\partial u}{\partial x_i} \right) \\
= p \Delta u + \nabla p \cdot \nabla u.
\]

\[
\frac{\partial u}{\partial n} = \nabla u \cdot n.
\]

So we have a second order PDE. In order to derive the variational formulation we used integration by parts. Let us revise some formulae related to the integration by parts.

**Notation:**

\[
\nabla v = \left( \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \right)^T
\]

\[
\nabla \cdot \nabla v = \nabla^2 v = \Delta v
\]

\[
\nabla \cdot A = \sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}
\]

\[
(D^2 v)_{ij} = \frac{\partial^2 v}{\partial x_i \partial x_j}
\]

**Theorem 4.7.1 (Divergence theorem).** Let \( A : \Omega \rightarrow \mathbb{R}^n \) be a \( C^1 \) vector field. The following equality holds.

\[
\int_{\Omega} \nabla \cdot A \, dx = \int_{\partial \Omega} A \cdot n \, ds.
\]

**Remark.** Suppose \( A = f e_i \) with the coordinate vector \( e_i = (0, \ldots, 1, \ldots, 0)^T \), i.e. the \( j \)th component is \( \{ e_i \}_j = \delta_{i,j} \). Then we see

\[
\nabla \cdot A = \frac{\partial}{\partial x_i} f.
\]

So by the Divergence theorem

\[
\int_{\Omega} f \, dx = \int_{\partial \Omega} f n \, ds.
\]

In one dimensional case where \( \Omega = (a, b) \), \( \partial \Omega = \{a, b\} \), Divergence theorem becomes

\[
\int_{a}^{b} \frac{\partial f}{\partial x} \, dx = f(b) - f(a).
\]

Let us derive the integration by parts formula.

**Proposition 4.7.1 (Integration by parts).**

For \( A \in C^1(\Omega, \mathbb{R}^n) \), \( g \in C^1(\Omega) \),

\[
\int_{\Omega} A \cdot \nabla g \, dx = \int_{\Omega} g A \cdot \nabla \, dx - \int_{\partial \Omega} g A \cdot n \, ds.
\]

**Proof.** By Divergence theorem we see that

\[
\int_{\Omega} \nabla \cdot (Ag) \, dx = \int_{\Omega} A \cdot (g \nabla) \, dx.
\]

Alternatively,

\[
\nabla \cdot (Ag) = g \nabla + A \cdot \nabla g.
\]

By combining these equality we get the desired formula.

For example, if \( A = \nabla u \) and \( g = v \), we have by integration by parts formula that

\[
\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} v \nabla u \cdot \nabla \, dx - \int_{\Omega} v \nabla \cdot \nabla u \, dx.
\]

Noting that

\[
\nabla u \cdot n = \frac{\partial u}{\partial n} \quad \text{and} \quad \nabla \cdot \nabla u = \Delta u.
\]
where $\Delta$ is Laplacian, we obtain
\[
\int_\Omega v\Delta u \, dx = \int_\Omega \nabla v \cdot \nabla u \, dx.
\]
Looking at our boundary value problem $-\varepsilon(p\nabla u) + qu = f$, we have that
\[
-\int_\Omega \varepsilon(p\nabla u) \, dx = \int_\Omega p\nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} p\frac{\partial u}{\partial n} v \, ds.
\]
Let $v$ be a sufficiently smooth test function. Multiply (NBVP) by $v$ and integrate using Divergence theorem.
\[
\int_\Omega fv \, dx = \int_\Omega (-\nabla \cdot (p\nabla u) + qu) \, dx
= \int_\Omega p\nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} p\frac{\partial u}{\partial n} v \, ds
\]
Since $\partial u/\partial n = 0$ on $\partial\Omega$, we do not need to place a restriction on the test function $v$. So if $u$ solve (BVP), then
\[
\int_\Omega (p\nabla u \cdot \nabla v + quv) \, dx = \int_\Omega fv \, dx,
\]
for any sufficiently smooth function $v$.

