NAPDE Sheet 2

January 2010

1. Given that $f \in L^2(0,1)$, state the weak (variational) formulation of each of the following boundary value problems:

   - a)
   $$-u'' + u = f \text{ for } x \in (0,1),\ u(0) = u(1) = 0$$
   - b)
   $$-u'' + u = f \text{ for } x \in (0,1),\ u(0) = 0, u'(1) = 0$$
   - c)
   $$-u'' + u = f \text{ for } x \in (0,1),\ u(0) = 0, u'(1) + u(1) = 0.$$

   Apply the Lax-Milgram lemma to show that each of the three weak formulations has a (corresponding) unique weak solution. You may wish to use the inequality from part b) of question Sheet 1 Question 3 to prove that the bilinear form associated with part c) is bounded in the function space $H^1_{E_0}(0,1)$.

2. Formulate the **Robin problem**
   $$\begin{aligned} -\Delta u &= f, \quad x \in \Omega, \\ u + \frac{\partial u}{\partial \nu} &= g, \quad x \in \partial\Omega \end{aligned}$$
   where ν is the unit outward pointing normal to $\partial\Omega$ in a variational form and state an equivalent minimization problem. Show that there can be at most one solution.

3. Let $\mathcal{V}$ denote the space $H^1_0(\Omega)$. Show that, for sufficiently smooth functions $v \in \mathcal{V}$
   $$\|v\|^2_{L_2} \le C_p \|\nabla v\|^2_{L_2},$$
   by using the eigenfunctions and eigenvalues of the problem
   $$\begin{aligned} -\Delta\phi &= \lambda\phi, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned}$$
   Hence deduce that
   $$\|\cdot\|_{\mathcal{V}} := \|\nabla \cdot\|_{L^2}$$
   is equivalent to the H^1 norm on $\mathcal{V}$.

4. Give a weak formulation of the **Neumann problem**
   $$\begin{aligned} -\Delta u &= f, \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= g, \quad x \in \partial\Omega, \\ \int_\Omega u\,dx &= 0. \end{aligned}$$
   State a necessary condition on f and g for this equation to have a solution. Deduce that there can be at most one solution if this condition is satisfied.

5. Consider the nonlinear elliptic partial differential equation

$$\begin{aligned} -\Delta u &= f(u), \quad x \in \Omega, \\ u &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{1}$$

Assume that $f(u) = F'(u)$ where

$$\exists C > 0 : \; F(a) - F(b) \leq f(b)(a-b) - C|a-b|^2 \quad \forall a, b \in \mathbb{R}. \tag{2}$$

By use of Green's formula show that weak solutions of (1) satisfy the problem

$$(W) \text{ Find } u \in H_0^1(\Omega) : a(u,v) = \langle f(u), v \rangle, \quad \forall v \in H_0^1(\Omega). \tag{3}$$

Carefully define all the notation contained in the statement of problem (W).

Now define the functional $G : H_0^1(\Omega) \mapsto R$ by

$$G(u) = \frac{1}{2} a(u,u) - (F(u), 1)$$

and the minimization problem

$$(M) \text{ Find } u \in H_0^1(\Omega) : G(u) \leq G(v), \quad \forall v \in H_0^1(\Omega) \tag{4}$$

Using (2) prove that (W) and (M) are equivalent.

6. For given $\alpha \geq 0$, state the weak formulation of the boundary value problem

$$-u'' + u = f \text{ for } x \in (0,1), \; u(0) = 0, u'(1) + \alpha u(1) = 0.$$

Using continuous piece-wise linear basis functions on a uniform mesh of size $h = 1/N, N \geq 2$, write down the finite element approximation to this problem and show that this has a unique solution u_h. Writing

$$u_h(x) = \Sigma_{i=1}^{N} U_i \phi_i(x)$$

where $\phi_i(x) = (1 - |x - x_i|/h)_+, i = 1, ...N$, obtain a system of linear algebraic equations for the vector of unknowns $(U_1, ..., U_N)^T$.

Suppose that $\alpha = 0$, $f(x) = 1$ and $h = 1/3$. Solve the resulting system of linear equations and compare the nodal values of the numerical and exact solutions