Now to use Lax-Milgram, we have to set up $V$, $a(\cdot, \cdot)$ and $l(\cdot)$. In order for the two inner products on the left hand side to make sense, we take
\[
V = H^1(\Omega),
\]
\[
a(u, v) = \int_\Omega (p\nabla u \cdot \nabla v + quv) \, dx,
\]
\[
l(v) = \int_\Omega fv \, dx,
\]
for all $u, v \in V$. Note that $V$ is a real Hilbert space with the norm
\[
\|v\|_V = \|v\|_{H^1(\Omega)} = \left(\int_\Omega |\nabla v|^2 \, dx + \int_\Omega v^2 \, dx\right)^{1/2}
\]
and obviously $a(\cdot, \cdot) : V \times V \to \mathbb{R}$ is bilinear and $l(\cdot) : V \to \mathbb{R}$ is linear. Moreover

we observe
\[
a(v, v) = \int_\Omega (p|\nabla v|^2 + qu^2) \, dx
\geq \varepsilon \int_\Omega |\nabla v|^2 \, dx + q_0 \int_\Omega \varepsilon^2 \, dx
\geq \min\{p_0, q_0\} \int_\Omega |v|^2 \, dx
= \alpha \int_\Omega f^2 \, dx,
\]
where we have put $\alpha := \min(p_0, q_0)$. Thus $a(\cdot, \cdot)$ is coercive.
\[
|a(v, w)| = \left|\int_\Omega (p\nabla v \cdot \nabla w + quw) \, dx\right|
\leq \int_\Omega \left(|p\nabla u \cdot \nabla w| + |quw|\right) \, dx
\leq C \int_\Omega \left(||\nabla v|^2 + v^2||^2\right)^{1/2} \left(||\nabla w|^2 + w^2||^2\right)^{1/2} \, dx
\leq C \int_\Omega \left(||\nabla v|^2 + v^2\right)^{1/2} \left(||\nabla w|^2 + w^2\right)^{1/2} \, dx
= C\|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}.
\]
Therefore, $a(\cdot, \cdot)$ is bounded. Finally let us check the boundedness of $l(\cdot)$.
\[
|l(v)| = \left|\int_\Omega fv \, dx\right|
\leq \int_\Omega |f| |v| \, dx
\leq \left(\int_\Omega f^2 \, dx\right)^{1/2} \left(\int_\Omega v^2 \, dx\right)^{1/2}
= ||f||_{L^2(\Omega)} \|v\|_{L^2(\Omega)}
\leq ||f||_{L^1(\Omega)} \|v\|_{H^1(\Omega)}
= C\|v\|_{H^1(\Omega)}.
\]
Thus, $l(\cdot)$ is bounded. Now we can apply Lax-Milgram to prove that there exists a unique solution $u \in V$ to the following problem (P).

(P) Find $u \in V$ such that
\[
a(u, v) = l(v)
\]
for all $v \in V$. 47
4.8 Variational Formulation of Elliptic Equation: Dirichlet Problem

On the same assumptions on \( \Omega, p, q, f \), we consider the following problem.

Find \( u : \Omega \to \mathbb{R} \) such that

\[
\begin{align*}
-\nabla \cdot (p \nabla u) + qu &= f, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

(DBVP)

Let us derive the variational form of (DBVP) as in the section 2.5. Multiply (DBVP) by a sufficiently smooth test function \( v \) and integrate. Then we see

\[
\int_\Omega (p \nabla u \nabla v + qwv) dx - \int_{\partial \Omega} p_u v du dx = \int_\Omega f v dx.
\]

Since we have \( u = 0 \) on \( \partial \Omega \), we have to force our test function \( v \) to satisfy the same condition; \( v = 0 \) on \( \partial \Omega \). Then we obtain

\[
\int_\Omega (p \nabla u \nabla v + qwv) dx = \int_\Omega f v dx
\]

for any sufficiently smooth function \( v \) with \( v = 0 \) on \( \partial \Omega \). Set

\[
V := \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \partial \Omega \} = H^1_0(\Omega).
\]

Note that \( V \) is a real Hilbert space with the inner product

\[
(v, w) := (\nabla v, \nabla w)_{L^2(\Omega)} + (v, w)_{L^2(\Omega)}
\]

and \( \|v\|_V = \|v\|_{H^1(\Omega)} \). As before we define

\[
a(u, v) := \int_\Omega (p \nabla u \nabla v + qwv) dx,
\]

\[
l(v) := \int_\Omega f v dx.
\]

The same argument as the previous section shows that \( a(\cdot, \cdot) \) is a coercive and bounded bi-linear form and \( l(\cdot) \) is a bounded linear functional on \( V \). Therefore Lax-Milgram’s theorem tells us that there uniquely exists a solution to the variational problem of (DBVP).

4.9 Inhomogeneous Boundary Condition

Let \( V \) be a Hilbert space and \( a(\cdot, \cdot) \) be a bilinear coercive form on \( V \times V \), let \( l(\cdot) \) be linear, let \( V_0 \) be a closed subspace of \( V \) and \( g \in V \). Set \( V'_e = \{ v \in V : v = v_0 + g, v_0 \in V_0 \} \) and consider the problem: find \( u \in V'_e \) such that \( a(u, v) = l(v) \forall v \in V_0 \). We can show that there exists a unique solution.

Let \( u_0 = u - g \). Then the problem becomes: find \( u_0 \in V_0 \) such that

\[
a(u_0) = l(v) - a(g, v) \quad \forall v \in V_0.
\]

(4.5)

For the finite element method it becomes: find \( u_h \) such that:

\[
u_h = g_0\phi_0 + \sum_{j=1}^{M-1} a_j\phi_j + g_1\phi_M.
\]

(4.6)

This means that \( A \) is the same as the homogeneous case but \( b \) now has contributions from \( g_0, g_1 \).

4.10 Second Order Elliptic Problems

Consider the problem

\[
- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f(x) \quad \forall x \in \Omega
\]

(4.7)

with \( u = 0 \) on \( \partial \Omega \). Multiply by a test function and integrate by parts in the second order terms using the divergence theorem. The result is the weak (variational) form of the BVP: find \( u \in V \) such that \( a(u, v) = l(v) \forall v \in V \) where \( V = H^1_0(\Omega) \) and

\[
a(u, v) := \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{n} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v dx + \int_\Omega c(x) u v dx,
\]

\[
l(v) := \int_\Omega f(x) v(x) dx.
\]

We seek to apply the Lax-Milgram theorem. Recall \( (v, w)_{H^1_0(\Omega)} = \int_\Omega vw + \nabla v \nabla w = (v, w) + (\nabla v, \nabla w) \). We have three conditions to check to satisfy the theorem.

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(1) Is \( l(\cdot) \) a bounded linear functional? Clearly
\[
\langle l(vw + \beta w), (f, vw + \beta w) \rangle = \langle l(v), f \rangle + \langle \beta l(w), w \rangle
\]
so \( l(\cdot) \) is a linear functional on \( V \) and
\[
|l(v)| = \left| \int_{\Omega} f(x)v(x) \, dx \right| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}
\]
where we have used the Cauchy-Schwartz inequality and thus \( l(\cdot) \) is bounded.

(2) Is \( a(\cdot, \cdot) \) bounded? Assume that \( |a_1|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \|E\|_{L^2(\Omega)} \) are all bounded for all \( i, j \) and that \( f \in L^2(\Omega) \). Then
\[
a(w, v) = \sum_{i,j=1}^n a_{ij} w_i v_j \; dx \leq \sum_{i,j=1}^n \|a_{ij}(x)\| w_i \|v_j\| \; dx + \sum_{i,j=1}^n b_{ij} w_i \|v_j\| \; dx
\]
\[
\leq \tilde{c} \left( \sum_{i,j=1}^n \|w_i\| \|v_j\| \right) + \sum_{i=1}^n \|w_i\| \|v\| \leq \tilde{c} \|w\| \|v\|
\]
\[
\leq \tilde{c} \left( \sum_{i=1}^n \|w_i\| \|v_i\| \right) \leq \tilde{c} \|w\| \|v\|
\]
where \( \tilde{c} = \max(\max_{x \in \Omega} |a_{ij}(x)|, \max_{x \in \Omega} |b_{ij}(x)|, \max_{x \in \Omega} |E(x)|) \) and \( \tilde{c} = \tilde{c}(n^2 + n + 1) \).

(3) Is \( a(\cdot, \cdot) \) coercive? The crucial assumption is that the \( a_{ij} \) satisfies the ellipticity assumption
\[
\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \xi^T A(x) \xi \geq \xi^T \xi
\]
i.e. for all \( x \in \Omega \) we must have
\[
\xi^T A(x) \xi \geq \xi^T \xi
\]
We also assume that
\[
c(x) - \frac{1}{2} \sum_{i=1}^n \frac{\partial b_i(x)}{\partial x_i} \geq \Theta \forall x \in \Omega.
\]
Then
\[
a(v, v) = \sum_{i,j=1}^n a_{ij}(x) v_i v_j + \sum_{i=1}^n \int_{\Omega} b_i(x) v_i v_i + \int_{\Omega} c(x) v(x)^2
\]
\[
\geq \tilde{c} \left( \sum_{i=1}^n v_i^2 \right) + \int_{\Omega} b_{ij}(x) \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} + \int_{\Omega} c v^2.
\]
The middle integral here is \( \frac{1}{2} \int_{\Omega} \nabla \cdot b(x) \nabla v^2 \), which after integration by parts equals
\[
\frac{1}{2} \int_{\Omega} b \nabla \cdot \nabla v^2
\]
so that
\[
a(v, v) \geq \tilde{c} \int_{\Omega} v_i^2 + \int_{\Omega} \nabla^2 v(x) \left( - \frac{1}{2} \nabla \cdot b(x) \right)
\]
\[
\geq \frac{1}{\tilde{c}} \int_{\Omega} v_i^2 = \frac{1}{\tilde{c}} \|v\|^2.
\]
Note that we will need \( \nabla \cdot b \in L^\infty(\Omega) \) for this to work. We wish to show that
\[
a(v, v) \geq c_0 \|v\|^2 = c_0 (\|v\| + \|\nabla v\|).
\]
Recall the Poincare-Friedrichs inequalities
\[
\|v\|^2 \leq c_0 \|\nabla v\|^2 \quad \forall v \in H^1_0(\Omega).
\]
Hence
\[
a(v, v) \geq c_0 \|\nabla v\|^2 \geq \frac{\tilde{c}}{c_0} \|v\|^2.
\]
4.10.1 Remarks on the Lax-Milgram Result

1. Uniqueness: by our usual methods this follows from the linearity of \( l(\cdot) \), the bilinearity of \( a(\cdot, \cdot) \) and the coercivity of \( a(\cdot, \cdot) \).
2. Stability estimate: we know that
\[ c_0|u|^2 \leq a(u, u) = l(u) \leq c_2|u|^2 v \]
so we can deduce that the solution to our BVP satisfies
\[ |u|_{H^1(\Omega)} \leq \frac{1}{c_0}^\frac{1}{2} \|l\|_{L^2(\Omega)}. \quad (4.12) \]

3. Continuity with respect to \( l() \). Consider the two problems
\[ u_1 \in V \quad \text{s.t.} \quad a(u_1, v) = l_1(v) \quad \forall v \in V \]
\[ u_2 \in V \quad \text{s.t.} \quad a(u_2, v) = l_2(v) \quad \forall v \in V \]
Then
\[ a(u_1 - u_2) = l_1(v) - l_2(v) = \hat{l}(v). \quad (4.13) \]
Choosing \( v = u_1 - u_2 \):
\[ c_0\|u_1 - u_2\|_V^2 = \hat{l}(u_1 - u_2) \leq \|l_1 - l_2\|_V \cdot \|u_1 - u_2\|_V \]
\[ \Rightarrow \|u_1 - u_2\|_V \leq \frac{\|l_1 - l_2\|_V}{c_0} \]
In terms of our original elliptic bvp’s we have that
\[ |u_1 - u_2|_{H^1(\Omega)} \leq \frac{1}{c_0}^{\frac{1}{2}} \|l_1 - l_2\|_{L^2(\Omega)}. \quad (4.14) \]

4. If \( l \) is the zero element of \( V^* \) (i.e. \( l(v) = 0 \) \( \forall v \in V \)) then \( 0 = a(u, u) \Rightarrow |u|_V = 0 \) by coercivity and \( u = 0 \).

### 4.10.2 Inhomogeneous Boundary Conditions

Consider the elliptic problem
\[ -\nabla \cdot (p \nabla u) + qu = f \quad x \in \Omega \]
\[ u = g \quad x \in \partial \Omega \]
where \( \Omega \) is a bounded open subset of \( \mathbb{R}^2 \). We assume that the data \( p, q, f, g \) are sufficiently smooth and that
\[ p(x) \geq p_0 > 0 \quad \forall x \in \Omega \]
\[ q(x) \geq q_0 > 0 \quad \forall x \in \Omega \]
Let \( v \) be a test function. Multiply by \( v \) and integrate:
\[ 0 = -\int_{\Omega} \nabla \cdot (p \nabla u) v + \int_{\Omega} q u v - \int_{\Omega} f v \]
\[ = I_1 + I_2 + I_3 \]
Now choosing \( \varphi = v, f = p \nabla u \) in:
\[ \nabla \cdot (vp) = \nabla \cdot f + \nabla \varphi \cdot f \]
\[ \nabla \cdot (vp) = \nabla \cdot (p \nabla u) + \nabla v \cdot p \nabla u \]
\[ I_1 = -\int_{\Omega} \nabla \cdot (vp) - \nabla v \cdot p \nabla u \]
\[ = \int_{\Omega} p \nabla v \cdot \nabla u - \int_{\Omega} p v \nabla u \cdot v \]
Choosing \( v = 0 \) on \( \partial \Omega \) we have \( I_1 = \int_{\Omega} p \nabla v \cdot \nabla u \). Thus
\[ 0 = \int_{\Omega} p \nabla v \cdot \nabla u + q uv - \int_{\Omega} f v \]
Set \( V_0 = H^1_0(\Omega), a(u, v) = \int_{\Omega} p \nabla v \cdot \nabla u + quv, l(v) = \int_{\Omega} f v \) Note that \( u \in V_0 \). However, \( g \in H^1(\Omega) \) so \( u - g \in H^1(\Omega) \) and \( u - g = 0 \) \( \in V_0 \). \( \forall v \in V \in V = H^1(\Omega) \) so \( w = g + v \in V_0 \).

Thus our variational problem \((P)\) is to find \( u \in V_0 \) such that \( a(u, v) = l(v) \) \( \forall v \in V_0 \). Observe that \( V_0 \) is a linear space but \( V = g + V_0 \) is an affine space. We can’t apply Lax-Milgram directly. Consider \( u^* = u - g \in V_0 \):
\[ a(u^*, v) = a(v, v) - l(v) \quad \forall v \in V_0 \]
so \( u^* \in V_0 \) solves
\[ a(u^*, v) = l(v) - a(g, v) =: l^*(v) \quad \forall v \in V_0 \]
Now we just need to check Lax-Milgram for this problem. Clearly \( a(\cdot, \cdot) \) is bilinear (and symmetric). Coercivity:
\[ a(u, v) = \int_{\Omega} p |\nabla u|^2 + quv \geq \int_{\Omega} |\nabla u|^2 + q_0 \int_{\Omega} |v|^2 \geq \min(p_0, q_0) \int_{\Omega} |\nabla u|^2 + |v|^2 \geq c_0 \|v\|_{H^1(\Omega)}^2. \]
Boundeness: using the Cauchy-Schwarz inequality we have
\[ |a(u, v)| = | \int_{\Omega} p \nabla u \cdot \nabla v + quv | \leq p \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + q \|u\|_{L^2} \|v\|_{L^2} \leq c_0 \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \leq c_0 \|\nabla u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \]
Clearly $l^*$ is linear and

$$l^*(v) = l(v) - a(g, v) \leq l(v) + |a(g, v)| \leq \|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)} \leq L^* \|v\|_{H^1(\Omega)}.$$ 

Thus there exists a unique $u^*$ and we conclude therefore that there exists a unique $u = u^* + g$.

The bilinear form is symmetric so there is an energy and associated minimisation problem:

$$J(v) = \frac{1}{2} l^*(v, v) - l(v)$$

Find $u \in V_g \text{ s.t. } J(u) \leq J(v) \forall v \in V_g$.

**Exercise:** Prove that these two problems are equivalent.

### 4.11 Finite Element Method in 2D

Take $\Omega$ to be a polygon. Let $\mathcal{T}_h$ be a triangulation of $\Omega$, $\mathcal{T}_h = \{e\}$ and set $h_h = \text{diam } e$ (the length of the longest side), $h = \max_{e \in \mathcal{T}_h} \text{diam } e$. We assume that $|\mathcal{T}_h| < \infty$.

Any triangles in $\mathcal{T}_h$ must intersect along a complete edge, at a vertex of not at all.

Note that any linear function on $\mathbb{R}^2$ is of the form $v(x, y) = ax + by + c$ and is defined by three parameters. Thus any function $v_h \in V_h := \{\eta \in C(\overline{\Omega}) : \eta|_e \text{ is linear}\}$ is uniquely determined by it values at the vertices of the triangulation:

$$\phi_i(x_j) = \delta_{ij} \quad i, j = 1, \ldots, N_h, \ x_j \text{ is a triangle vertex.}$$

The support of the basis functions is local, so $A$ will again be sparse.

**Example 4.11.1.** Find $u \in H^1(\Omega)$ such that

$$\int_{\Omega} p \nabla u \nabla v + q u v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H^1(\Omega).$$

A finite element method applied to this yields the problem: find $u_h \in H^1(\Omega)$ such that

$$\int_{\Omega} p \nabla u_h \nabla v_h + q u_h v_h \, dx = \int_{\Omega} f v_h \, dx \quad \forall v_h \in V_h.$$